The Auslander-Reiten Correspondence of
Wakamatsu-Silting Complexes

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Abstract

We introduce the notion of relative coresolving subcategories and bounded Hom-projective generator and then give the Auslander-Reiten correspondence of Wakamatsu-silting complexes.

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1 Introduction

The tilting theory is well known, and plays an important role in the representation theory of Artin algebra. The classical notion of tilting and cotilting modules was first considered in the case of finite dimensional algebras by Brenner and Butler [3] and by Happel and Ringel [4]. A very important notion in the derived category is the notion of tilting complexes [7] since they characterize derived equivalences, see [6][7]. M. Auslander and I. Reiten [2] gave the correspondence between tilting or cotilting modules and suitable subcategories of modR. A slightly general notion is the notion of silting complexes [1][5][9]. The notion of silting complexes is far generalization of tilting modules. Later, F. Mantese and I. Reiten gave the correspondence between Wakamatsu tilting

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modules and suitable subcategories of mod$R$. Wei [9] gave the Auslander-Reiten correspondence of silting complexes. Then Zhang and Wei [11] gave the Auslander-Reiten correspondence of cosilting complexes. Another far generalization of tilting modules and silting (tilting) complexes. And the author proved Wakamatsu silting complexes is self-dual, in sense that a complex $T$ is Wakamatsu-silting if and only if $DT$ is Wakamatsu-silting, where $D$ is the usual duality functor for artin algebras. So it is natural to consider whether there is a correspondence between Wakamatsu silting complexes and suitable subcategories of $D^b(modR)$.

Throughout this paper, $R$ always denotes an artin algebra. We denote by mod$R$ the category of all finitely generated left $R$-modules and by $D$ the derived category of mod$R$ and $D^b(modR)$ the bounded derived category of mod$R$.

$D$ is a triangulated category with [1] the shift functor. Assume that $C$ is a full subcategory of $D$. Recall that $C$ is closed under extension if for any triangle $X \to Y \to Z \to$ in $D$ with $X, Z \in D$, we have $Y \in D$. The subcategory $C$ is cosuspended (resp., suspended) if it is closed under extension and under the functor $[-1]$ (resp., [1]). It is easy to prove that $C$ is cosuspended (resp., suspended) if and only if for any triangle $X \to Y \to Z \to$ (resp., $Z \to Y \to X \to$) in $C$ with $Z \in C$, one has that $X \in C \iff Y \in C$.

For any object $L \in D$ has a $C$-resolution (resp., $C$-coresolution) with the length at most $n$ ($n > 0$), if there are triangles $L_{i+1} \to X_i \to L_i \to$ (resp., $L_i \to X_i \to L_{i+1} \to$) with $0 \leq i \leq n$ such that $L_0 = L$, $L_{n+1} = 0$ and each $X_i \in C$. In the case, we denoted by $C$-res.$\dim(L) \leq n$ (resp., $C$-cores.$\dim(L) \leq n$). One may compare such notions with the usual finite resolutions and coresolutions respectively in the module category.

Associated with a subcategory $C$, we have the following notations which are widely used in the tilting theory (see for instance [2]), where $n \geq 0$ and $m$ is an integer.

$$C^{\perp_{\geq 0}} = \{ N \in D \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in C \text{ and all } i \neq 0 \}. $$

$$C^{\perp_{> 0}} = \{ N \in D \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in C \text{ and all } i \neq 0 \}. $$

$$C^{\perp_{\geq m}} = \{ N \in D \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in C \text{ and all } i \geq m \}. $$

$$C^{\perp_{> m}} = \{ N \in D \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in C \text{ and all } i > m \}. $$

$$C^{\perp_{\geq 0}} = \{ N \in D \mid N \in C^{\perp_{\geq m}} \text{ for some } m \}. $$

Note that $C^{\perp_{\geq m}}$ (resp., $C^{\perp_{> m}}$) is suspend (resp., cosuspend) and closed under direct summands and that $C^{\perp_{\geq 0}}$ is a triangulated subcategory of $D$.

The subcategory $C$ is said to be semi-selforthogonal (resp., selforthogonal) if $C \subseteq C^{\perp_{\geq 0}}$ (resp., $C \subseteq C^{\perp_{\geq 0}}$). For instance, both subcategories Proj$R$ and
Inj\(R\) are selforthogonal, where \(R\) is a ring.

Let \(T\) be a complex. Denote by \(\text{add}_DT\) the subcategory of all direct summands of finite coproducts of \(T\). \(T\) is said to be semi-selforthogonal provided that \(M \otimes_P T \neq 0\).

We introduce the following subcategories associated with the semi-selforthogonal complex \(T\).

\[
\mathcal{X}^I = \{ X \mid N \in \mathcal{X}^I_T \text{ for some finite interval } I \text{ of integers such that } T \in \mathcal{D}(\text{mod}R) \}.
\]

\[
\tau \mathcal{X}^I = \{ N \mid N \in \mathcal{X}^I_T \text{ for some finite interval } I \text{ of integers such that } T \in \mathcal{D}(\text{mod}R) \}.
\]

Let \(C\) be a subcategory.

We denote \(\langle C \rangle_+ = \{ N \mid N = C[n] \text{ for some finite } C \in \mathcal{C} \text{ and some integer } n \geq 0 \} \) and

\[
\langle C \rangle_- = \{ N \mid N = C[n] \text{ for some finite } C \in \mathcal{C} \text{ and some integer } n \leq 0 \}.
\]

In the following, we give some definitions which are useful for the main results.

**Definition 1.1** A subcategory \(\mathcal{X}\) of \(\mathcal{D}(\text{mod}R)\) is said to be coresolving if \(\mathcal{X}\) satisfies the following two conditions:

1. \(\mathcal{X}\) is closed under extension,
2. \(\mathcal{X}\) is closed under the functor \([1]\).

If \(\mathcal{X}\) is coresolving and satisfies \(DR[n] \in \mathcal{X}\) for some \(n > 0\), then we call it relative coresolving.

We give the dual definition of relative resolving subcategories.

**Definition 1.2** A subcategory \(\mathcal{Y}\) of \(\mathcal{D}(\text{mod}R)\) is said to be resolving if \(\mathcal{Y}\) satisfies the following two conditions:

1. \(\mathcal{Y}\) is closed under extension,
2. \(\mathcal{Y}\) is closed under the functor \([-1]\).

If \(\mathcal{Y}\) is resolving and satisfies \(R[-n] \in \mathcal{Y}\) for some \(n > 0\), then we call it relative resolving.
We give two useful definitions.

**Definition 1.3** A complex $T$ in a subcategory $\mathcal{X}$ of $\mathcal{D}^b(\text{mod} R)$ is said to be a bounded Hom-projective generator of $\mathcal{X}$ if $\mathcal{X} \subseteq T^{>1}$ and for any $X \in \mathcal{X}$, there are triangles $X_{i+1} \rightarrow T_i \rightarrow X_i \rightarrow$, where $X_0 = X$, $X_i \in \mathcal{X} \cap \mathcal{D}^I(\text{mod} R)$ for some finite interval $I$ relative to $X$, where $i \geq 0$.

Dually, we can define a bounded Hom-injective cogenerator.

**Definition 1.4** A complex $T$ in a subcategory $\mathcal{Y}$ of $\mathcal{D}^b(\text{mod} R)$ is said to be a bounded Hom-injective cogenerator of $\mathcal{Y}$ if $\mathcal{Y} \subseteq T^{1<0}$ and for any $Y \in \mathcal{Y}$ and there are triangles $Y_i \rightarrow T_i \rightarrow Y_{i+1} \rightarrow$, where $Y_0 = Y$, $Y_i \in \mathcal{Y} \cap \mathcal{D}^I(\text{mod} R)$ for some finite interval $I$ relative to $Y$, where $i \geq 0$.

**Corollary 1.5** If a subcategory $\mathcal{X}$ of $\mathcal{D}^b(\text{mod} R)$ has a bounded Hom-projective generator, then $\mathcal{X} \subseteq T^b$. Dually, If a subcategory $\mathcal{Y}$ of $\mathcal{D}^b(\text{mod} R)$ has a bounded Hom-injective cogenerator, then $\mathcal{Y} \subseteq X^b$.

**Example 1.6** If $T$ is semi-selforthogonal, then these categories $\mathcal{T}^b$ and $\mathcal{X}^I$ have a bounded Hom-projective generator $T$. Dually, $\mathcal{X}^b$ and $\mathcal{X}^I$ have a bounded Hom-injective cogenerator $T$.

**Lemma 1.7** [10] (1) The subcategory $\tau \mathcal{X}^b$ of $\mathcal{D}^b(\text{mod} R)$ is coresolving and closed under direct summands.

(2) The subcategory $\mathcal{X}^b_T$ of $\mathcal{D}^b(\text{mod} R)$ is resolving and closed under direct summands.

## 2 Wakamatsu-silting complexes

In the following, we recall the definition and some characterizations of Wakamatsu-silting complexes in [10].

**Definition 2.1** [10] A complex $T \in \mathcal{D}^b(\text{mod} R)$ is said to be a Wakamatsu-silting complex provided $T$ is semi-selforthogonal and $R \in \langle \mathcal{X}^b_T \rangle_+$.

The following lemmas give some useful characterizations of Wakamatsu-silting complexes.

**Lemma 2.2** [10] Assume that $T \in \mathcal{D}^{[-r,0]}(\text{mod} R)$ for some suitable integer $r$. Then the following are equivalent.

(1) $T$ is Wakamatsu-silting.

(2) $T$ is semi-selforthogonal and $R \in \mathcal{X}^{[-r,0]}_T$. 

Lemma 2.3 [10] Let $T$ be a complex in $\mathcal{D}^b(\text{mod}R)$. The following are equivalent.

1. $T$ is Wakamatsu-silting.
2. $T$ is semi-selforthogonal and $DR \in \langle T, \mathcal{X}^b \rangle$. 
3. $DT$ (in $\mathcal{D}^b(\text{mod}R)$) is Wakamatsu-silting.

So we obtain a result which is dual to Lemma 2.2.

Lemma 2.4 Assume that $T \in \mathcal{D}^{[0,r]}(\text{mod}R)$ for some suitable integer $r$. Then the following are equivalent.

1. $T$ is Wakamatsu-silting.
2. $T$ is semi-selforthogonal and $DR \in \mathcal{T}\mathcal{X}^{[0,r]}$.

Proof. The proof is dual to [10] Theorem 3.4.

Proposition 2.5 Let $T \in \mathcal{D}^b(\text{mod}R)$ be a Wakamatsu-silting complex. Then $T^b \cap (\mathcal{X}^b)^{>0} = \text{add}_D T$.

Proof. It is obvious that $\text{add}_D T \subseteq T^b \cap (\mathcal{X}^b)_{>0}$. Take $X \in T^b \cap (\mathcal{X}^b)^{>0}$. Then there is a triangle $X' \rightarrow T' \rightarrow X \rightarrow$ where $X' \in T^b$ and $T' \in \text{add}_D T$. Since $X \in (\mathcal{X}^b)^{>0}$, the triangle splits. So, we have $X \in \text{add}_D T$.

The other condition can be proved dually.

3 Auslander-Reiten correspondence

In this section, we will give a characterization of Wakamatsu-silting complexes in term of subcategories of $\mathcal{D}^b(\text{mod}R)$.

Lemma 3.1 Let $T$ and $T'$ be two bounded Hom-projective generators of a relative coresolving subcategory $\mathcal{X} \subseteq \mathcal{D}^b(\text{mod}R)$. Then $T \simeq T'$.

Proof. Since $T'$ is a bounded Hom-projective generator of $\mathcal{X}$, there is a triangle $Y \rightarrow T_1' \rightarrow T \rightarrow$ where $T_1' \in \text{add}_D T'$ and $Y \in \mathcal{X}$. Since $\mathcal{X} \subseteq T^{>0}$, the triangle splits. Then we obtain that $T \in \text{add}_D T'$. Dually, we have $T' \in \text{add}_D T$. So, we have $T \simeq T'$.

Proposition 3.2 Let $T \in \mathcal{D}^b(\text{mod}R)$. Then the map $\phi : T \rightarrow T^b$ is an injective map between isomorphism classes of Wakamatsu-silting complexes and relative coresolving subcategories with a bounded Hom-projective generator.

Proof. By Example 1.6, Lemma 2.3 and [[10] Proposition 2.1], we obtain that the map is well defined. By Lemma 3.1, the map is injective.
Proposition 3.3 There exists a surjective map between relative coresolving subcategories $\mathcal{X} \subseteq \mathcal{D}^b(\text{mod} R)$ with a bounded Hom-projective generator and isomorphism classes Wakamatsu-silting complexes. This map is defined by $\psi : \mathcal{X} \rightarrow T$, where $\text{add}_T T = \mathcal{X} \cap \perp \mathcal{X}$.

Moreover, $(\psi \circ \phi)(T) = T$ for any Wakamatsu-silting complex $T$ and $\mathcal{X} \subseteq (\phi \circ \psi)(T)$ for any relative coresolving subcategory $\mathcal{X} \subseteq \mathcal{D}^b(\text{mod} R)$ with a bounded Hom-projective generator $T$.

Proof. Since $\mathcal{X}$ has a bounded Hom-projective generator $T$, we have $\text{add}_T T \subseteq \mathcal{X} \cap \perp \mathcal{X}$. If $X \in \mathcal{X} \cap \perp \mathcal{X}$, there is a split triangle $X' \rightarrow T' \rightarrow X \rightarrow$ where $T' \in \text{add}_T T$ and $X' \in \mathcal{X}$. So the triangle splits. Then we have $\mathcal{X} \cap \perp \mathcal{X} \subseteq \text{add}_T T$.

It is easy to see that $T$ is semi-selforthogonal. Now, we prove that $T$ is a Wakamatsu-silting complex. By Lemma 2.3, we need to show that $\text{DR} \in \langle T, \mathcal{X} \rangle$. Since $\mathcal{X}$ is relative coresolving, we obtain $\text{DR}[n] \in \mathcal{X}$ for some $n > 0$ by definition. By Corollary 1.5, $\text{DR}[n] \in T, \mathcal{X}$, i.e. $\text{DR} \in \langle T, \mathcal{X} \rangle$. So we obtain that $T$ is a Wakamatsu-silting complex. Then the map is well-defined.

By Proposition 2.5, we obtain that $\tau \mathcal{X} \cap \perp \mathcal{X} = \text{add}_T T$ for a Wakamatsu-silting complex. According to Example 1.6, Lemma 1.7, and Lemma 2.3, we obtain that $\tau \mathcal{X}$ is a relative coresolving subcategory with a bounded Hom-projective generator $T$. It follows that $(\psi \circ \phi)(T) = T$ for any Wakamatsu-silting complex.

Let $\mathcal{X}$ be a relative coresolving subcategory with a bounded Hom-projective generator $T$. By Corollary 1.5, we obtain $\mathcal{X} \subseteq (\phi \circ \psi)(\mathcal{X}) = \tau \mathcal{X}$.

We collect the results in Proposition 3.2 and 3.3 and give the following main results.

Theorem 3.4 Let $T \in \mathcal{D}^b(\text{mod} R)$ and $\mathcal{X} \subseteq \mathcal{D}^b(\text{mod} R)$. Then $\phi : T \rightarrow \tau \mathcal{X}$ and $\psi : \mathcal{X} \rightarrow T$, where $\text{add}_T T = \mathcal{X} \cap \perp \mathcal{X}$, are inverse bijection isomorphism classes of Wakamatsu-silting complexes and relative coresolving subcategories with a bounded Hom-projective generator, maximal among those with the same bounded Hom-projective generator.

Proof. By Proposition 3.3, for a relative coresolving subcategories $\mathcal{X}$ with a bounded Hom-projective generator $T$, we have $\mathcal{X} \subseteq (\phi \circ \psi)(\mathcal{X}) = \tau \mathcal{X}$. Thus, for any Wakamatsu-silting complex $T$, $\phi(T) = \tau \mathcal{X}$ is maximal among those with the same bounded Hom-projective generator $T$.

On the other hand, if $\mathcal{X}$ is a relative coresolving subcategory with a bounded Hom-projective generator, maximal among those with the same bounded Hom-projective generator, then $\mathcal{X} = (\phi \circ \psi)(\mathcal{X}) = \tau \mathcal{X}$.

We give the dual version of Theorem 3.4.
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**Theorem 3.5** Let \( T \in \mathcal{D}^b(\text{mod}R) \) and \( \mathcal{Y} \subseteq \mathcal{D}^b(\text{mod}R) \). Then \( \sigma : T \rightarrow \mathcal{X}_T^b \) and \( \xi : \mathcal{Y} \rightarrow T \), where \( \text{add}_T \mathcal{Y} = \mathcal{Y} \cap \mathcal{Y}^{\perp_0} \) are inverse bijection isomorphism classes of Wakamatsu-silting complexes and relative resloving subcategories with a bounded Hom-injective cogenerator, maximal among those with the same bounded Hom-injective cogenerator.

**References**


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