Real Division Algebras with Central Idempotent Satisfying \((x^2, y^2, x^2) = 0\)

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Abstract

We show that every real division algebra, with a non-zero central idempotent, satisfying \((x^2, y^2, x^2) = 0\) is flexible and isomorphic to either a commutative division algebra of dimension \(\leq 2\), a scalar isotope of a mutation of the quaternion algebra \(\mathbb{H}\) or a kind of isotope of the octonion algebra \(\mathbb{O}\).
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1. Introduction

One of the fundamental results about real division algebras is the \((1, 2, 4, 8)\)-theorem. It is proved partially by Hopf [H 40] then finished by Kervaire [K 58] and, independently, Milnor-Bott [BM 58]. It states that if the real space \(\mathbb{R}^n\) possess a bilinear product without divisors of zero, then \(n = 1, 2, 4\) or 8 [HKR 91].

It is well known that \(\mathbb{R}\) (real numbers), \(\mathbb{C}\) (complex numbers), \(\mathbb{H}\) (quaternions) classify all associative real division algebras [Fr 1878], [HKR 91]. Also \(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\) (octonions) classify all alternatives real division algebras [Z 31], [HKR 91]. These last two results have been established by purely algebraic arguments.

A result established by means of differential geometry reinforced the theory. Let \(\mathcal{A}\) be a real division algebra of dimension \(\geq 2\) with a norm \(\|\cdot\|\) and let \(S\) be its unit-sphere. It is proved that if \(\mathcal{A}\) contains an unit-element then the mapping \(S \to S, \ x \mapsto x^2 \|x^2\|\) has degree 2 [Y 81, Lemma 1] and therefore surjective [Ha 02, 2.2 (b) p. 134].

A classification of all quadratic flexible real division algebras has been given by means of the \((1, 2, 4, 8)\)-theorem and vector isotopy from algebras \(\mathbb{H}, \mathbb{O}\) [CDKR 99]. Such an algebras are isomorphic to either \(\mathbb{R}, \mathbb{C}\), a mutation \(\mathbb{H}(\lambda)\) \((\lambda \in \mathbb{R} \setminus \{\frac{1}{2}\})\) of the quaternion algebra or a vector isotope \(\mathbb{O}(\varphi)\) of the octonion algebra.

Motivated by these results, we set out to study those real division algebras without unit satisfying the identity \((x^2, y^2, x^2) = 0\).

We introduce, in section 2, the basic tools especially a reminder on quadratic algebras.

Section 3 is devoted for the definition and properties of a notion of isotopy over quadratic algebras.

Our paper ends in section 4 with our main result.

2. Notations and definitions

Let \(\mathcal{A}\) be a non-associative algebra over the field \(\mathbb{R}\) of real numbers. We denote by \(L_x, (x, y, z), [x, y], \mathcal{A}(x)\), respectively, the operator of left-multiplication
by $x \in \mathcal{A}$, the associator of $x, y, z \in \mathcal{A}$, the commutator of $x, y \in \mathcal{A}$ and the subalgebra generated by $x \in \mathcal{A}$.

An element $x \in \mathcal{A}$ is said to be central if $[x, \mathcal{A}] = 0$. The algebra $\mathcal{A}$ is said to be flexible (resp. third-power associative, power-commutative) if it satisfies $(x, \mathcal{A}, x) = 0$ (resp. $(x, x, x) = 0$, $\mathcal{A}(x)$ is commutative) for all $x \in \mathcal{A}$. It is said to be a division algebra if it is finite-dimensional and $L_x$ is bijective for all non-zero $x \in \mathcal{A}$.

Let $\lambda$ be a real number. The algebra $\mathcal{A}(\lambda)$ obtained from the space $\mathcal{A}$ by putting $x^\lambda y = \lambda xy + (1 - \lambda)yx$ is called the mutation $\lambda$ of $\mathcal{A}$.

The algebra $\mathcal{A}$ is said to be quadratic if it contains an unit-element $e$ and $e, x, x^2$ are linearly dependent for all $x \in \mathcal{A}$.

It is well known [Os 62] that every quadratic real algebra $\mathcal{A}$, with unit $e$, is obtained from an anti-commutative one, $(V, \wedge)$ and a bilinear form $(\ldots, \ldots)$ over $V$ by providing space $\mathbb{R}e \oplus V$ with the product:

$$(\alpha e + x)(\beta e + y) = (\alpha \beta + (x, y))e + (\alpha y + \beta x + x \wedge y).$$

Space $V$ is none other than $\{x \in \mathcal{A} : x^2 \in \mathbb{R}e, x \notin (\mathbb{R}e \setminus \{0\})\}$ [Os 62], [BBO 82] noted $Im(\mathcal{A})$ [HKR 91]. The bilinear form is extended to all of $\mathcal{A}$ by defining $(e, e) = 1$ and $(e, x) = 0$ for $x \in Im(\mathcal{A})$ [BBO 82]. $(Im(\mathcal{A}), \wedge)$ is called the associated anti-commutative algebra of $\mathcal{A}$. The algebra $\mathcal{A}$ is denoted by $(Im(\mathcal{A}), \wedge, (\ldots, \ldots))$ [Os 62]. If, in addition, $\mathcal{A}$ has no zero divisors then $Im(\mathcal{A}) = \{x \in \mathcal{A} : x^2 = -\omega e, \omega > 0\}$ [HKR 91, p. 224].

3. Scalar vector isotopy

Let $\mathcal{A}$ be a real quadratic algebra with unit $e$ and let $\lambda$ be fixed in $\mathbb{R} \setminus \{0\}$ then the linear mapping

$$\Psi : \mathcal{A} = \mathbb{R}e \oplus Im(\mathcal{A}) \to \mathcal{A} \quad ae + u \mapsto ae + \lambda u$$

is bijective. The algebra $\lambda \mathcal{A}$ with underlying space $\mathcal{A}$ and product given by

$$x \circ y = \Psi(x)\Psi(y)$$

is called the scalar isotope of $\mathcal{A}$ determined by $\lambda$. It is showed that if $\mathcal{A}$ is flexible then $\lambda \mathcal{A}$ is flexible [BBO 82].

Let $\lambda, \lambda' \in \mathbb{R} \setminus \{0\}$. It is clear that $\lambda \mathcal{A} = \mathcal{A}$ and $\lambda'(\lambda' \mathcal{A}) = \lambda' \mathcal{A}$. The following proposition solves the isomorphism problem for scalars isotopes of quadratic real algebras:

**Proposition 1.** Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a real quadratic division algebra with unit element $e$ (resp. $e'$) and let $\lambda, \lambda' \in \mathbb{R} \setminus \{0\}$. We assume that the only central element of $\mathcal{A}$ (resp. $\mathcal{B}$) is a scalar multiple of its unit element. Then the following two statements are equivalent:

1. $\lambda \mathcal{A}, \lambda' \mathcal{B}$ are isomorphic,
2. $\lambda' = \lambda$ and $\mathcal{A}, \mathcal{B}$ are isomorphic.
Proof. (1) ⇒ (2). Let \( \Psi_\lambda \) (resp \( \Lambda_{\lambda'} \)) be the automorphism of the space \( \mathcal{A} \) (resp. space \( \mathcal{B} \)) defined by \( \Psi_\lambda(\alpha e + u) = \alpha e + \lambda u \) (resp. \( \Lambda_{\lambda'}(\alpha e' + v) = \alpha e' + \lambda' v \)) and let \( \ast \) (resp. \( \ast' \)) be the product in algebra \( \lambda \mathcal{A} \) (resp. \( \lambda' \mathcal{B} \)). The juxtaposition refers to the products of algebras \( \mathcal{A} \) and \( \mathcal{B} \).

Clearly, \( e \) is a non-zero central element of algebra \( \lambda \mathcal{A} \).

Now, if \( \Phi : \lambda \mathcal{A} \to \lambda' \mathcal{B} \) is an isomorphism of algebras then \( \Phi(e) \) is a non-zero central element of algebra \( \lambda' \mathcal{B} \). By hypothesis, there is a non-zero \( \gamma \in \mathbb{R} \) such that \( \Phi(e) = \gamma e' \). We deduce easily, from equality

\[
\Phi(e \ast e) = \Phi(e) \ast' \Phi(e)
\]

that \( \gamma = 1 \) and then \( \Phi(e) = e' \).

Now, let \( u \) be non-zero in \( \text{Im}(\mathcal{A}) \) then \( u^2 = -\omega e \) with \( \omega > 0 \) and we have:

\[
\left( \Lambda_{\lambda'}\Phi(u) \right)^2 = \Phi(u) \ast' \Phi(u) \\
= \Phi(u \ast u) \\
= \Phi(\lambda^2 u^2) \\
= -\lambda^2 \omega e'.
\]

Consequently \( \Lambda_{\lambda'}\Phi(\text{Im}(\mathcal{A})) \subseteq \text{Im}(\mathcal{B}) \) and we also have:

\[
\Phi(\text{Im}(\mathcal{A})) \subseteq \Lambda_{\lambda'}^{-1}\text{Im}(\mathcal{B}) = \text{Im}(\mathcal{B}).
\]

Thus \( \Phi \) commutes with \( \Psi_\lambda \). Now, for every \( u \in \text{Im}(\mathcal{A}) \) we have:

\[
\Phi(\lambda u) = \Phi(\Psi_\lambda(u)) \\
= \Lambda_{\lambda'}(\Phi(u)) \text{ because } \Phi(e) = e' \\
= \lambda'\Phi(u) \text{ because } \Phi(u) \in \text{Im}(\mathcal{B}).
\]

So \( \lambda' = \lambda \). Now, for every \( x \in \mathcal{A} \), we have:

\[
\Phi(\Psi_\lambda(x)) = \Phi(x \ast e) \\
= \Phi(x) \ast \Phi(e) \\
= \Lambda_{\lambda}(\Phi(x)) \text{ because } \Phi(e) = e'.
\]

So \( \Phi \circ \Psi_\lambda = \Lambda_{\lambda} \circ \Phi \). Now, for every \( x, y \in \mathcal{A} \), we have:

\[
\Phi(\Psi_\lambda(x)\Psi_\lambda(y)) = \Phi(x \ast y) \\
= \Phi(x) \ast \Phi(y) \\
= \Lambda_{\lambda}(\Phi(x))\Lambda_{\lambda}(\Phi(y)) \\
= \Phi(\Psi_\lambda(x))\Phi(\Psi_\lambda(y)).
\]

This shows that \( \Phi(x'y') = \Phi(x')\Phi(y') \) for all \( x', y' \in \mathcal{A} \) because \( \Psi_\lambda \) is bijective.
Real division algebras with central idempotent satisfying \((x^2, y^2, x^2) = 0\)

(2) ⇒ (1). If \(\lambda' = \lambda\) and \(\Phi : A \to B\) is an isomorphism of algebras then we show in the same way that \(\Phi : \lambda A \to \lambda B\) be an isomorphism of algebras. □

Now, let \((Im(A), \wedge), (\cdot, \cdot)\) be the associated anti-commutative algebra of \(A\) and a negative definite symmetric bilinear form over \(Im(A)\), respectively. We consider an automorphism \(\varphi\) of the euclidean space \((Im(A), -(\cdot, \cdot))\) and we put

\[ x\Delta y = \varphi^*(\varphi(x) \wedge \varphi(y)), \quad x, y \in Im(A) \]

where \(\varphi^*\) is the adjoint automorphism of \(\varphi\) and denote by \((Im(A), (\cdot, \cdot), \wedge), (Im(A), (\cdot, \cdot), \Delta)\), respectively, the real quadratic algebras constructed from the anti-commutative algebras \((Im(A), \wedge)\) \((Im(A), \Delta)\) and the symmetric bilinear form \((\cdot, \cdot)\). The algebra \((Im(A), (\cdot, \cdot), \Delta)\) is said to be obtained from \(A = (Im(A), (\cdot, \cdot), \wedge)\) and \(\varphi\) by vector isotopy \([CDKR 99, Remark 3.4. 1]\). It is denoted by \(A(\varphi)\). We will denote by \(\tilde{\varphi}\) the endomorphism which prolongs \(\varphi\) in all \(A\) defined by \(\tilde{\varphi}(ae + u) = ae + \varphi(u)\).

**Definition 1.** Let \(\lambda, (Im(A), \wedge), (\cdot, \cdot), \varphi\) be, respectively, a non-zero real number, the associated anti-commutative algebra of \(A\), a negative definite symmetric bilinear form over \(Im(A)\) and an automorphism of the euclidean space \((Im(A), (\cdot, \cdot))\). By composing vector isotopy and scalar isotopy we obtain, from \(A = (Im(A), (\cdot, \cdot), \wedge)\) and \((\lambda, \varphi)\) a new algebra \(\lambda A(\varphi)\) called scalar vector isotope of \(A\) determined by \((\lambda, \varphi)\).

### 4. Main result

Let now \(A\) be a real division algebra of dimension \(n\) whose underlying real space can be taken to be equal to \(\mathbb{R}^n\). We will denote by \(||\cdot||\) any norm over the space \(\mathbb{R}^n\) and by \(S^{n-1}\) its unit-sphere.

We have the following key result:

**Lemma 1.** Assume that \(A\) contains a non-zero central idempotent \(e\). Then the well defined continuous mapping \(\Phi : S^{n-1} \to S^{n-1}\) \(x \mapsto \|x\|^2 \cdot x^2\) is surjective.

**Proof.** We define on the vector space \(A\) a product \(x \circ y = L_e^{-1}(x)L_e^{-1}(y)\) and obtain a new real division algebra \((A, \circ)\) with unit element \(e\). So the continuous mapping \(\Psi : S^{n-1} \to S^{n-1}\) \(x \mapsto \|x \circ x\|^2 \cdot x \circ x\) is of degree 2 \([Y 81, Lemma 1]\). It follows that \(\Psi\) is surjective \([Ha 02, 2.2 (b) p. 134]\) and so is \(\Phi = \Psi \circ L_e\). □

**Corollary 1.** Every real division algebra \(A\) having a non-zero central idempotent and satisfying \((x^2, y^2, x^2) = 0\) is flexible.

**Proof.** Let \(x, y \in A\). According to Lemma 1 there exists \(a_x, a_y \in A\) such that \(a_x^2 = x\) and \(a_y^2 = y\). Thus \((x, y, x) = (a_x^2, a_y^2, a_x^2) = 0\). So \(A\) is flexible. □

It is convenient to state the following main result taken from \([BBO 82]\):
Theorem 1. If \( \mathcal{A} \) is a finite-dimensional real algebra, then \( \mathcal{A} \) is a flexible division algebra if and only if \( \mathcal{A} \) has one of the following forms:

1. \( \mathcal{A} \) is a commutative division algebra of dimension 1 or 2,
2. \( \mathcal{A} \) is isomorphic to a scalar isotope \( \lambda \mathcal{B} \) of some quadratic real division algebra \( \mathcal{B} \) which is flexible (and hence noncommutative Jordan), or
3. \( \mathcal{A} \) is a generalized pseudo-octonion algebra.

We will use the notation \( \mathcal{O} = (\text{Im}(\mathcal{O}), \wedge, (\cdot,\cdot)) \) expressing the quadratic algebra \( \mathcal{O} \) with its anti-commutative associated algebra and its defined negative symmetric bilinear form. We denote by \( E \) the euclidian space \( (\text{Im}(\mathcal{O}), -(\cdot,\cdot)) \).

We now state our main result:

Theorem 2. Every real division algebra \( \mathcal{A} \) having a non-zero central idempotent and satisfying \((x^2, y^2, x^2) = 0\) is flexible. It is isomorphic to either a commutative division algebra of dimension \( \leq 2 \) or a scalar isotope \( \lambda \mathcal{B} \) (\( \lambda \in \mathbb{R} \setminus \{0\} \)) of some quadratic flexible real division algebra \( \mathcal{B} \). Concretely

1. If \( \dim \mathcal{A} = 4 \) then \( \mathcal{A} \) is isomorphic to \( \lambda \mathcal{H}(\alpha) \) for some real numbers \( \lambda, \alpha \) with \( \lambda \neq 0 \) and \( \alpha \neq \frac{1}{2} \). Moreover, \( \lambda \mathcal{H}(\alpha), \lambda \mathcal{H}(\alpha') \) are isomorphic \((\lambda, \alpha' \in \mathbb{R} \setminus \{0\}, \alpha, \alpha' \in \mathbb{R} \setminus \{\frac{1}{2}\})\) if and only if \( \lambda' = \lambda \) and either \( \alpha' = \alpha \) or \( \alpha' = 1 - \alpha \).
2. If \( \dim \mathcal{A} = 8 \) then \( \mathcal{A} \) is isomorphic to \( \lambda \mathcal{O}(s) \) with \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( s \) a positive definite symmetric endomorphism of euclidian space \( E \). Moreover, \( \lambda \mathcal{O}(s), \lambda \mathcal{O}(s') \) are isomorphic \((\lambda, \alpha' \in \mathbb{R} \setminus \{0\}, s, s' \) being positive definite symmetric endomorphisms of euclidian space \( E \)\) if and only if \( \lambda' = \lambda \) and there is an automorphism \( \Phi \) of algebra \( \mathcal{O} \) such that \( s' = \Phi^{-1} \circ s \circ \Phi \).

Proof. \( \mathcal{A} \) is flexible by Corollary 1. The well-known generalized pseudo-octonion algebras do not contain any non-zero central idempotent \([BBO 82, \text{Theorem 6.42}]\), thus the second statement follows from Theorem 1. It remains to concretize the structure of \( \mathcal{A} \) in dimensions 4 and 8:

1. If \( \dim \mathcal{A} = 4 \) then \( \mathcal{B} \) is isomorphic to a mutation \( \mathcal{H}(\alpha) \) \((\alpha \in \mathbb{R} \setminus \{\frac{1}{2}\})\) of quaternion algebra \( \mathcal{H} \) \([CDKR 99, \text{Remark 3.4 2}]\). Proposition 1 shows that \( \lambda \mathcal{B} \) is isomorphic to \( \lambda \mathcal{H}(\alpha) \). Thus this case in dimension 4 is concluded by Proposition 1 and \([CDKR 99, \text{Remark 3.4 2}]\).
2. If \( \dim \mathcal{A} = 8 \) then, according to \([CDKR 99, \text{Theorem 5.7}]\), \( \mathcal{B} \) is isomorphic to a vector isotope \( \mathcal{O}(s) \) with \( s \) a positive definite symmetric endomorphism of euclidian space \( E \). Proposition 1 shows that \( \lambda \mathcal{B} \) is isomorphic to \( \lambda \mathcal{O}(s) \). The proof is completed under Proposition 1 and \([CDKR 99, \text{Theorem 5.7}]\).

\[ \square \]

Remark 1. Every 8-dimensional real quadratic flexible division algebra \( \mathcal{A} \) contains 4-dimensional subalgebras \([CDKR 99, \text{Theorem 4.7}]\) and so is \( \lambda \mathcal{A} \) for any non-zero real number \( \lambda \). It is then relevant to know whether an 8-dimensional
real quadratic division algebra contains 4-dimensional subalgebras. This issue has already been raised in [CDKR 99, Remark 4.8].

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