The Minimal Degree Standard Identity on $E \otimes E$

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Abstract

Let $K$ be a field of characteristics $p > 2$. Giambruno and Koshlukov have proved [2] that Grassmann Algebra $E$ satisfies the standard identity of degree $m$ if, and only if, $m \geq p + 1$. The object of this paper is to extend such result to $E \otimes E$, more precisely proving that $E \otimes E$ satisfies the standard identity of degree $m$ if, and only if, $m \geq 2p$.

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1 Introduction

In this paper we address with algebras over a field $K$ of positive characteristic $p > 2$. We denote by $K\langle X \rangle$ the free associative algebra of rank freely generated over $K$ by the set $X = \{x_1, x_2, \ldots\}$.

Given an algebra $A$, it is said that polynomial $p(x_1, \ldots, x_n) \in K\langle X \rangle$ (or the expression $p(x_1, \ldots, x_n) = 0$) is a polynomial identity for algebra $A$ when
\[ p(a_1, \ldots, a_n) = 0 \] for whichever \( a_1, \ldots, a_n \in A \). The set of all polynomial identities of \( A \) is denoted by \( T(A) \). It is said that \( A \) is an algebra with a polynomial identity (PI-algebra) when \( T(A) \neq \{0\} \). When \( p(x_1, \ldots, x_n) \in T(A) \), it is also said that algebra \( A \) satisfies \( p(x_1, \ldots, x_n)( \text{or} p(x_1, \ldots, x_n) = 0) \).

We denote by

\[ s_m(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(m)} \]

and

\[ w_m(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(m)} \]

such as standard and symmetric polynomials, respectively, of degree \( m \) where \((-1)^\sigma\) is the sign of permutation \( \sigma \).

Let \( V \) be a vector space over \( K \) with a fixed basis \( \{e_1, e_2, \ldots\} \). The Grassmann algebra \( E \) of \( V \) over \( K \) is the vector space with a basis consisting of 1 and all products the form \( e_{j_1} \cdots e_{j_k} \) where \( j_1 < \ldots < j_k, k \geq 1 \). The multiplication in \( E \) is induced by \( e_i e_j + e_j e_i = 0 \). Clearly \( E = E_0 \oplus E_1 \), where \( E_0 = \text{spanned}(\{1, e_{j_1} \cdots e_{j_m} \mid m \text{ even}\}) \) and \( E_1 = \text{spanned}(\{e_{j_1} \cdots e_{j_m} \mid m \text{ odd}\}) \).

The non-unitary Grassmann algebra \( E' \) is the sub algebra of \( E \) generated by \( \{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \ldots < i_k, k \geq 1\} \).

In [2] Giambruno e Koshlukov it is proved that \( s_m \in T(E) \) if, and only if, \( m \geq p + 1 \). In this paper, we extend this result for algebra \( E \otimes E \). Remembering that \( s_m(x_1, \ldots, x_m) = \sum_{i=1}^{m} (-1)^{i+1} x_i s_{m-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \).

Thus, if \( s_k \in T(A) \) so \( s_m \in T(A) \) for every \( m \geq k \). Therefore, to determine the minimal degree of the standard polynomial that makes it a polynomial identity for algebra \( A \), it is only necessary to find a \( k \) such as \( s_k \in T(A) \) and \( s_{k-1} \notin T(A) \). On section 2, it will be proved that algebra \( E \otimes E \) satisfies the standard polynomial of degree \( 2p \), and, on section 3, it will be proved that algebra \( E \otimes E \) does not satisfy the standard polynomial of degree \( 2p - 1 \). Thereby, it is concluded that the minimal degree of a standard identity that is a polynomial identity of the \( E \otimes E \) is equal to \( 2p \).

## 2 Algebra \( E \otimes E \) satisfies the standard polynomial of degree \( 2p \)

Regev shows in [4] that \( x^p \in T(E') \), using analog arguments, Machado shows in [3] that \( x^{2p} \in T(E' \otimes E') \). In this section, similar ideas will be used to show that \( x^{2p} \in T(E' \otimes E' \otimes E_1) \) and from it, conclude that \( s_{2p} \in T(E \otimes E) \).

**Lemma 2.1** \( E'_r \otimes E'_s \otimes E_1 \) with \( r, s \in \{0, 1\} \) satisfies \( x^p = 0 \).
Proof: First, it will be proved that $E'_r \otimes E'_s \otimes E_1$ with $r, s \in \{0, 1\}$ satisfies $w_p = 0$. Thus, given $b_i, b_j \in E'_r \otimes E'_s \otimes E_1$ it is known that

\[
\begin{align*}
    b_ib_j &= -b_jb_i \quad \text{if } r=s=0 \\
    b_ib_j &= -b_jb_i \quad \text{if } r=s=1 \\
    b_ib_j &= b_jb_i \quad \text{if } r=0, s=1 \\
    b_ib_j &= b_jb_i \quad \text{if } r=1, s=0
\end{align*}
\]

Thus, if $b_1, ..., b_p \in E'_r \otimes E_1 \otimes E_1$ (or $E_1 \otimes E'_s \otimes E_1$) therefore

\[
w_p(b_1, ..., b_p) = p!b_1...b_p = 0.
\]

And if $b_1, ..., b_p \in E_1 \otimes E_1 \otimes E_1$ (or $E'_0 \otimes E'_0 \otimes E_1$) then

\[
w_p(b_1, ..., b_p) = (\sum_{\sigma \in S_p} (-1)^\sigma) b_1...b_p = 0.
\]

Now we will demonstrate that $E'_r \otimes E'_s \otimes E_1$ satisfies $x^p = 0$. Therefore, if $b = \sum_{i=1}^t \alpha_ib_i$, where $b_i$ are elements of the basis of $E'_r \otimes E'_s \otimes E_1$. Consequently that:

\[
b^p = \sum_{1 \leq i_1, ..., i_p \leq t} \alpha_{i_1}...\alpha_{i_p}b_{i_1}...b_{i_p}.
\]

If $t < p$, then at least two of the $b_i$, in each part, are equal. Therefore $b_{i_1}...b_{i_p} = 0$, and from this, resulting in $b^p = 0$. If $t \geq p$, then

\[
b^p = \sum_{1 \leq i_1 < ... < i_p \leq t} \alpha_{i_1}...\alpha_{i_p}w_p(b_{i_1}, ..., b_{i_p}) = 0.
\]

□

Lemma 2.2 $E'_r \otimes E'_s \otimes E_1$ satisfies $x^{2p} = 0$

Proof: Notice that if $ab = -ba$ then $(a+b)^2 = a^2 + b^2$, and if $ab = ba$ then $(a+b)^p = a^p + b^p$.

Given that $b \in E'_r \otimes E'_s \otimes E_1$, it can be written as $b = b_{001} + b_{111} + b_{101} + b_{011}$ where $b_{rs1} \in E'_r \otimes E'_s \otimes E_1$. As $b_{001}$ and $b_{111} + b_{101} + b_{011}$ anticommute then $b^2 = b_{001}^2 + (b_{111} + b_{101} + b_{011})^2$. As $b_{001}^2 \in E_0 \otimes E_0 \otimes E_0$ then $b_{001}$ and $(b_{111} + b_{101} + b_{011})^2$ commute, therefore $b^{2p} = b_{001}^{2p} + (b_{111} + b_{101} + b_{011})^{2p}$. According to lemma 2.1, $b_{001}^{2p} = 0$, therefore $b^{2p} = (b_{111} + b_{101} + b_{011})^{2p}$. As $b_{111}$ and $b_{101} + b_{011}$ commute then $b^{2p} = [b_{111}^{2p} + (b_{101} + b_{011})^{2p}]^2$. According to lemma 2.1, $b_{111}^{2p} = 0$, therefore $b^{2p} = (b_{101} + b_{011})^{2p}$. As $b_{101}$ and $b_{011}$ anticommute then $b^{2p} = (b_{101}^2 + b_{011}^2)^p$. As $b_{101}^2, b_{011}^2 \in E_0 \otimes E_0 \otimes E_0$ then $b^{2p} = b_{101}^{2p} + b_{011}^{2p}$. According to lemma 2.1, $b_{101}^{2p} = b_{011}^{2p} = 0$, therefore $b^{2p} = 0$. □

Lemma 2.3 The algebras $E \otimes E$ and $E'_r \otimes E'_s$ satisfy the same multilinear identities.
Proof: As $E' \otimes E' \subset E \otimes E$ follows that $E' \otimes E'$ satisfies all multilinear identities of $E \otimes E$.

On the other hand, let $f(x_1, ..., x_n)$ be a multilinear identity of $E' \otimes E'$. It can be written as $f(x_1, ..., x_n) = \sum_{i=1}^{n} \alpha_{\sigma} x_{\sigma(1)} ... x_{\sigma(n)}$ where $\alpha_{\sigma} \in K$. Besides that, it can be verified that $E \otimes E$ satisfies $f = 0$ it is only necessary to check that $f$ nullifies under elements $a_1, ..., a_n$ of the basis of $E \otimes E$ which are elements of shape $1 \otimes 1$, $e_i \otimes 1$, $1 \otimes e_i$ or $e_i \otimes e_j$. If $a_1 ... a_n = 0$, there is nothing to be done. So, $e_i$ will be taken in a way that $a_1 ... a_n \neq 0$.

Now, consider that $a_1' = e_{j_1} e_{j_2} ... e_{j_n}$ such as any of $e_{j_1}, ..., e_{j_n}$ are shown as terms of $a_i$. As $a_i a_i' \in E' \otimes E'$ for every $i = 1, ..., n$, so $f(a_1 a_1', ..., a_n a_n') = 0$. Besides that, as $a_i' \in E_0 \otimes E_0$ so $0 = f(a_1 a_1', ..., a_n a_n') = a_1' ... a_n' f(a_1, ..., a_n) = \alpha a_1' ... a_n' a_1 ... a_n$. Therefore $\alpha = 0$ and so $f(a_1, ..., a_n) = 0$.

Eventually, the main result from this section has been proved.

Theorem 2.4 Algebra $E \otimes E$ satisfies $s_{2p} = 0$.

Proof: It will be shown that with algebra $E' \otimes E'$ satisfies $s_{2p} = 0$ and the result comes from lemma 2.3. Let $a_1, ..., a_{2p} \in E' \otimes E'$, and take $b = a_1 \otimes e_1 + \cdots + a_{2p} \otimes e_{2p}$. As $b \in E' \otimes E' \otimes E_1$ comes from lemma 2.2 that $b^{2p} = 0$. On the other hand, $b^{2p} = s_{2p}(a_1, ..., a_{2p}) \otimes e_1 ... e_{2p}$. Therefore, $s_{2p}(a_1, ..., a_{2p}) = 0$.

3 Algebra $E \otimes E$ doesn’t satisfy standard polynomial of degree $2p - 1$

In the previous section, it was proved that $s_{2p} \in T(E \otimes E)$. In this section it will be proved that $s_{2p-1} \notin T(E \otimes E)$. Thus, it will be concluded that $2p$ is precisely the smallest degree from standard polynomial which makes it polynomial identity for algebra $E \otimes E$. In order to show this last result, the lemma approved by Berele in [1] will be needed.

Let $g_1, ..., g_k \in E_0 \otimes E_1$ and $h_1, ..., h_l \in E_1 \otimes E_0$ such that $g_1 ... g_k h_1 ... h_l \neq 0$. Notice that $g_i g_j = -g_j g_i$, $h_i h_j = -h_j h_i$ and $g_i h_j = h_j g_i \forall i, j$. Now $f(k, l)$ must be defined, considering that

$$s_{k+l}(g_1, ..., g_k, h_1, ..., h_l) = f(k, l) g_1 ... g_k h_1 ... h_l.$$
Applying the formula 
\[ s_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i s_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \]
it is easy to conclude that \( f(k, l) \) satisfies the relation
\[ f(k, l) = k f(k - 1, l) + (-1)^k l f(k, l - 1). \]

**Lemma 3.1 ([1], lemma 6.2)** The solution for the equation above is
\[ f(k, l) = \frac{1 + (-1)^k l}{2} k!(\left\lfloor \frac{k+l}{2} \right\rfloor). \]

**Theorem 3.2** Algebra \( E \otimes E \) does not satisfy \( s_{2p-1} = 0 \).

**Proof:** Let \( g_1, \ldots, g_p \in E_0 \otimes E_1 \) and \( h_1, ..., h_l \in E_1 \otimes E_0 \) such as the product is not null. Making \( k = l = p - 1 \) and using lemma 3.1 we have that
\[ f(k, l) = (p-1)! 2\left(\frac{p-1}{2}\right) \equiv (-1)^{p-1} mod p. \]

Therefore, \( s_{2p-2}(g_1, \ldots, g_{p-1}, h_1, ..., h_{p-1}) = (-1)^{p-1} g_1 \cdots g_{p-1} h_1 \cdots h_{p-1} \neq 0 \). As \( 2p - 1 \) is an odd number, so:
\[ s_{2p-1}(g_1, \ldots, g_{p-1}, h_1, ..., h_{p-1}, 1) = s_{2p-2}(g_1, \ldots, g_{p-1}, h_1, ..., h_{p-1}) \]
\[ = (-1)^{p-1} g_1 \cdots g_{p-1} h_1 \cdots h_{p-1} \neq 0. \]

□

Now, putting together the theorems 2.4 and 3.2 we have the following result.

**Theorem 3.3** \( s_m \in T(E \otimes E) \) if, and only if, \( m \geq 2p \).

**References**

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