A Note on Finite Group 1-Cohomology via
Semi-Direct Products with Applications to
Permutation Modules

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Abstract

Section 17 of the important textbook [1] of M. Aschbacher studies Finite Group 1-Cohomology with a field coefficient ring via semi-direct products. This approach yields new structures and results to this basic subject.

Here we assume that the coefficient ring is any commutative ring and we obtain all of the results of [1, Section 17] excluding Theorem 17.12. Via duality, this theorem extends the previous main result Theorem 17.11. In our final main results we assume that the coefficient ring is a discrete valuation ring, so that [1, Theorem 17.12] is a special case. Thus all of our results are applicable to Finite Group Modular Representation Theory. We conclude with applications to finite group permutation modules.

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1. Introduction, Analyses, and Main Results

Section 17 of the important textbook [1] “Finite Group Theory” by M. Aschbacher is devoted to a study of Group 1-cohomology with field coefficients via semi-direct products. This approach yields new structures and results in this basic area.
In this article, we assume that the coefficient ring is any commutative ring and we obtain all of the results of [1, Section 17] except for the final main results [1, Theorem 17.12]. This theorem uses duality to extend the previous main result [1, Theorem 17.11]. By restricting the coefficient ring to a discrete valuation ring, our final main result implies [1, Theorem 17.12]. Consequently all of our results are applicable to Finite Group Modular Representation Theory.

In this section, we present our main results and applications. Section 2 contains some required general results.

Our notation and terminology are standard and tend to follow [1], [2], and [3]. All rings have identities and all modules over a ring $R$ are unitary. Also $\text{mod-}R$ will denote the additive category of finitely generated right $R$-modules. Also if $V, W$ are modules in $\text{mod-}R$, then $V | W$ will signify that $V$ is isomorphic in $\text{mod-}R$ to a direct summand of $W$ and similarly for $R$-mod. Unless specifically mentioned, $R$-modules are right $R$-modules.

If $X, Y$ are sets, then $F(X, Y)$ denotes the set of functions from $X$ to $Y$. If $f \in F(X, Y)$ and $x \in X$, then $xf$ will denote the element of $Y$ determined by $x$ and $f$.

In this article, $G$ will denote a finite group and $\mathcal{O}$ will denote a commutative ring (possibly $\mathcal{O} = \mathbb{Z}$ or $\mathcal{O}$ is a field). Also $\mathcal{O}G$ will denote the group algebra of $G$ over $\mathcal{O}$.

Let $X, Y$ be (right) $\mathcal{O}G$-modules. Then

(1.1) The additive group $\text{Hom}_{\mathcal{O}}(X, Y)$ becomes an $\mathcal{O}G$-module where if $\phi \in \text{Hom}_{\mathcal{O}}(X, Y)$, $g \in G$, $a \in \mathcal{O}$ and $x \in X$, then $x(a(\phi^g)) = x((a\phi)^g) = (xg^{-1})\phi ga$.

With $G$ acting trivially on $\mathcal{O}$,

(1.2) $X^* = \text{Hom}_{\mathcal{O}}(X, \mathcal{O})$ becomes an $\mathcal{O}G$-module where if $\phi \in X^*$, $g \in G$, $a \in \mathcal{O}$, and $x \in X$, then $(a\phi)(x) = (xg^{-1})\phi a$.

(1.3) The additive group $F(G, X)$ becomes an ($\mathcal{O}G$)-module where if $\phi \in F(G, X)$, $g, h \in G$ and $a \in \mathcal{O}$, then $g((a\phi)^h) = (gh^{-1})\phi ha$.

Let $W$ be an $\mathcal{O}G$-module and let $GW = G \times W$ denote the semi-direct product of $G$ over $(W, +)$ so that if $g_1, g_2 \in G$ and $w_1, w_2 \in W$, then

$$(g_1, w_1)(g_2, w_2) = (g_1g_2, w_1g_2 + w_2).$$

Here $(1, 0)$ is the identity of $GW$ and if $(g, w) \in GW$, then $(g, w)^{-1} = (g^{-1}, -(wg^{-1}))$.

(1.4) $1 \times W \leq GW$, $(G \times 0)$ is a complement to $1 \times W$ in $GW$ and $GW = (G \times 0)(1 \times W)$.

(1.5) $Z(GW) = C_{GW}((G, 0)) = (Z(G) \cap C_G(W)) \times C_W(G)$. 
Proof. Let \( g \in G \) and \((h, w) \in GW\). Then \((g, 0)(h, w)(g^{-1}, 0) = (ghg^{-1}, wgw^{-1})\). Hence \( C_{GW}(G, 0) \leq (Z(G) \cap C_G(W)) \times C_W(G) \leq C_{GW}(GW) = Z(GW)\) and we are done.

(1.6) \( N_{GW}((G, 0)) = (G, 0)(1 \times C_W(G)) = G \times C_W(G)\) and if \( T \) is a transversal of \( W \) in \( C_W(G) \) so that \( W = \bigoplus_{t \in T} (C_W(G) + t)\), then \( \{(G, 0)^{(1,t)} \mid t \in T\} \) is the \( GW \)-conjugacy class of the subgroup \((G, 0)\) of \( GW\).

Let \([W, G]\) denote the \( OG \)-submodule of \( W \) generated by the elements \( w - wg \) for all \( w \in W \) and \( g \in G \) so that \([W, G] = \{w - wg \mid w \in W \text{ and } g \in G\}\).

We shall frequently identify the group \( G \) with the subgroup \( G \times 0 \) of \( GW \) via the map \( i_G: G \to G \times 0 \) such that \( g \mapsto (g, 0)\) for all \( g \in G \) and identify the group \((W, +)\) with the (multiplicative) group \( 1 \times W \) via the map \( i_W: W \to 1 \times W \) such that \( w \mapsto (1, w)\) for all \( w \in W\).

Let \( \Gamma(G, W) \) denote the subgroup of the (additive) group \( F(G, W) \) of 1-cocycles (cf. [1, p. 65]). That is, \( \gamma \in \Gamma(G, W) \) if \((gh)\gamma = (g\gamma)h + h\gamma\) for all \( g, h \in G\). Here \( 1\gamma = 0 \) for all \( \gamma \in \Gamma(G, W)\).

As on [1, p. 65]:

(1.7) \( \Gamma(G, W) \) is an \( OG \)-submodule of \( F(G, W)\).

From [1, Theorem 17.1] we have:

(1.8) If \( \gamma \in \Gamma(G, W)\), then \( S(\gamma) = \{(g, g\gamma) \mid g \in G\} \) is a complement to \( 1 \times W \) in \( GW \) and the map \( S\) sending \( \Gamma(G, W) \) into the set of complements to \( 1 \times W \) in \( GW \) such that \( \gamma \to S(\gamma) \) is a bijection. Thus \( \Gamma(G, W) \) acts regularly on the set of complements to \( 1 \times W \) in \( GW \).

(1.9) Let \( \alpha_W: W \to F(G, W) \) be such that \( g(w\alpha_W) = w - wg \) for all \( g \in G \) and \( w \in W\). Then \( \alpha_W \) is an \( OG \)-module homomorphism such that \( \text{Ker}(\alpha_W) = C_W(G), W\alpha_W \leq \Gamma(G, W) \) and \( G(W\alpha_W) = [W, G]\).

Lemma 1.1. Let \( g \in G \) and \( \gamma \in \Gamma(G, W)\). Then

(a) \( g^{-1}\gamma g = -g\gamma\);
(b) \( \gamma^g - \gamma = -(g\gamma)\alpha_W\); and
(c) \( \gamma^g = \gamma \) if and only if \( g\gamma \in C_W(G)\).

Proof. Here \((g^{-1}g)\gamma = 0 = (g^{-1})g + g\gamma\) and (a) is proved. Let \( h \in G\). Then \( h(\gamma^g - \gamma) = (ghg^{-1})\gamma g - h\gamma = (gh)\gamma g^{-1}g + g^{-1}\gamma g - h\gamma = (gh)\gamma + g^{-1}\gamma g - h\gamma = g\gamma h + h\gamma + g^{-1}\gamma g - h\gamma = g\gamma h - g\gamma = h(-(g\gamma)\alpha_W)\) and (b) is proved. Also \( h(-(g\gamma)\alpha_W) = g\gamma h - g\gamma = 0\) for all \( h \in H\) if and only if \( g\gamma \in C_W(G)\), and we are done.

Set \( A = \text{aut}(GW) \) and \( U(G, W) = c_A(1 \times W) \cap C_A((GW)/(1 \times W)) \) so that \( U(G, W) \leq A\). Let \( c \in GW \to A \) denote the conjugation group homomorphism with \( \text{Ker}(c) = Z(GW) = (Z(G) \cap C_G(W)) \times C_W(G)\).
Lemma 1.2. (a) $\operatorname{Res}^{GW}_{G \times 0}(c) : G \times 0 \to N_A(U(G,W))$ is a group homomorphism;
(b) Via $\iota_G \operatorname{Res}^{GW}_{G \times 0}(c) : G \times 0 \to N_A(U(G,W))$, $U(G,W)$ becomes a $G$-group;
(c) If $w \in W$ and $(g,v) \in GW$, then $(1,-w)(g,v)(1,w) = (g,v + w - wg)$; and
(d) $c(1 \times W) \leq U(G,W)$.

Proof. Let $\phi \in U(G,W)$, $(g,w) \in GW$ and $h \in G$. Then
\[(1,w)(c(h^{-1},0)\phi(c(h,0)) = (1,wh^{-1})(\phi(c(h,0)) = (1,wh^{-1})c(h,0) = (1,w).\]

Also
\[(g,w)((c(h^{-1},0)\phi(c(h,0)) = (gh^{-1},wh^{-1})\phi(c(h,0)) = (gh^{-1},w^*)(c(h,0) = (g,\bar{w})\]
for unique $w^*, \bar{w} \in W$. Thus $c(h^{-1},0)\phi(c(h,0)) \in U(G,W)$ and the rest is clear.

Corollary 1.3. If $C_W(G) = 0$, then $\alpha_W$ is injective.

From [1, Theorem 17.2] we have:

(1.10) The map $\Phi_W : \Gamma(G,W) \to U(G,W)$ such that if $\gamma \in \Gamma(G,W)$ and $(g,w) \in GW$, then $(g,w)(\gamma \Phi_W) = (g,w + g\gamma)$ is a group isomorphism.

Lemma 1.4. (a) $\Phi_W : \Gamma(G,W) \to U(G,W)$ is a $G$-group isomorphism; and
(b) If $w \in W$, then $(w\alpha_W)\Phi_W = c(1,w)$.

Proof. Let $(g,w) \in GW$, $\gamma \in \Gamma(G,W)$ and $h \in G$. Then
\[(g,w)(\gamma^{h}\Phi_W) = (g,w + (h^{-1}gh))\gamma h)\]
and
\[(g,w)((\gamma \Phi_W)^h) = (g,w)c(h^{-1},0)(\gamma \Phi_W)c(h,0) = (gh^{-1},wh^{-1})(\gamma \Phi_W)c(h,0) = (gh^{-1},wh^{-1} + (gh^{-1})\gamma)c(h,0) = (g,w + (gh^{-1})\gamma h).\]

The proof is complete.

Let $\gamma \in \Gamma(G,W)$, $g \in G$ and $a \in O$. Then
\[(1.11) \ g(\gamma a) = (g\gamma)a. \]
Consequently via $\Phi_W : \Gamma(G,W) \to U(G,W)$, $U(G,W)$ becomes an $OG$ module. That is: if $\gamma \Phi_W \in U(G,W)$ where $\gamma \in \Gamma(G,W)$ and $a \in O$, then $(\gamma \Phi_W)a = (\gamma a)\Phi_W$.

Since if $w \in W$ and $(g,v) \in GW$, then $(g,v)((w\alpha_W)\Phi_W) = (g,v + w - vg) = (g,v)c(1 \times w)$. Thus we have proved:
Corollary 1.6. (a) \([U(G,W), G] \leq \text{Res}_{1\times W}^{GW}(c)\)

(b) \(\Gamma(G,W)/(W\alpha_W) \cong U(G,W)/(W\text{Res}_{1\times W}^{GW}(c))\) in \(\mathcal{O}G\text{-mod}\)

\[H^1(G,W) \cong \Gamma(G,W)/(W\alpha_W) \cong U(G,W)/(1 \times W)c;\]

and

(c) \(G\) acts trivially on \(H^1(G,W)\).

Let \(V\) be an \(\mathcal{O}G\)-submodule of \(W\). Thus \(GV = G \times V \leq GW = G \times W\).

Also \(\Phi_V : \Gamma(G, V) \rightarrow U(G, V)\) is an \(\mathcal{O}G\)-module isomorphism and \(\alpha_V : V \rightarrow \Gamma(G, V)\) is an \(\mathcal{O}G\)-module homomorphism with \(\text{Ker}(\alpha_V) = G\text{V}(G)\) and \(GW = (GV)(1 \times W)\).

Set \(B = \text{aut}(GV)\) so that \(U(G,V) \leq B\) and conjugation \(c^* : GV \rightarrow B\) is a group homomorphism with \(\text{Ker}(c^*) = Z(GV) = (Z(G) \cap C_G(V)) \times (C_V(G))\).

Set \(U(V) = \{w \in W \mid w - wg \in V\} \)for all \(g \in G\). Thus \(C_{W}(G) + V \leq U(V)\) and \(U(V)\) is an \(\mathcal{O}G\)-submodule of \(W\). Also \(U(V) = \{w \in W \mid w + V \in C_{W/V}(G)\}\). Thus:

\[(1.12)\quad U(V)/V = C_{W/V}(G);\]

\[(1.13)\quad U(V) = W\text{ if and only if } [W,G] \leq V;\text{ and}\]

\[(1.14)\quad [U(V), G] \leq V.\]

Note that \(GW = (G \times V)(1 \times W)\).

If \(w \in W\) and \((g,v) \in GV\), then \((1,-w)(g,v)(1,w) = (g,v+w-wg) = (g,v)((w\alpha)(\Phi_V)) = (g,v)c(1 \times v)\). Thus:

Proposition 1.7. (a) \(N_{GW}(GV) = (GV)(1 \times U(V)) = G \times U(V)\); and

(b) \(\alpha_W(U(V)) \leq \Gamma(G,V)\) and \(\text{Res}^W_{U(V)}(\alpha_W) : U(V) \rightarrow \Gamma(G,V)\) is an \(\mathcal{O}G\)-module homomorphism with \(\text{Ker}(\text{Res}^W_{U(V)}(\alpha_W)) = C_W(G)\).

Theorem 1.8. (a) \(\iota_{U(V)} \text{Res}^W_{1\times U(V)}(c) = \text{Res}^W_{U(V)}(\alpha_W\Phi_V) : U(V) \rightarrow U(G,W)\)

is an \(\mathcal{O}G\)-module homomorphism with \(\text{Ker}(\text{Res}^W_{U(V)}(\alpha_W\Phi_V)) = C_W(G)\). Also \(\text{Res}^W_{U(V)}(\alpha_W\Phi_V) = \iota_V c^* : V \rightarrow U(G,V)\);

(b) \(H^1(G,V) \cong \text{coker}(\alpha_V) \cong \Gamma(G,V)/(V\alpha_V)\) in \(mod-\mathcal{O}G\) so that \(\text{Res}^W_{U(V)}(\alpha_W)\)

induces an \(\mathcal{O}G\)-module injection of \(U(V)/(c_W(G) + V)\) into \(H^1(G,V)\); and

(c) \(H^1(G,V) \cong U(G,V)/(V(\iota_V c^*))\) and \(\alpha_W\Phi_V\) induces an \(\mathcal{O}G\)-module injection of \(U(V)/(C_W(G) + V)\) into \(H^1(G,V)\).

Remark 1.9. If the commutative coefficient ring \(\mathcal{O}\) is specialized to a field, the following corollary yields a proof of [1, Theorem 17.11]

Corollary 1.10. Let \(V\) be an \(\mathcal{O}G\) submodule of the \(\mathcal{O}G\)-module \(W\). Assume also that \([W,G] \leq V\) and \(C_W(G) = (0)\) so that \(C_V(G) = (0)\). Then:

(a) \(U(V) = W\);
\[ \text{(b) The map } \alpha_W \Phi_V : W \to U(G, V) \text{ is any } \mathcal{O}_G\text{-module injection such that } \iota_V \Res_{1 \times V}^G(c^*) = \Res_V^W(\alpha_V \Phi_V) : V \to U(G, V) \text{ is an } \mathcal{O}_G\text{-module injection; and} \]

\[ \text{(c) } \alpha_W \text{ induces an } \mathcal{O}_G\text{-module injection of } W / V \text{ into } H^1(G, V). \]

Our final results extend [1, Theorem 17.12].

For the remainder of this section, assume that \( \mathcal{O} \) is a discrete valuation ring.
Thus these results apply to Finite Group Modular Representation Theory.

Let \( \mathcal{C} \) denote the full additive subcategory of \( \text{mod-} \mathcal{O}_G \) of modules that are finitely generated and \( \mathcal{O} \)-free.

Let \( W \) and \( V \) be modules in \( \mathcal{C} \) and let \( \beta : W \to V \) ba a surjective homomorphism in \( \mathcal{C} \) so that the exact sequence

\[ (1.16) \quad (0) \to \Ker(\beta) \xrightarrow{I} W \xrightarrow{\beta} V \to (0) \text{ in } \mathcal{C} \text{ splits in } \mathcal{O}\text{-mod where } I \text{ is the inclusion map.} \]

Here \( V^* = \text{Hom}_\mathcal{O}(V, \mathcal{O}) \) and \( W^* = \text{Hom}_\mathcal{O}(W, \mathcal{O}) \) are modules in \( \mathcal{C} \). Thus the exact sequence

\[ (1.17) \quad (0) \to V^* \xrightarrow{\beta^*} W^* \xrightarrow{I^*} \Ker(\beta)^* \to (0) \text{ in } \mathcal{C} \text{ splits in } \mathcal{O}\text{-mod.} \]

For simplicity of notation, we shall identify (via \( \beta^* \)) \( V^* \) with \( V^* \beta^* \) so that \( V^* \) is a submodule of \( W^* \) in \( \mathcal{C} \).

Here \( U(V^*) = \{ \phi \in W^* \mid \phi - \phi g \in V^* \text{ for all } g \in G \} \) and let \( \tilde{c} : GW^* \to A^* = \text{aut}(GW^*) \) denote the conjugation group homomorphism with \( \Ker(\tilde{c}) = Z(GW^*) \).

Let \( t_W : W^{**} \to W \) and \( t_V : V^{**} \to V \) be the natural isomorphisms in \( \mathcal{C} \) of [3, Theorem 8.4(iii)].

From Theorem 1.8 we obtain:

**Theorem 1.11.** Assume the conditions above. Then:

(a) \( i_{U(V^*)} \Res_{1 \times V}^G(\tilde{c}) = \Res_{U(V^*)}^W(\alpha_W \Phi_V^*) : U(V^*) \to U(G, V^*) \) is a homomorphism in \( \mathcal{C} \) with \( \Ker(\Res_{U(V^*)}^W(\alpha_W \Phi_V^*)) = C_{W^*}(G) \). Also

\[ \Res_{U(V^*)}^W(i_{U(V^*)} \Res_{1 \times U(V^*)}^G(\tilde{c})) = \iota_V c^* \]

where \( c^* : GV^* \to \text{aut}(GV^*) \) is the conjugation group homomorphism; and

(b) \( \Res_{U(V^*)}^W(\alpha_W \Phi_V^*) \) induces a group injection of \( U(V^*)/(V^*) + (C_{W^*}(G)) \) into \( H^1(G, V^*) \) where \( G \) acts trivially on \( H^1(G, V^*) \).

**Corollary 1.12.** Assume that \( \Ker(\beta) \leq C_W(G) \) and that \( W = [W, G] \). Then:

(a) \( W^* = U(V^*) \) and \( C_{W^*}(G) = (0) \);

(b) \( \iota_{W^*} \Res_{1 \times W^*}^G(\tilde{c}) = \alpha_{W^*} \Phi_V^* : W^* \to U(G, V^*) \) is an injection in \( \mathcal{C} \) such that \( \Res_{W^*}^W(\alpha_{W^*} \Phi_V^*) = \iota_V c^* : V^* \to U(G, V^*) \) is an injection in \( \mathcal{C} \) where \( \tilde{c} : GV^* \to \text{aut}(GV^*) \) is the group conjugation homomorphism;

(c) \( G \) acts trivially on

\[ H^1(G, V^*) \cong U(G, V^*)/(V^* c^*) \cong \Gamma(G, V^*)/(V^* \alpha_V^*); \]

and
(d) \( \alpha_W: W^* \to \Gamma(G,V^*) \) is an injection in \( \mathcal{C} \) that induces an injection \( \pi_W*: W^*/V^* \to H^1(G,V^*) \) in \( \mathcal{O}G\)-mod.

**Proof.** Clearly Lemma 2.5 implies that \( C_{W^*}(G) = (0) \). Also (1.16), (1.17), and [3, Lemma 2.8.5] imply that Ker(\( \beta^* \)) \( \leq C_{W^*}(G) \). Thus Lemma 2.6 implies that \( [W^*, G] \leq V^* \) and (a) is proved. Theorem 1.11 implies (b)–(d) and we are done. \( \square \)

**Remark 1.13.** Assume that \( \mathcal{O} \) is a field. Then the following theorem yields a proof of [1, Theorem 17.12].

**Theorem 1.14.** Assume that Ker(\( \beta \)) \( \leq C_{W}(G) \) and that \( W = [W, G] \). Set \( \delta = \alpha_{W^*}\Phi_{V^*} \) so that \( \delta: W^* \to U(G,V^*) \) is an injection in \( \mathcal{C} \) and

\[
\beta^*\delta = \iota_{V^*}\tilde{c}^* : V^* \to U(G,V^*)
\]

is also an injection in \( \mathcal{C} \). Also \( C_{W^*}(G) = (0) \). Assume that \( W^*\delta | U(G,V^*) \) in \( \mathcal{O}\)-mod. Then:

(a) \( V(\iota_{V^*}\tilde{c}^*) | U(G,V^*) \) in \( \mathcal{O}\)-mod and \( H^1(G,V^*) \cong U(G,V^*)/V(\iota_{V^*}\tilde{c}^*) \) is a module in \( \mathcal{C} \) on which \( G \) acts trivially. Also

\[
(1.18) \quad (0) \to V^* \xrightarrow{\iota_{V^*}\tilde{c}^*} U(G,V^*) \xrightarrow{\pi} H^1(G,V^*) \to (0)
\]

is an \( \mathcal{O}\)-split exact sequence in \( \mathcal{C} \) where \( \pi \) is the natural group surjection. Thus

\[
(1.19) \quad (0) \to H^1(G,V^*)^* \xrightarrow{\pi^*} U(G,V^*)^* \xrightarrow{(\iota_{V^*}\tilde{c}^*)^*tV} V \to (0)
\]

is an \( \mathcal{O}\)-split exact sequence in \( \mathcal{C} \) where \( G \) acts trivially on \( H^1(G,V^*)^* \);

(b) \( U(G,V^*)/(W^*\delta) \) is a module in \( \mathcal{C} \) on which \( G \) acts trivially and;

\[
(1.20) \quad (0) \to W^* \xrightarrow{\delta} U(G,V^*) \xrightarrow{\sigma} U(G,V^*)/(W^*\delta) \to (0) \text{ is an \( \mathcal{O}\)-split exact sequence in \( \mathcal{C} \)}
\]

where \( \sigma \) is the natural group surjection. Thus:

\[
(1.21) \quad (0) \to (U(G,V^*)/W^*\delta)^* \xrightarrow{\sigma^*} U(G,V^*)^* \xrightarrow{\delta^*tW} W \to (0) \text{ is an \( \mathcal{O}\)-split exact sequence in \( \mathcal{C} \)}; \text{ and}
\]

(c) \( G \) acts trivially on \( H^1(G,V^*)^* \) and on \( (U(G,V^*)/W^*\delta)^* \).

**Proof.** Here \( W^*/V^* \) is an \( \mathcal{O}\)-pure \( \mathcal{O}G \)-module by Lemma 2.7 and \( U(G,V^*)/W^*\delta \) is also an \( \mathcal{O}\)-pure \( \mathcal{O}G \)-module by hypothesis. Thus (a) holds. Since \( G \) acts trivially on \( H^1(G,V^*) \) and \( V^*(i_{V^*}\tilde{c}^*) = V^*\beta^*\delta \leq W^*\delta \); both (b) and (c) follow and we are done. \( \square \)

We conclude this section with applications to finite group permutation module theory ([2, p. 12] and [1, Chapter 4 Exercises 5 and 6]). We only assume that \( \mathcal{O} \) is a commutative ring.

Let \( I \) be a finite set and let \( \pi: G \to \text{Sym}(I) \) be a group homomorphism with \( G \) transitive on \( I \). Let \( V \) be a free \( \mathcal{O} \)-module with an \( \mathcal{O} \)-basis \( B \{ x_i \ | \ i \in I \} \). Thus \( \pi \) induces a unique group homomorphism \( \alpha: G \to \text{End}_\mathcal{O}(V) \) such that \( x_i(g\alpha) = x_i(g\pi) \) for all \( i \in I \) and \( g \in G \).
Let \( f : V \to \mathcal{O} \) denote the \( \mathcal{O} \)-module epimorphism such that \( \sum_{i \in I} (x_i a_i) \mapsto \sum_{i \in I} a_i \) for all \( a_i \in \mathcal{O} \) and \( i \in I \). Viewing \( \mathcal{O} \) as a trivial \( \mathcal{O}G \)-module, \( f \) is an epimorphism in \( \text{mod-}\mathcal{O} \). Let \( W = \ker(f) \) so that \( W \) is a submodule of \( V \) in \( \text{mod-}\mathcal{O} \). Also set \( z = \sum_{i \in I} x_i \) and \( Z = z\mathcal{O} \). Then:

\[
(1.22) \quad C_V(G) = Z; \quad \text{and}
\]

\[
(1.23) \quad [V,G] = W.
\]

Indeed, \((1.22)\) is trivial and \([V,G] \leq W\) is clear. Fix \( i \in I \), set \( H = \text{Stab}_G(i) \) and let \( Y \) be a right transversal of \( H \) in \( G \) with \( 1 \in Y \) so that \( G = \cup_{y \in Y} (Hy) \) is disjoint. Thus \( B = \{ i(y\pi) \mid y \in Y \} \). Let \( \Gamma = \sum_{y \in Y} (x_i(y\pi) a_i(y\pi)) \in W \). Thus \( \Gamma = \sum_{y \in Y} ((x_i(y\pi) - x_i) a_i(\pi y)) \in [V,G] \) and \((1.23)\) is proved.

Thus:

\[
(1.24) \quad U(W) = V.
\]

From part (b) of Theorem 1.8, we conclude

\[
(1.25) \quad \text{There is an injection in } \text{mod-}\mathcal{O}G \text{ of } V/(Z+W) \text{ into } H^1(G,V) \text{ where } V/(Z+W) \cong \mathcal{O}/(|I|\mathcal{O}) \text{ in } \text{mod-}\mathcal{O}G
\]

Finally, we restrict to the context of modular group representation theory so that in the remainder of this section we assume that \( p \) is a prime integer such that \( p\mathcal{O} = (0) \) and \( p \) divides \(|I|\). Thus:

\[
(1.26) \quad Z \leq W \text{ so that } \overline{W} = W/Z \text{ is a submodule of } \overline{V} = V/Z \text{ in } \text{mod-}\mathcal{O}G \text{ and } [\overline{V},G] = \overline{W}.
\]

We conclude with an extension of [1, Chapter 4 Exercise 6(3)]. If \( \mathcal{O}^p(G) = G \), then

\[
\begin{align*}
(\text{a}) & \quad C_{\overline{V}}(G) = O = C_{\overline{W}}(G); \quad \text{and} \\
(\text{b}) & \quad \text{there is an } \mathcal{O}G\text{-module injection of } \overline{V}/\overline{W} \text{ into } H^1(G,\overline{W}) \text{ in } \text{mod-}\mathcal{O}G
\end{align*}
\]

where \( \overline{V}/\overline{W} \cong \mathcal{O}/(p\mathcal{O}) \) in \( \text{mod-}\mathcal{O}G \).

Indeed, let \( \gamma \in W \) and \( g \in G \) be such that \( \gamma(g\alpha) = \gamma + t \) for a unique \( t \in Z \). Then for any positive integer \( j \), \( \gamma(g\alpha)^j = \gamma + j t \). Thus (a) is proved and Corollary 1.10 yields (b) and we are done.

2. Required General Results

We shall frequently utilize [3, Section 2.8] in the following.

Let \( \mathcal{O} \) be a commutative ring, let \( A \) be a finitely generated \( \mathcal{O} \)-free \( \mathcal{O} \)-algebra. Let \( W \) be a finitely generated right \( A \)-module so that \( W^* = \text{Hom}_\mathcal{O}(W,\mathcal{O}) \) is a finitely generated \( \mathcal{O} \)-free left \( A \)-module such that \( \text{rank}_\mathcal{O}(W) = \text{rank}_\mathcal{O}(W^*) \) ([3, Theorem 2.8.4]).

In contrast to [3, Exercise 2.8.2(ii)]:
Lemma 2.1. Assume that $V$ is an $\mathcal{O}$-free $A$-submodule of the $\mathcal{O}$-free module $W$ in mod-$A$ such that $W = V \oplus X$ in $\mathcal{O}$-mod for some $\mathcal{O}$-free submodule $X$ of $W$ in mod-$\mathcal{O}$. Also let $I: V \to W$ denote the inclusion map in mod-$A$. Then $I^* = \text{Res}^W_V: W^* \to V^*$ is surjective in $A$-mod.

Proof. Here $W^* = V^* \oplus X^*$ in $\mathcal{O}$-mod by [3, Lemma 2.8.6]. Thus the first component projection map $I^*: W^* \to V^*$ is surjective. Let $v \in V$, $a \in A$, and $\phi \in W^*$. Then $v((a\phi)I^*) = v\phi = v(a\phi I^*)$ so that $a(\phi I^*) = (a\phi)I^*$ and we are done. \qed

Lemma 2.2. Let $(0) \to U \xrightarrow{\alpha} V \xrightarrow{\beta^*} W \to (0)$ be an $\mathcal{O}$-split exact sequence in mod-$A$ where $U$, $V$, and $W$ are free in $\mathcal{O}$-mod. Then $(0) \to W^* \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} U^* \to (0)$ is an $\mathcal{O}$-split exact sequence in $A$-mod.

Proof. Clearly $(0) \to W^* \to V^* \xrightarrow{\alpha^*} U^*$ is exact by [3, Exercise 2.8.2(ii)]. Here there is a homomorphism $\gamma: V \to U$ in $\mathcal{O}$-mod such that $\alpha \gamma = \text{Id}_U$. Hence $\gamma^* \alpha^* = \text{Id}_V$. Let $\phi \in U^*$. Then $(\phi \gamma^*) \in V^*$ and $(\phi \gamma^*) \alpha^* = \phi$ so that $\alpha^*$ is surjective and we are done. \qed

Lemma 2.3. Let $W$ be an $\mathcal{O}$-free module in mod-$A$, let $V$ be an $\mathcal{O}$-free submodule of $W$ in mod-$A$ and let $X$ be an $\mathcal{O}$-free submodule of $W$ in $\mathcal{O}$-mod such that $W = V \oplus X$ in $\mathcal{O}$-mod. Let $I: V \to W$ be the inclusion map in mod-$A$ and let $\pi: W \to V$ denote the first component projection map in mod-$\mathcal{O}$. Then;

(a) $I^*: W^* \to V^*$ is surjective in $A$-mod; and

(b) $(0) \to V \xrightarrow{J^*} W \xrightarrow{J} W/V \to (0)$ is an exact sequence in mod-$A$ where $W/V$ is $\mathcal{O}$-free.

Proof. Lemma 2.1 implies (a). Thus $(0) \to \text{Ker}(I^*) \xrightarrow{J} W^* \xrightarrow{I^*} V^* \to (0)$ is an $\mathcal{O}$-split exact sequence in $A$-mod where $J$ is the inclusion map. Thus [3, Lemma 2.8.5] implies that $(0) \to V \xrightarrow{J^*} W \xrightarrow{J} \text{Ker}(I^*)^* \to (0)$ is an $\mathcal{O}$-split exact sequence in mod-$A$ for a suitable surjection $J$ in mod-$A$ and the proof is complete. \qed

For the remainder of this section let $\mathcal{O}$ be a discrete valuation ring. Thus we may utilize [2, Theorems 4.24-4.26].

Lemma 2.4. Let $W$ be a finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module. Then $C_W(G)$ is an $\mathcal{O}$-pure $\mathcal{O}G$-submodule of $W$.

Proof. Let $0 \neq \alpha \in \mathcal{O}$ and $0 \neq w \in W$ be such that $w\alpha \in C_W(G)$. Then $(w\alpha)g = w\alpha = (wg)\alpha$ for all $g \in G$. Since $W$ is torsion free, $w \in C_W(G)$ and we are done. \qed

Lemma 2.5. Let $W$ be a finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module such that $W = [W,G]$. Then $C_{W^*}(G) = (0)$. 
Proof. Assume that $C_{W^*}(G) \neq (0)$. Then we have the exact $\mathcal{O}$-split sequence in mod-$\mathcal{O}G$.

\begin{equation}
(2.1) \quad (0) \to C_{W^*}(G) \xrightarrow{\alpha} W^* \xrightarrow{\pi} W^*/C_{W^*}(G) \to (0); \quad \text{and}
\end{equation}

\begin{equation}
(2.2) \quad (0) \to (W^*/C_{W^*}(G))^* \xrightarrow{\tilde{\pi}} W \xrightarrow{\tilde{\alpha}} (C_{W^*}(G))^* \to (0)
\end{equation}

where $\alpha$ is the injection map, $\pi$ is the canonic group surjection and, since $W^{**} \cong W$ in mod-$\mathcal{O}G$ and by [3, Theorem 2.8.4(iii)], we have the injection $\tilde{\pi}$ and surjection $\tilde{\alpha}$ in mod-$\mathcal{O}G$ induced by $\pi$ and $\alpha$, resp. Here $G$ acts trivially on $C_{W^*}(G)^*$ so that $[W,G] \leq (W^*/C_{W^*}(G))^*\tilde{\pi}$. Here $\text{rank}_{\mathcal{O}}(W^*/C_{W^*}(G))^*\tilde{\pi} = \text{rank}_{\mathcal{O}}(W^*/C_{W^*}(G)) = \text{rank}_{\mathcal{O}}(W^*) - \text{rank}_{\mathcal{O}}(C_{W^*}(G))$. However $C_{W^*}(G)$ is a non-zero $\mathcal{O}$-free pure submodule of $W^*$ and so $[W,G] \leq W$. This contraction completes the proof. □

Lemma 2.6. Let $V$ be an $\mathcal{O}G$-submodule of the finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module $W$ such that, in mod-$\mathcal{O}$, $W = V \oplus X$ for an $\mathcal{O}$-module $X$ so that $V$ and $X$ are finitely generated $\mathcal{O}$-free $\mathcal{O}$-modules. Let $I: V \to W$ denote the inclusion map. Then $[W,G] \leq V$ if and only if $\text{Ker}(I^*) \leq C_{W^*}(G)$ where $I^* = \text{Res}^W_V: W^* \to V^*$.

Proof. Note that $(0) \to V \xrightarrow{I} W \xrightarrow{\pi} W/V \to (0)$ and $(0) \to (W/V)^* \xrightarrow{\pi^*} W^*/V^* \to (0)$ are $\mathcal{O}$-split exact sequences in mod-$\mathcal{O}G$ where $\pi$ is the canonic group surjection.

From [3, Theorem 2.8.4] we conclude that the following four conditions are equivalent: (i) $[W,G] \leq V$, (ii) $G$ acts trivially on $W/V$, (iii) $G$ acts trivially on $(W/V)^*$, (iv) $\text{Ker}(I^*) \leq G_{W^*}(G)$. The proof is complete. □

Lemma 2.7. Let $W$, $V$, $X$, and $I$ be as in Lemma 2.3. Then $U(V) = \{ w \in W \mid w - wg \in V \text{ for all } g \in G \}$ is a pure submodule of $W$.

Proof. Here $W/V$ is an $\mathcal{O}$-pure $\mathcal{O}G$-module and $U(V)/V = C_{W/V}(G)$. Now Lemma 2.4 implies that $U(V)/V$ is an $\mathcal{O}$-pure submodule of $W/V$ and we are done. □

References


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