On the Exponential Diophantine Equation

\[(4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z\]

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Abstract

Let \( m \) be a positive integer. Then we show that the exponential Diophantine equation \((4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z\) has only the positive integer solution \((x, y, z) = (1, 1, 2)\) under some conditions. The proof is based on elementary methods and Baker’s method.

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1 Introduction.

In 2012, Terai[T1] showed that if \( m \) is a positive integer such that \( 1 \leq m \leq 20 \) or \( m \not\equiv 3 \pmod{6} \), then the Diophantine equation

\[(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z\]

(1.1)
has only the positive integer solution \((x, y, z) = (1, 1, 2)\). The proof is based on elementary methods and Baker’s method. Suy-Li\[SL\] proved that if \(m \geq 90\) and \(3 \mid m\), then equation (1.1) has only the positive integer solution \((x, y, z) = (1, 1, 2)\) by means of the result of Bilu-Hanrot-Voutier [BHV] concerning the existence of primitive prime divisors in Lucas-numbers. Finally, Bertók[?] has completely solved equation (1.1) including the remaining cases \(20 < m < 90\). His proof can be done by the help of exponential congruences. This is a nice application of Bertók and Hajdu[BH].

Then several authors have studied the exponential Diophantine equation
\[
(pm^2 + 1)^x + (qm^2 - 1)^y = (rm)^z
\]
under some conditions, where \(p, q, r\) are positive integers satisfying \(p + q = r^2\):

- (Miyazaki-Terai[MT], 2014) \((m^2 + 1)^x + (qm^2 - 1)^y = (rm)^z\)
- (Terai-Hibino[TH1], 2015) \((12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^z\)
- (Terai-Hibino[TH2], 2017) \((3pm^2 - 1)^x + (p(p - 3)m^2 + 1)^y = (pm)^z\)
- (Fu-Yang[FY], 2017) \((pm^2 + 1)^x + (qm^2 - 1)^y = (rm)^z\), \(r \mid m\)
- (Pan[P], 2017) \((pm^2 + 1)^x + (qm^2 - 1)^y = (rm)^z\), \(m \equiv \pm 1 \pmod{r}\)
- (Murat[M], 2018) \((18m^2 + 1)^x + (7m^2 - 1)^y = (5m)^z\)
- (Kizildere et al.[KMS], 2018) \[((q + 1)m^2 + 1)^x + (qm^2 - 1)^y = (rm)^z\)
- (Terai[T2], 2020) \((4m^2 + 1)^x + (21m^2 - 1)^y = (5m)^z\)
- (Terai-Shinsho[TS], 2020) \((3m^2 + 1)^x + (qm^2 - 1)^y = (rm)^z\)

In this paper, we consider the exponential Diophantine equation
\[
(4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z
\]
with \(m\) positive integer. Our main result is the following:

**Theorem 1.** Let \(m\) be a positive integer. When \(m\) is odd, we suppose that
\[
m \equiv -1 \pmod{3}, \quad m \equiv 2 \pmod{5} \quad \text{or} \quad m \equiv \pm 1, \pm 2 \pmod{7}.
\]
Then equation (1.2) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

In Section 2, we quote a result on a lower bound for linear forms in two logarithms due to Laurent [L]. In Section 3, we use elementary methods and Baker’s method to show Theorem 1 under the condition (1.3).
2 Preliminaries.

In order to obtain an upper bound for a solution of Pillai’s equation, we need a result on lower bounds for linear forms in the logarithms of two algebraic numbers. We will introduce here some notations. Let \( \alpha_1 \) and \( \alpha_2 \) be real algebraic numbers with \(|\alpha_1| \geq 1\) and \(|\alpha_2| \geq 1\). We consider the linear form

\[
\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,
\]

where \( b_1 \) and \( b_2 \) are positive integers. As usual, the logarithmic height of an algebraic number \( \alpha \) of degree \( n \) is defined as

\[
h(\alpha) = \frac{1}{n} \left( \log |a_0| + \sum_{j=1}^{n} \log \max \{1, |\alpha^{(j)}|\} \right),
\]

where \( a_0 \) is the leading coefficient of the minimal polynomial of \( \alpha \) (over \( \mathbb{Z} \)) and \( (\alpha^{(j)})_{1 \leq j \leq n} \) are the conjugates of \( \alpha \). Let \( A_1 \) and \( A_2 \) be real numbers greater than 1 with

\[
\log A_i \geq \max \left\{ h(\alpha_i), \frac{|\log \alpha_i|}{D}, 1, \frac{1}{D} \right\},
\]

for \( i \in \{1, 2\} \), where \( D \) is the degree of the number field \( \mathbb{Q}(\alpha_1, \alpha_2) \) over \( \mathbb{Q} \). Define

\[
b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.
\]

We choose to use a result due to Laurent [[L], Corollary 2] with \( m = 10 \) and \( C_2 = 25.2 \).

**Proposition 1** ([Laurent][L]). Let \( \Lambda \) be given as above, with \( \alpha_1 > 1 \) and \( \alpha_2 > 1 \). Suppose that \( \alpha_1 \) and \( \alpha_2 \) are multiplicatively independent. Then

\[
\log |\Lambda| \geq -25.2 D^4 \left( \max \left\{ \log b' + 0.38, \frac{10}{D} \right\} \right)^2 \log A_1 \log A_2.
\]

3 Proof of Theorem 1.

In this section, we give a proof of Theorem 1.

3.1 the case \( m = 1 \).

We first show that when \( m = 1 \), equation (1.2) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).
Lemma 1. The equation
\[ 5^x + 44^y = 7^z \]  
has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

Proof. Taking equation (3.1) modulo 3 implies that
\[ 2^x + 2^y \equiv 1 \pmod{3}, \]
so \(x\) and \(y\) are odd. Hence equation (3.1) has only the positive integer solution \((x, y, z) = (1, 1, 2)\) by Lemma 6 of Scott [S], which states that if \(a\) and \(b\) are relatively prime integers greater than one and \(p\) is an odd prime, then the equation \(a^x + b^y = p^z\) has at most one solution in positive integers \((x, y, z)\) when the parities of \(x\) and \(y\) are preassigned, except for the case \((a, b, p) = (3, 10, 13)\) (taking \(a < b\)).

By Lemma 1, we may suppose that \(m \geq 2\).

Lemma 2. In (1.2), \(y\) is odd.

Proof. It follows that \(z \geq 2\) from (1.2). Taking (1.2) modulo \(m^2 \geq 4\) implies that \(1 + (-1)^y \equiv 0 \pmod{m^2}\) and hence \(y\) is odd.

3.2 the case where \(m\) is even.

Lemma 3. If \(m\) is even, then equation (1.2) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

Proof. If \(z \leq 2\), then \((x, y, z) = (1, 1, 2)\) from (1.2). Hence we may suppose that \(z \geq 3\). Taking (1.2) modulo \(m^3\) implies that
\[ 1 + 4m^2x - 1 + 45m^2y \equiv 0 \pmod{m^3}, \]
so
\[ 4x + 45y \equiv 0 \pmod{m}, \]
which is impossible, since \(y\) is odd and \(m\) is even. We therefore conclude that if \(m\) is even, then equation (1.2) has only the positive integer solution \((x, y, z) = (1, 1, 2)\).

3.3 the case where \(m\) is an odd number satisfying (1.3).

By Lemma 3, we may suppose that \(m\) is odd and \(m \geq 3\). Let \((x, y, z)\) be a solution of (1.2).

Lemma 4. If \(m\) is an odd number satisfying (1.3), then \(y = 1\) and \(x\) is odd.
On the exponential Diophantine equation \((4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z\)

**Proof.** Suppose that \(m\) is an odd number satisfying (1.3). We first show that \(x\) is odd and \(z\) is even. The proof is divided into three cases:

(a) \(m \equiv -1 \pmod{3}\), (b) \(m \equiv 2 \pmod{5}\), (c) \(m \equiv \pm1, \pm2 \pmod{7}\).

(a) \(m \equiv -1 \pmod{3}\). Taking (1.2) modulo 3 implies that \((-1)^x + (-1)^y \equiv (-1)^z \pmod{3}\).

Since \(y\) is odd from Lemma 2, we see that \(x\) is odd and \(z\) is even.

(b) \(m \equiv 2 \pmod{5}\). Taking (1.2) modulo 5 implies that \(2^x + (-1)^y \equiv (-1)^z \pmod{5}\).

Since \(y\) is odd, we have \(2^x \equiv 1 + (-1)^z \pmod{5}\). This shows that \(z\) is even.

Then \(1 = \left(\frac{2}{5}\right)^{x-1} = (-1)^{x-1}\), where \(\left(\frac{\star}{\star}\right)\) denotes the Jacobi symbol. Hence \(x\) is odd.

(c) \(m \equiv \pm1, \pm2 \pmod{7}\). Since \(m^2 \equiv 1 \pmod{7}\) or \(m^2 \equiv 4 \pmod{7}\), taking (1.2) modulo 7 implies that

\[
5^x + 2^y \equiv 0 \pmod{7} \quad \text{or} \quad 3^x + 4^y \equiv 0 \pmod{7},
\]

that is, \(\left(\frac{5}{7}\right)^x = \left(\frac{-2^y}{7}\right)\) or \(\left(\frac{3}{7}\right)^x = \left(\frac{-4^y}{7}\right)\). This shows that \(x\) is odd.

In these cases, we see that \(\left(\frac{45m^2 - 1}{4m^2 + 1}\right) = 1\) and \(\left(\frac{7m}{4m^2 + 1}\right) = -1\).

Indeed,

\[
\left(\frac{45m^2 - 1}{4m^2 + 1}\right) = \left(\frac{m^2 - 12}{4m^2 + 1}\right) = \left(\frac{4m^2 + 1}{m^2 - 12}\right) = \left(\frac{7^2}{m^2 - 12}\right) = 1
\]

and

\[
\left(\frac{7m}{4m^2 + 1}\right) = \left(\frac{7}{4m^2 + 1}\right) \left(\frac{m}{4m^2 + 1}\right) = \left(\frac{4m^2 + 1}{7}\right) \left(\frac{4m^2 + 1}{m}\right) = -1,
\]

since \(m^2 \equiv 1, 4 \pmod{7}\). Hence \(z\) is even from (1.2).

Suppose that \(y \geq 2\). Taking (1.2) modulo 8 implies that

\[
5^x \equiv (7m)^z \equiv 1 \pmod{8},
\]

so \(x\) is even, which contradicts the fact that \(x\) is odd as seen from the above. Consequently we obtain \(y = 1\). \(\square\)
From Lemma 4, it follows that \( y = 1 \) and \( x \) is odd. If \( x = 1 \), then we obtain \( z = 2 \) from (1.2). From now on, we may suppose that \( x \geq 3 \). Hence our theorem is reduced to solving Pillai’s equation

\[ c^z - a^x = b \]  \hspace{1cm} (3.2)

with \( x \geq 3 \), where \( a = 4m^2 + 1 \), \( b = 45m^2 - 1 \) and \( c = 7m \).

We now want to obtain a lower bound for \( x \).

**Lemma 5.** \( x \geq \frac{1}{4} (m^2 - 45) \).

**Proof.** Since \( x \geq 3 \), equation (3.2) yields the following inequality:

\[ (7m)^z = (4m^2 + 1)^z + 45m^2 - 1 \geq (4m^2 + 1)^3 + 45m^2 - 1 > (7m)^3. \]

Hence \( z \geq 4 \). Taking (3.2) modulo \( m^4 \) implies that

\[ 1 + 4m^2 x + 45m^2 - 1 \equiv 0 \pmod{m^4}, \]

so \( 4x + 45 \equiv 0 \pmod{m^2} \). Hence we obtain our assertion. \( \square \)

We next want to obtain an upper bound for \( x \).

**Lemma 6.** \( x < 2521 \log c \).

**Proof.** From (3.2), we now consider the following linear form in two logarithms:

\[ \Lambda = z \log c - x \log a \quad (> 0). \]

Using the inequality \( \log(1 + t) < t \) for \( t > 0 \), we have

\[ 0 < \Lambda = \log(\frac{c^z}{a^x}) = \log(1 + \frac{b}{a^x}) < \frac{b}{a^x}. \] \hspace{1cm} (3.3)

Hence we obtain

\[ \log \Lambda < \log b - x \log a. \] \hspace{1cm} (3.4)

On the other hand, we use Proposition 1 to obtain a lower bound for \( \Lambda \). It follows from Proposition 1 that

\[ \log \Lambda \geq -25.2 \left( \max \{ \log b' + 0.38, 10 \} \right)^2 (\log a) (\log c), \] \hspace{1cm} (3.5)

where \( b' = \frac{x}{\log c} + \frac{z}{\log a} \).

We note that \( a^{x+1} > c^z \). Indeed,

\[ a^{x+1} - c^z = a(c^z - b) - c^z = (a-1)c^z - ab \geq 4m^2 \cdot 49m^2 - (4m^2+1)(45m^2-1) > 0. \]
On the exponential Diophantine equation \((4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z\)

Hence \(b' < \frac{2x + 1}{\log c} \).

Put \(M = \frac{2x + 1}{\log c}\). Combining (3.4) and (3.5) leads to

\[
x \log a < \log b + 25.2 \left( \max \left\{ \log \left( 2M + \frac{1}{\log c} \right) + 0.38, 10 \right\} \right)^2 (\log a) (\log c),
\]

so

\[
M < 1 + 25.2 \left( \max \left\{ \log \left( 2M + \frac{1}{2} \right) + 0.38, 10 \right\} \right)^2,
\]

since \(\log c = \log(7m) \geq \log 21 > 2\). We therefore obtain \(M < 2521\). This completes the proof of Lemma 6.

We are now in a position to prove Theorem 1. It follows from Lemmas 5, 6 that

\[
\frac{1}{4} (m^2 - 45) < 2521 \log(7m).
\]

Hence we obtain \(m \leq 276\). From (3.3), we have the inequality

\[
\left| \frac{\log a}{\log c} - \frac{z}{x} \right| < \frac{b}{xa^2 \log c},
\]

which implies that \(\left| \frac{\log a}{\log c} - \frac{z}{x} \right| < \frac{1}{2x^2} \), since \(x \geq 3\). Thus \(\frac{z}{x}\) is a convergent in the simple continued fraction expansion to \(\frac{\log a}{\log c}\).

On the other hand, if \(\frac{p_r}{q_r}\) is the \(r\)-th such convergent, then

\[
\left| \frac{\log a}{\log c} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1} + 2)q_r^2},
\]

where \(a_{r+1}\) is the \((r + 1)\)-st partial quotient to \(\frac{\log a}{\log c}\) (see e.g. Khinchin [K]).

Put \(\frac{z}{x} = \frac{p_r}{q_r}\). Note that \(q_r \leq x\). It follows, then, that

\[
a_{r+1} > \frac{a^x \log c}{bx} - 2 \geq \frac{a^{q_r} \log c}{bq_r} - 2. \quad (3.6)
\]

Finally, we checked by Magma [BC] that inequality (3.6) does not hold for any \(r\) with \(q_r < 2521 \log(7m)\) in the range \(3 \leq m \leq 276\).

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On the exponential Diophantine equation $(4m^2 + 1)^x + (45m^2 - 1)^y = (7m)^z$


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