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(α, 1)-Reverse Derivations on Prime Near-Rings

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Abstract

Let N be a prime left near- ring with multiplicative center Z and d be a (α, 1)-Reverse derivation. During this paper we have a tendency to generalize a number of the results on near- rings admitting a special style of derivation particularly (α, 1)- Reverse derivation, where α, 1 are automorphism of near- rings. Finally, we have a tendency to prove commutatively theorems in prime near- rings with (α, 1)- Reverse derivation.

Keywords: Prime near- rings, Reverse derivation, (α, 1)- Reverse derivation, (α, 1) commuting derivation

1. Introduction

The concept of reverse derivation was first time introduced [6] by Herstien. Bresar and Vukman [5] have introduced the notion of a reverse derivation. The reverse derivation on prime rings [7] and left (α, 1)- derivation in prime rings [8] by C.Jaya Subba Reddy et.al. H.E.Bell and G.Manson have established some results on commututity of prime near- rings with derivations (See in [2], [3], [4]) and Ahmed et.al. in [9]. Many authors have investigated the properties and results of prime, semiprime and near- rings with (σ, τ)- derivations (See in [1], [11], [12]). Recently some results regarding commutativity in prime near- rings with derivation are generalized in many ways in which.

2. Preliminaries

Throughout this paper, N will denote a zero symmetric left near-ring with multiplicative Centre Z. N is called a prime near- ring if aNb = {0} implies a = 0 or b = 0. Let α and 1 be two near- ring automorphisms of N. An additive mapping d: N→ N is called a (α, 1) reverse derivation if d(xy) = d(y) α(x) + yd(x) holds for all x, y∈ N. For x, y ∈ N, the symbol [x, y] will denote the commutator xy − yx while the symbol (x, y) will denote the additive commutator x + y – x − y. Given x, y ∈N, we write $[x, y]_{\alpha,1}$ = xα(y) – yx. The symbol $[x, y]_{\alpha,1}$ will denote the (α, 1) commutator xα(y) – yx whereas (α, 1)- reverse derivation d are called (α, 1)-commuting if $[x, d(x)]_{\alpha, 1}$ = 0 for all x ∈ N. An element x ∈ N for which d(x) = 0 is called a constant. Some recent results on rings were down commutativity of prime and semi-prime rings admitting appropriately affected reverse derivations. It is natural to seem for comparable results on near-rings and this has been tired [1], [3], [4] and [11]. It is our purpose to increase a number of these results on prime near-rings admitting appropriately affected (α, 1)- reverse derivation.

We begin with the subsequent lemmas that are helpful in sequel.

Lemma 1: An additive endomorphism d on a near-ring N is a (α, 1)- reverse derivation if and only if $d(xy) = yd(x) + d(y)\alpha(x)$, for all x, y ∈ N.

Proof: Let d be a (α, 1)- reverse derivation on a near-ring N.
We have $d(xy) = d(y)\alpha(x) + yd(x)$, for all x, y ∈ N. (1)
Substitute (y + y) within the place of y, we get
$d(x(y+y)) = d(y)\alpha(x) + d(y)\alpha(x) + yd(x) + yd(x)$ (2)
On the other hand, we have $d(xy + xy) = d(xy) + d(xy)$
$= d(y)\alpha(x) + yd(x) + d(y)\alpha(x) + yd(x)$, for all x, y ∈ N. (3)
Comparing (2), (3) and using (1), we get $d(xy) = yd(x) + d(y)\alpha(x)$, for x, y ∈ N.
Conversely, let $d(xy) = yd(x) + d(y)\alpha(x)$, for all x, y ∈ N.
Then $d(x(y+y)) = yd(x) + yd(x) + d(y)\alpha(x) + d(y)\alpha(x)$, for all x, y ∈ N. (4)
Also, $d(xy + xy) = d(y)\alpha(x) + yd(x) + d(y)\alpha(x) + yd(x)$, for all x, y ∈ N. (5)
Comparing (4) and (5), using hypothesis we have $d(xy) = d(y)\alpha(x) + yd(x)$, for all x, y ∈ N. That is (α, 1)- reverse derivation on a near- ring N.

Lemma 2: Let d be a (α, 1)- reverse derivation on the near-ring N. Then N satisfies the following partial distributive laws:

(i) $z\big(d(y)\alpha(x) + yd(x)\big) = zd(y)\alpha(x) + zyd(x)$, for all x, y, z ∈ N.

(ii) $z\big(yd(x) + d(y)\alpha(x)\big) = zyd(x) + zd(y)\alpha(x)$, for all x, y, z ∈ N.

Proof: Let d be a (α, 1)- reverse derivation on the near-ring N.
We have $d(xy) = d(y)\alpha(x) + yd(x)$, for all x, y ∈ N. For any z ∈ N, then
$d((xy)z) = d(z)\alpha(x)\alpha(y) + zd(y)\alpha(x) + zyd(x)$, for all x, y, z ∈ N. (6)

On the other hand, we have $d((xy)z) = d(z)\alpha(y)\alpha(x) + zd(y)\alpha(x) + yzd(x)$ (7)
Comparing (6) and (7), we obtain $zd(y)\alpha(x) + yzd(x) = zd(y)\alpha(x) + zyd(x)$
$z\big(d(y)\alpha(x) + yd(x)\big) = zd(y)\alpha(x) + zyd(x)$, for all x, y, z ∈ N.
In the similar manner, (ii) is established.
Lemma 3: Let d be a (α, 1)- reverse derivation on N and suppose u ∈ N is not a left zero divisor. If $[u, d(u)]_{\alpha, 1} = 0$, then (x, u) is a constant for every x ∈ N.

Proof: Since $u(u + x) = u^2 + ux$, so we obtain $d(u(u + x)) = d(u^2) + d(ux)$
$d(x)\alpha(u) + ud(u) = ud(u) + d(x)\alpha(u)$ (8)
By the hypothesis $[u, d(u)]_{\alpha, 1} = 0 \Longrightarrow x\alpha(d(u)) - d(u)x = 0$
That is $\alpha(d(u)) = d(\alpha(u)) = d(u)$ and $\alpha(u) = u$, for all u ∈ N. (9)
Using (9) in (8), we get $\alpha(u)\big(d(x) + d(u) - d(x) - d(u)\big) = 0$, for all x ∈ N.
That is $\alpha(u)d(x, u) = 0$.
Since α is an automorphism of N, α(u) is not a left-zero divisor. Thus d(x, u) = 0.
Hence (x, u) is constant, for all x ∈ N.

Theorem 1: Let N have no non-zero divisors of zero. If N admits a non-trivial (α, 1)-commuting (α, 1)- reverse derivation d, then (N, +) is abelian.

Proof: Let c be any additive commutator. Then the application of Lemma 3 yields that c is a constant. Moreover, for any x ∈ N, xc is also an additive commutator. Hence xc is a constant.
Thus, 0 = d(xc) = d(c) α(x) + cd(x). i.e., cd(x) = 0, for all x ∈ N and additive commutators c. Since d(x) ≠ 0, for some x ∈ N, so c = 0. Thus c = 0 for all additive commutators c. Hence, (N, +) is abelian.

3. Main Results

Lemma 4: Let N be a prime near-ring.

(i) If z is a non-zero element in Z, then z is not a zero divisor.
(ii) If there exists a non-zero element z of Z such that z + z ∈ Z, then (N, +) is abelian.
(iii) Let d be a non-trivial (α, 1)- reverse derivation on N. Then xd(N) = (0) or d(N)x =(0), implies x = 0.
(iv) If N is 2-torsion free and d is a (α, 1)- reverse derivation on N such that $d^2 = 0$ and α, 1 commute with d, then d = 0.
(v) If N admits a non-trivial (α, 1)- reverse derivation d for which d(N) ⊆ Z, then c ∈ Z for each constant element c of N.

Proof: (i) and (ii) are already proved in [2].
(iii) Let xd(r) = 0, for all r ∈ N. (10)

Replace r by yr in the equation (10), to get xd(r)α(y) + xrd(y) = 0, for all x, y, r ∈ N.
Using (10) in the above relation we get xrd(y) = 0. That is xNd(N) = (0). Since N is prime and d(N) ≠ 0, so we have x = 0.
Arguing as above, we can show that d(r)x = 0, for all r ∈ N, implies that x = 0.
(iv) For arbitrary x, y ∈ N, we have $d^2(xy) = 0$. After a simple calculation, we obtain 2d(α(x))d(y) = 0. Since N is 2-torsion free, so d(α(x))d(N) = (0), for each x ∈ N. Since N is prime and d(N) ≠ 0, so we have d(α(x)) = 0.
Hence d = 0, by using (iii) and the fact that α is an automorphism.
(v) Let c be an arbitrary constant and let x be a non-constant element of N. Then d(x)c = d(xc) ∈ Z for each non-constant element x∈ N.
This implies that d(x) cy = y d(x)c, for all y ∈ N . Since d(x) ∈ Z- {0}, it follows that d(x)cy = d(x)yc, for all y ∈ N and we conclude that d(x)(yc − cy) = 0; for all y ∈ N and additive commutator c. Hence, using (i), we get the required result.

Theorem 2: Let N be a prime near-ring admitting a non-trivial (α, 1)- reverse derivation d for which d(N) ⊆ Z. Then (N, +) is abelian. Moreover, if N is 2-torsion free and α, 1 commute with d, then N is a commutative ring.

Proof: Since d(N) ⊆ Z and d is non-trivial, there exists a non-zero element x in N such that z = d(x) ∈ Z \ {0} and z + z = d(x + x) ∈ Z. Hence (N, +) is abelian by Lemma 4(ii).
Assume now that, N is 2-torsion free and α, 1 commute with d. Application of Lemma 2(i) yields that,

$$z\big(d(y)\alpha(x) + yd(x)\big) = zd(y)\alpha(x) + zyd(x), \text{ for all } x, y, r \in N. \quad (11)$$

Since d(N) ⊆ Z, it follows that d(xy) ∈ Z, for all x, y ∈ N. Thus, $rd(xy) = d(xy)r$, for all x, y, r ∈ N.

$$r\big(d(y)\alpha(x) + yd(x)\big) = \big(d(y)\alpha(x) + yd(x)\big)r = d(y)\alpha(x)r + yd(x)r \quad (12)$$

Equating (11) and (12), we get

$$d(y)(\alpha(x)r - r\alpha(x) = d(x)(ry - yr), \text{ for all } x, y, r \in N. \quad (13)$$

Suppose on contrary that N is not commutative and choose r, y ∈ N with $ry - yr \neq 0$. Let x = d(a), for all a∈ N.
This yields that $\alpha(x) = \alpha\big(d(a)\big) = d(\alpha(a)) \in Z$. (14)
Now using (14) in (13), we get $d(y)(d(\alpha(a))r - rd(\alpha(a))) = d(d(a))(ry - yr)$
i.e., $d^2(a)(ry - yr) = 0$, for all a ∈ N. By Lemma 4(i), we see that the central element $d^2(a)$ cannot be a divisor of zero, we conclude that $d^2(a) \neq 0$, for all a ∈ N. Thus, $ry - yr = 0$, for all r, y ∈ N. Hence N is a commutative ring.

Theorem 3: Let N be a prime near-ring admitting a non-trivial (α, 1)- reverse derivation d such that d(x)d(y) = d(y)d(x), for all x, y ∈ N. Then (N, +) is abelian. Moreover, if N is 2-torsion free and α, 1 commute with d, then N is a commutative ring.

Proof: By our hypothesis d(x)d(y) = d(y)d(x), for all x, y ∈ N. For this we have d(x + x) d(x + y) = d(x + y) d(x + x), for all x, y ∈ N. Implies that d(x) d(x) + d(x) d(y) = d(x) d(x) + d(y) d(x) and hence d(x) d(x, y) = 0, for all x, y ∈ N. i.e., d(x) d(c) = 0, for all x ∈ N and additive commutator c.
Now, application of Lemma 4(iii) yields that d(c) = 0, for all additive commutators c. Since N is a left near-ring and c is an additive commutator, xc is also an additive commutator for any x ∈ N. Hence d(xc) = 0, for all x ∈ N and additive commutator c. Thus by Lemma 4(iii), we have c = 0 and hence (N, +) is abelian.
Now assume that N is 2-torsion free and α, 1 commute with d. Then applications of Lemmas 1 and 2(i) yield that,

$$d(d(x)y)d(z) = yd^2(x)d(z) + d(y)\alpha\big(d(x)\big)d(z). \tag{15}$$

Other hand

$$d(d(x)y)d(z) = d(z)d(d(x)y) = d(z)yd^2(x) + d(z)d(y)\alpha\big(d(x)\big) \tag{16}$$

From (15) and (16), we obtained

$$d^2(x)(yd(z) - d(z)y) = 0, \text{ for all x, y, z} \in N \tag{17}$$

Replacing y by yr in (17), we get $d^2(x)y(rd(z) - d(z)r) = 0$, for all x, y, r ∈ N. Thus $d^2(x)N(rd(z) - d(z)r) = 0$, for all x, y, r ∈ N. Since N is prime, so we have $d^2(x) = 0$ or $rd(z) - d(z)r = 0$. But $d^2(x) \neq 0$ by lemma 4(iv), so we have $rd(z) - d(z)r = 0$. This implies that d(N) ⊆ Z.
Hence N is a commutative ring by theorem 2. Hence the proof is completed.

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