Some Characterizations of Ternary Semigroups

by Bi-Ideals

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Abstract

In this paper we have presented some properties of bi-ideals in ternary semigroups. We have presented some definitions and prepositions on bi-ideals and minimal bi-ideals in a ternary semigroup.

Keywords: Bi-Ideal, 0 –Minimal Bi-Ideal, Ternary Semigroup

1 Introduction and Elementary Concepts

D. H. Lehmer [8] gave the definition of a ternary semigroup as follows:

Definition 1.1. A ternary semigroup is a non-empty set $S$ together with a ternary operation which has the property of association:

$$(abc)de = a(bcd)e = ab(cde)$$

for all $a, b, c$ and $d$ in $S$. 

Definition 1.2. A non-empty set $T$ of a ternary semigroup $S$ is called a subsemigroup of $S$ if $a \in T$, $b \in T$ and $c \in T$ imply $abc \in T$.

Definition 1.3. An element $e$ of a ternary semigroup $S$ is called:
(i) left identity of $S$ if $eea = a$ for all $a$ in $S$
(ii) right identity of $S$ if $aee = a$ for all $a$ in $S$
(iii) lateral identity of $S$ if $eae = a$ for all $a$ in $S$
(iv) two sided identity of $S$ if $e$ is left and right identity of $S$
(v) identity of $S$ if $e$ is left, right and lateral identity of $S$.

Definition 1.4. An element $z$ of a ternary semigroup $S$ is called a zero element of $S$ if $zab = zza = zaz = azb = abz = azz = z$ for all $a, b$ in $S$.

Let $S$ be any ternary semigroup and 1 a fixed element in $S$. We extend the ternary operation of $S$ to $S \cup 1$ defining $111 = 1$ and $11a = 1a1 = a11 = a$ for all $a$ in $S$. In this way we have attached to $S$ the identity element 1. Similarly we attach to $S$ the zero element 0 defining $000 = 0ab = a0b = ab0 = 0$ for all $a, b, c$ in $S$. So we have:

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup 1 & \text{otherwise} \end{cases}$$
$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup 0 & \text{otherwise} \end{cases}$$

F. M. Sioson [7] gave the following definitions of ideals as follows:

Definition 1.5. A non-empty subset $A$ of a ternary semigroup $S$ is called:
(i) left ideal of $S$ if $SSA \subseteq A$
(ii) right ideal of $S$ if $ASA \subseteq A$
(iii) lateral ideal of $S$ if $SAS \subseteq A$
(iv) two sided ideal of $S$ if $A$ is a left and right ideal of $S$
(v) ideal of $S$ if $A$ is a left, right and lateral ideal of $S$.

Proposition 1.6. Let $S$ be a ternary semigroup, $L$ a left ideal of $S$, $R$ a right ideal of $S$ and $M$ a lateral ideal of $S$. Then, $LMR$ is a two sided ideal of $S$. Moreover $RML \subseteq R \cap M \cap L$.

Proof. Let $a \in L, b \in M, c \in R$ and $s_1, s_2 \in S$. Then, $s_1s_2(abc) = (s_1s_2a)bc \in LMR$ since $s_1s_2a \in L$ due to the fact that $L$ is a left ideal of $S$. Thus $LMR$ is a left ideal of $S$. We have also that $(abc)s_1s_2 = ab(cs_1s_2) \in LMR$ since $cs_1s_2 \in R$ due to the fact that $R$ is a right ideal of $S$. Thus $LMR$ is a right ideal of $S$. Now let we show that $RML \subseteq R \cap M \cap L$. Let $a \in R, b \in M$ and $c \in L$. Then, $abc \in R$ since $R$ is a right ideal of $S$. We have also that $abc \in M$ since $M$ is a lateral ideal of $S$. On the other side $abc \in L$ since $L$ is a left ideal of $S$. This implies that $abc \in R \cap M \cap L$.

**Definition 1.7.** A non-empty subset $Q$ of a ternary semigroup $S$ is called a quasi-ideal of $S$ if $QSS \cap SQS \cap SSS \subseteq Q$ and $QSS \cap SQS \cap SSQ \subseteq Q$.

The notion of regularity was introduced and studied by J. von Neumann \cite{10} in 1936.

**Definition 1.8.** A ternary semigroup $S$ is called regular if for all $a \in S$ exist $x, y \in S$ such that $a = axya$.

## 2 Bi-Ideals in Ternary Semigroups

Good and Hughes \cite{11} introduced the notion of bi-ideal.

**Definition 2.1.** A ternary subsemigroup $B$ of a ternary semigroup $S$ is called bi-ideal of $S$ if $BSBSB \subseteq B$.

**Preposition 2.2.** Every bi-ideal $B$ of a ternary semigroup $S$ with zero $0$ contains $0$.

**Proof.** For all $a \in B$, $a0a0a = 0a = 0000a = 0 \in B$ since $BSBSB \subseteq B$.

**Preposition 2.3.** Every left, right and lateral ideal of a ternary semigroup $S$ is a bi-ideal of $S$.

**Proof.** Let $L$ a left ideal of $S$. Then $LSL \subseteq (SSS)SL \subseteq SSL \subseteq L$. Similarly, if $R$ is a right ideal of $S$ we have $RSR \subseteq RS(SSS) \subseteq RSS \subseteq R$. We also have that if $M$ is a lateral ideal of $S$ then $MSM \subseteq SSMSS \subseteq SMS \subseteq M$.

**Preposition 2.4.** Let $Q$ be a quasi-ideal of a ternary semigroup $S$. Then $Q$ is a bi-ideal of $S$.

**Proof.** $QSSQ \subseteq SQS \cap SQS \cap QSS \subseteq Q$. This implies that $Q$ is a bi-ideal of $S$.

**Note 2.5.** The converse does not hold, in general, that is, a bi-ideal of a ternary semigroup $S$ may not be a quasi-ideal of $S$.

**Remark 2.6.** Since every left, right, and lateral ideal of $S$ is a quasi-ideal of $S$, it follows that every left, right, and lateral ideal of $S$ is a bi-ideal of $S$, but the converse is not true, in general.

**Preposition 2.7.** The intersection of two quasi-ideals $Q_1$ and $Q_2$ of a ternary semigroup $S$ is a bi-ideal of $S$. 

**Note 2.5.** The converse does not hold, in general, that is, a bi-ideal of a ternary semigroup $S$ may not be a quasi-ideal of $S$.
Proof. By Preposition we have $Q_1SQ_1SQ_1 \subseteq Q_1$ and $Q_2SQ_2SQ_2 \subseteq Q_2$. Thus $(Q_1 \cap Q_2)S(Q_1 \cap Q_2)S(Q_1 \cap Q_2) \subseteq Q_1SQ_1SQ_1 \subseteq Q_1$ and $(Q_1 \cap Q_2)S(Q_1 \cap Q_2)S(Q_1 \cap Q_2) \subseteq Q_2SQ_2SQ_2 \subseteq Q_2$. Therefore $(Q_1 \cap Q_2)S(Q_1 \cap Q_2)S(Q_1 \cap Q_2) \subseteq Q_1 \cap Q_2$. This implies that $Q_1 \cap Q_2$ is a bi-ideal of $S$.

Proposition 2.8. The intersection of any set of bi-ideals of a ternary semigroup $S$ is either empty or a bi-ideal of $S$.

Proof. Let $(B_i)_{i \in I}$ be any non-empty set of bi-ideals of $S$. Then $(\cap_{i \in I} B_i)S(\cap_{i \in I} B_i)S(\cap_{i \in I} B_i) \subseteq B_iSB_iSB_i \subseteq B_i$, for all $i \in I$. Thus $(\cap_{i \in I} B_i)S(\cap_{i \in I} B_i)S(\cap_{i \in I} B_i) \subseteq \cap_{i \in I} B_i$ which implies that $\cap_{i \in I} B_i$ is a bi-ideal of $S$.

Preposition 2.9. The intersection of any set of bi-ideals of a ternary semigroup $S$ with zero $0$ is a bi-ideal of $S$.

Proof. Let $(B_i)_{i \in I}$ be any set of bi-ideals of $S$. Since $0 \in B_i$ for all $i \in I$ we have $0 \in \cap_{i \in I} B_i$. Thus $\cap_{i \in I} B_i \neq \emptyset$. Furthermore, the proof can be continued in the same way as in Preposition 2.8.

Preposition 2.10. If $B$ is a bi-ideal of a ternary semigroup $S$ and $T$ is a ternary subsemigroup of $S$, then $B \cap T$ is a bi-ideal of $T$.

Proof. $(B \cap T)T(B \cap T)T(B \cap T) \subseteq BTBTB \subseteq BSBSB \subseteq B$ and $(B \cap T)T(B \cap T)T(B \cap T)T(B \cap T) \subseteq TT TT \subseteq TTTT \subseteq TTT \subseteq T$. These assertions imply that $(B \cap T)T(B \cap T)T(B \cap T)T \subseteq B \cap T$. Thus $B \cap T$ is a bi-ideal of $T$.

Lemma 2.11. If $B$ is a bi-ideal of a ternary semigroup $S$ and $T_1, T_2$ are two ternary subsemigroups of $S$, then $B T_1 T_2, T_1 B T_2$ and $T_1 T_2 B$ are bi-ideals of $S$.

Proof. $(B T_1 T_2)S(B T_1 T_2)S(B T_1 T_2) = B(T_1 T_2 B)(T_1 T_2 B)T_1 T_2 \subseteq BSBSB T_1 T_2 \subseteq BT_1 T_2$. This implies that $B T_1 T_2$ is a bi-ideal of $S$. We also have that $(T_1 T_2)S(T_1 T_2)S(T_1 T_2) = T_1 B(T_2 B T_1 B)(T_2 B T_1 B)T_1 B \subseteq T_1 BSBSB T_1 B \subseteq T_1 BT_2$. This implies that $T_1 BT_2$ is a bi-ideal of $S$. Finally, we have $(T_1 T_2 B)S(T_1 T_2 B)S(T_1 T_2 B) = T_1 T_2 B(T_2 B T_1 B)(T_2 B T_1 B)B \subseteq T_1 T_2 BSBSB \subseteq T_1 T_2 B$. This implies that $T_1 T_2 B$ is a bi-ideal of $S$.

Corollary 2.12. If $B_1, B_2$ and $B_3$ are three bi-ideals of a ternary semigroup $S$, then $B_1 B_2 B_3$ is a bi-ideal of $S$.

Proof. $(B_1 B_2 B_3)S(B_1 B_2 B_3)S(B_1 B_2 B_3) = B_1 B_2 B_3(SB_1 B_2 B_3)(SB_1 B_2 B_3)B_3 \subseteq B_1 B_2 B_3 SB_3 B_3 \subseteq B_1 B_2 B_3$. This implies that $B_1 B_2 B_3$ is a bi-ideal of $S$.
Corollary 2.13. If \( Q_1, Q_2 \) and \( Q_3 \) are three quasi-ideals of a ternary semigroup \( S \), then \( Q_1 Q_2 Q_3 \) is a bi-ideal of \( S \).

Proof. It follows by Preposition 2.4. and Corollary 2.12.

In general, if \( B \) is a bi-ideal of a ternary semigroup \( S \), and \( T \) is a bi-ideal of \( B \), then \( T \) is not a bi-ideal of \( S \), but in particular, we have the following result.

Theorem 2.14. Let \( B \) be a bi-ideal of a ternary semigroup \( S \), and \( T \) a bi-ideal of \( B \) such that \( T^3 = T \). Then \( T \) is a bi-ideal of \( S \).

Proof. Since \( B \) is a bi-ideal of \( S \) we have \( B \subseteq BBSB \subseteq B \), and since \( T \) is a bi-ideal of \( B \) we have \( B \subseteq TBSTBT \subseteq T \). Therefore, \( TSTST = (TTT)STS(TTT) = TT(TSTST)TT \subseteq TT(BBSB)TT \subseteq TTBTT = TTB(TTT) \subseteq T(TBSTBT)B \subseteq TTT = T \).

Preposition 2.15. Let \( A, B, \) and \( C \) be three ternary subsemigroups of a ternary semigroup \( S \) and \( T = ABC \). Then, \( T \) is a bi-ideal if at least one of \( A, B, C \) is a right, a lateral, or a left ideal of \( S \).

Proof. Let \( T = ABC \) and \( A \) a right ideal of \( S \). Then \((ABC)S(ABC)S(ABC) = A(SSS)(SSS)SSBC \subseteq A(SSS)SSBC \subseteq (ASS)BC \subseteq ABC \). Therefore, \( T = ABC \) is a bi-ideal of \( S \). Now, let \( B \) a right ideal of \( S \). Then \((ABC)S(ABC)S(ABC) \subseteq AB(SSS)(SSS)SSC \subseteq AB(SSS)SC \subseteq ABSSC \subseteq ABC \). This implies that \( T = ABC \) is a bi-ideal of \( S \). Finally, let \( C \) be a right ideal of \( S \). We have \((ABC)S(ABC)S(ABC) \subseteq ABC(SSS)(SSS)SS \subseteq ABC(SSS)SS \subseteq ABCSS \subseteq ABC \). Whence, \( T = ABC \) is a bi-ideal of \( S \). Similarly we prove the other two cases.

Let \( X \) be a non-empty set of a ternary semigroup \( S \). The bi-ideal of \( S \) generated by \( X \) we mean the intersection \( (X)_b \) of all bi-ideals of \( S \) containing \( X \) which actually is a bi-ideal of \( S \) in view of Preposition 2.9. If \( X \) is a finite subset of \( S \), then \( (X)_b \) \( [(X)_b, (X)_r, (X)_l, \) or \( (X) \) \] is called a finely generated bi-ideal [ quasi-ideal, left ideal, lateral ideal, right ideal or ideal ] of \( S \).

Preposition 2.16. Let \( T \) be an ideal [ left ideal, lateral ideal, right ideal, quasi-ideal or bi-ideal] of a ternary semigroup \( S \). If \( Y \) is a ternary subsemigroup of \( S \) such that \( SST \cup STS \cup TSS \subseteq Y \subseteq T \), \( SST \subseteq Y \subseteq T \), \( TST \subseteq Y \subseteq T \), \( TSS \subseteq Y \subseteq T \), \( SST \cap (STS \cup SSTSS) \cap TSS \subseteq Y \subseteq T \) or \( TSTST \subseteq Y \subseteq T \) then \( Y \) is an ideal [ left ideal, lateral ideal, right ideal, quasi-ideal or bi-ideal] of \( S \).

Proof. \( SSY \subseteq SST \subseteq STS \cup TSS \subseteq Y \), \( SYS \subseteq STS \subseteq STS \cup TSS \subseteq Y \), \( YSS \subseteq TSS \subseteq STS \cup TSS \subseteq Y \), \( SSY \subseteq ST \subseteq Y \), \( SYS \subseteq STS \subseteq Y \), \( YSS \subseteq TSS \subseteq Y \).
$Y, SSS \cap (SYS \cup SSYS) \cap YSS \subseteq SST \cap (STS \cup SSTSS) \cap TSS \subseteq Y$ or $YSYS \subseteq TSTST \subseteq Y$.

**Theorem 2.17.** If $S$ is a regular ternary semigroup then $BSBSB = B$ for every bi-ideal $B$ of $S$.

**Proof.** Let $S$ be a regular ternary semigroup and $B$ a bi-ideal of $S$. Then, it is evident that $BSBSB \subseteq B$. Let $a \in B$. Since $S$ is regular we have that exist $x, y \in S$ such that $a = axaya$. This implies that $a \in BSBSB$. It follows that $B \subseteq BSBSB$.

**Theorem 2.18.** Let $S$ be a regular ternary semigroup, and $B$ a bi-ideal of $S$. Then, $BSB \subseteq B$.

**Proof.** Let $a \in BSB$. Since $S$ is regular we have that exist $x, y \in S$ such that $a = axaya$. We also have that $a = b_1sb_2$ with $b_1, b_2 \in B$ and $s \in S$. Thus, $a = (b_1sb_2)x(b_1sb_2)y(b_1sb_2) = (b_1sb_2xb_1)(sb_2y)b_1sb_2 \subseteq B(SSS)BSB \subseteq BSBSB \subseteq B$. This implies that $BSB \subseteq B$.

**Theorem 2.19.** Let $S$ be a regular ternary semigroup, and $B$ a bi-ideal of $S$. Then, $B$ is a quasi-ideal of $S$.

**Proof.** $BSS \cap (SBS \cup SSBSS) \cap SSB = BSS(SBS \cup SSBSS)SSB = B(SSS)B(SSS)B \cup B(SSS)SB(SSS)SB \subseteq BSBSB \cup BSSBSSB \subseteq B \cup BSB \subseteq B \cup B = B$.

In view of Lemma 2.11. and Theorem 2.19., we have the following result.

**Theorem 2.20.** If $Q_1, Q_2$ are two ternary subsemigroups and $Q_3$ is a bi-ideal of a ternary semigroup $S$, then $Q_1Q_2Q_3, Q_1Q_3Q_2$ and $Q_3Q_1Q_2$ are quasi-ideals of $S$.

**Theorem 2.21.** If a ternary semigroup $S$ is regular, then the condition $B_g(a) = aSaSa$ holds for every element $a$ of $S$ ( $B_g(a)$ denotes the smallest bi-ideal of $S$ containing $a$).

**Proof.** $B_g(a)SB_g(a)SB_g(a) = (aSaSa)S(aSaSa)S(aSaSa) = a(SaS)SaS(aSaS)a \subseteq aSSaSSS \subseteq aSaSa = B_g(a)$. Consequently, $B_g(a)$ is a bi-ideal of $S$. Since $S$ is regular, there exist $x, y \in S, a = axaya \in aSaSa = B_g(a)$. Let $T$ be a bi-ideal of $S$ such that $a \in T$. Then, $B_g(a) = aSaSa \subseteq TSTST \subseteq T$.

**Proposition 2.22.** Let $R, M$ and $L$ be respectively right, lateral and left ideals of a ternary semigroup $S$. Then, any ternary subsemigroup $B$ of $S$ such that $RML \subseteq B \subseteq R \cap M \cap L$ is a bi-ideal of $S$. 
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Proof. \( BSBSB \subseteq (R \cap M \cap L)S(R \cap M \cap L)S(R \cap M \cap L) \subseteq \text{RSMSL} \subseteq \text{RML} \subseteq B \) so that \( B \) is indeed a bi-ideal.

Proof. Let \( S \) be a ternary semigroup and \( B \) a bi-ideal of \( S \). If the elements of \( B \) are regular, then \( B \) is a quasi-ideal.

Proof. If \( s_1s_2b_1 = s_3b_2s_4 = b_3s_5s_6 \in SSB \cap SBS \cap BSS \) then there are \( x, y \in S \) such that \( b_1x b_1y b_1 = b_1 \). Thus \( s_1s_2b_1 = s_1s_2(b_1x b_1y b_1) = (s_1s_2b_1)x b_1y b_1 = (b_3s_5s_6) x b_1y b_1 = b_3(s_5s_6x) b_1y b_1 \in B(SSS)BSB \subseteq BSBSB \subseteq B \). This implies that \( SSB \cap SBS \cap BSS \subseteq B \). Similarly we show that \( SSB \cap SSBS \cap BSS \subseteq B \). Whence \( SSB \cap (SBS \cup SSBS) \cap BSS \subseteq B \) and \( B \) is a quasi-ideal of \( S \).

3 Minimal and 0 – Minimal Bi-Ideals in Ternary Semigroups

Definition 3.1. A bi-ideal \( U \) of a ternary semigroup \( S \) is a minimal bi-ideal if there is no bi-ideal \( T \) such that \( T \subset U \) (We use \( \subset \) for proper containment.)

Definition 3.2. A non-zero bi-ideal \( U \) of a ternary semigroup \( S = S^0 \) is a minimal bi-ideal if there is no bi-ideal \( T \), with \( \{0\} \subset T \subset U \).

Definition 3.3. A bi-ideal \( B \) of a ternary semigroup \( S = S^0 \) is called nilpotent if there exists an odd positive integer \( n \geq 3 \) such that \( B^n = \{0\} \).

Proposition 3.4. Let \( B \) be a 0 – minimal bi-ideal of a ternary semigroup \( S = S^0 \). Then \( B \) is nilpotent if and only if \( B^3 = \{0\} \).

Proof. Let \( n \geq 3 \) be an odd positive integer. Then since the product of three bi-ideals is a bi-ideal \( B^{n-2} \) is a bi-ideal which is clearly contained in \( B \) and we have \( B^{n-2} = B \) if \( B^{n-2} \neq \{0\} \). Thus \( B^n = B^3 = \{0\} \) precisely when \( B \) is nilpotent.

Definition 3.5. We will call a 0 – minimal bi-ideal \( B \) of a ternary semigroup \( S = S^0 \) a nilpotent 0 – minimal bi-ideal if \( B \) is a zero ternary subsemigroup, i.e., \( B^3 = \{0\} \).

Theorem 3.6. Let \( B \) be a nilpotent 0 – minimal bi-ideal of a ternary semigroup \( S = S^0 \). Then the following statements are equivalent:

1. some non-zero element of \( B \) is irregular
2. no non-zero element of \( B \) is regular
3. for some \( b \in B \setminus \{0\} \), \( bSbSb = \{0\} \)
4. for each \( b \in B \), \( bSbSb = \{0\} \)
( in any of the above cases B = \{b, 0\};
5. each element in B is regular
6. some non-zero element of B is regular
7. bSbSb \neq \{0\} for each b \in B\{0\}
8. bSbSb \neq \{0\} for some b \in B
( in any of these cases B is a quasi-ideal ).

**Proof.** In any of the above cases one need consider only bSbSb for b \in B. We observe that bSbSb is a bi-ideal contained in B. Thus by the minimality of B either bSbSb = \{0\} or bSbSb = B. In cases 1 or 2 if b is irregular then bSbSb \subset B and hence bSbSb = \{0\}. Clearly \{b, 0\} is then a bi-ideal and hence B = \{b, 0\}. The equivalence of statements 1 – 4 should now be obvious.

Indeed, it is now clear that a non-zero element of B can be regular precisely when each element in B is regular. Furthermore b \neq 0 is regular iff bSbSb \neq \{0\} since in such a case bSbSb = B. It follows that each of the statements 5 – 8 are equivalent and for any of these cases B is a quasi-ideal.

**Definition 3.7.** For a, b \in S, a given ternary semigroup, we write aBb if 1) a = b or 2) there exist u, v, w, z \in S such that auava = b and bwbzb = a.

**Preposition 3.8.** The relation B defined in S is an equivalence relation.

**Preposition 3.9.** If A is a bi-ideal of a ternary semigroup S then A = \bigcup_{a \in A} B_a, i.e., any bi-ideal is the union of its B -classes.

**Preposition 3.10.** Let S be a ternary semigroup with zero 0. If a bi-ideal B is a non-zero B -class union, then it is a 0-minimal bi-ideal of S.

The converse of this preposition is also true as we show in the following:

**Theorem 3.11.** Let S be a ternary semigroup with zero 0. A bi-ideal B is 0-minimal if and only if it is a non-zero B -class union.

**Proof.** Let B be a 0-minimal bi-ideal of S = S^0. Let a, b \in B\{0\}. Since \{b, b^3\} \cup bSbSb and \{a, a^3\} \cup aSaSa are clearly non-zero bi-ideals contained in B we must have B = \{b, b^3\} \cup bSbSb = \{a, a^3\} \cup aSaSa.

Now assume a \neq b. We can proceed from the last equality by cases.
Suppose a = b^3. We have two sub-cases to consider.
1) If also b = a^3 then a = b^3 = aa^3aa^3a = b^3a^3b^3a^3b^3 = b(bba^3)b(bba^3bb)b and also b = a^3 = a(aab^3)a(aab^3aa)a. It follows that aBb.
2) If $b \neq a^3$ we must have $b \in aSaS$ and $b = auava$ for some $u, v \in S$. Then $a = b^3 = (auava)(auava)(auava) = b(bbuavaub)(bvaauavbb)b$. Again it follows that $aBb$.

Now if $a \neq b$ and $a \neq b^3$ we must have $a \in bSbSb$ so that $a = bwzb$ for some $w, z \in S$. again we examine $b$ by cases as above. If $b = a^3$ we have simply case 2) with the roles of $a$ and $b$ interchanged. If $b \in aSaSa$ then $b = auava$ for some $u, v \in S$. In either case it follows that $aBb$. By we may conclude that $B = B_b \cup \{0\}$. The converse is just Preposition 3.10.

**Preposition 3.12.** Let $S$ be a ternary semigroup with $0$. If $R$ is a $0$−minimal right ideal, $M$ is a $0$−minimal lateral ideal and $L$ is a $0$−minimal left ideal, then either $RML = \{0\}$ or $RML$ is a $0$−minimal bi-ideal of $S$.

**Proof.** Suppose $RML \neq \{0\}$ and that there is a bi-ideal $B$ with $\{0\} \subset B \subset RML$. Since $RML \subset R \cap M \cap L$ we have $BSS \subset (RML)SS \subset (R \cap M \cap L)SS \subset RSS \subset R$. Thus, by the minimality of $R$ we have $BSS = R$. We also have that $SBS \subset S(RML) \subset S(R \cap M \cap L)S \subset SMS \subset M$. By the minimality of $M$ it follows that $SBS = M$. Furthermore, we have $SSB \subset SS(RML) \subset SS(R \cap M \cap L) \subset SSL \subset L$ and since $L$ is minimal we have $SSB = L$. Thus $B \subset RML = (BSS)(SBS)(SSB) = B(SSS)B(SSS)B \subset BSBSB \subset B$ which is a contradiction. Hence $RML$ is $0$−minimal bi-ideal.

**Preposition 3.13.** Let $S$ be a ternary semigroup with $0$. If $B$ is a $0$−minimal bi-ideal of $S$ then for any right ideal $R$ contained in $BSS$, any lateral ideal $M$ contained in $SBS$ and any left ideal $L$ contained in $SSB$ we have either $RML = \{0\}$ or $RML = B$.

**Proof.** Let $R \subset BSS, M \subset SBS$ and $L \subset SSB$. Then $RML \subset (BSS)(SBS)(SSB) = B(SSS)B(SSS)B \subset BSBSB \subset B$. Since $RML$ is a bi-ideal and $B$ is $0$−minimal it follows that $RML = \{0\}$ or $RML = B$.

**Preposition 3.14.** Let $S$ be a ternary semigroup. If $B$ is a minimal bi-ideal of $S$ then $BSS$, $SBS$ and $SSB$ are minimal right, lateral and left ideals of $S$ respectively and we have $B = (BSS)(SBS)(SSB)$, i.e., $B$ is the product of a minimal right ideal, a minimal lateral ideal and a minimal left ideal.

**Proof.** Let $R$ be a right ideal, $M$ a lateral ideal and $L$ a left ideal of $S$ with $R \subset BSS, M \subset SBS$ and $L \subset SSB$. Since $RML$ is a bi-ideal with $RML \subset BSSBSSSB \subset BSBSB \subset B$ we must have $B = RML$. But $RML \subset R \cap M \cap L$ and thus $B \subset R$. Hence we have $BSS \subset RSS \subset R$ so that $R = BSS$. Therefore, $BSS$ is a minimal right ideal. Similarly $SSB$ is a minimal left ideal. Now $B \subset M$ implies $SBS \subset SMS \subset M$. Thus $SBS = M$. It follows that $SBS$ is a minimal lateral ideal. Since $B$
is a minimal bi-ideal which contains \((BSS)(SBS)(SSB)\), itself a bi-ideal, we must have \(B = (BSS)(SBS)(SSB)\).

**References**


Received: December 11, 2020; Published: January 8, 2021