Abstract

We explore the nature of the continued fraction expansion of the Hurwitz numbers \( H = \frac{(ae^{2/n} + b)}{(ce^{2/n} + d)} \), with \( D = |ad - bc| \neq 0 \). We prove some results for determinant \( p^k \) with \( p \) a prime number. Also, we conjecture families of ‘pure’ Hurwitz numbers with determinant 2.

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1 Introduction

It is well known that any real number may be expressed as a decimal, which can terminate or not. Indeed, the decimal for a real number terminates if and only if the number is a rational fraction whose denominator is of the form \( 2^a5^b \). The decimals which does not terminate may be periodic. The pure recurring decimals are those in which the periodic part starts just after the decimal point. For example \( 1/7 = 0.142857142857 \ldots = 0.\overline{142857} \) has pure decimal expansion. Numbers which has pure recurring decimals are completed characterized ([3, Theorem 135, p. 143].)
On the other hand, every real number can be expressed as a simple continued fraction. The continued fraction for a real number \( \xi \) is obtained by repeating the procedure of “taking the integral part to be the partial quotient and starting again with the reciprocal of the number minus its integral part (if it is nonzero)”. The convergents of a continued fraction of an irrational number provide the “best” approximations by rationals. Also, we know that the continued fraction expansion of a real number is essentially unique.

Let \([a_0, a_1, a_2, \ldots]\) denotes the continued fraction

\[
a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}
\]

It is well know that the continued fraction for \( \xi \) terminates if and only if \( \xi \) is rational. An infinite simple continued fraction \([a_0, a_1, a_2, \ldots]\) is said to be periodic if there is a positive integer \( k \) and a non-negative integer \( N \) such that \( a_n = a_{n+k} \) for all \( n \geq N \). A periodic continued fraction may be written

\[
[a_0, a_1, \ldots, a_{N-1}, \overline{a_N, a_{N+1}, \ldots, a_{N+k-1}}].
\]

It turns out that an infinite continued fraction is periodic if and only if the number which represents is a quadratic number. A continued fraction is purely periodic if it is of the form \([a_0, \ldots, a_k]\). The real quadratic irrationals that have pure periodic continued fraction expansions are completely characterized ([4, Theorem 7.20, p. 348].)

Hurwitz proved ([2, 5]) that transcendental numbers of the form \((ae^{2/n} + b)/(ce^{2/n} + d)\) with \( a, b, c, d, n \) integers, \( ad - bc \neq 0 \), eventually consist of a fixed number of arithmetic progressions. Also see [6].

Let \( q \) be a power of a prime \( p \) and let \( \mathbb{F}_q \) denote a finite field of \( q \) elements. In the function field case, the simple continued fractions for the analogues of the Hurwitz numbers, are obtained replacing the integers by polynomials in \( \mathbb{F}_q[t] \) and the usual exponential replaced by the Carlitz exponential. These analogues have patterns but of completely different kind ([7, 8].)

2 Hurwitz numbers

2.1 Equivalent numbers

If \( \xi \) and \( \eta \) are two real numbers such that

\[
\xi = \frac{a\eta + b}{c\eta + d},
\]
where \(a, b, c, d\) are integers such that \(ad - bc = \pm 1\), then \(\xi\) is said to be equivalent to \(\eta\), since this relation is reflexive, symmetrical and transitive. Two irrational numbers are equivalent if and only if they have continued fractions with the same “tail”. Thus, a number and its integral linear transformation of determinant \(\pm 1\) are equivalent.

2.2 Hurwitz numbers

A number of the form

\[
H = \frac{ae^{2/n} + b}{ce^{2/n} + d}
\]

where \(a, b, c, d, n\) are integers with \(n \neq 0, ad - bc \neq 0\) is said to be a Hurwitz number. Classical results of Hurwitz show that the sequence of partial quotients for the continued fraction for \(H\) eventually consist of a fixed number of arithmetic progressions, i.e., the continued fraction of a Hurwitz number is of the form

\[
[c_0, c_1, \ldots, c_r, \varphi_1(j), \ldots, \varphi_l(j)]_{j \geq 0}
\]

\[
= [c_0, c_1, \ldots, c_r, \varphi_1(0), \ldots, \varphi_l(0), \varphi_1(1), \ldots, \varphi_l(1), \ldots],
\]

where \(c_0\) is an integer, \(c_1, \ldots, c_r\) are positive integers, \(\varphi_i\) is of the form \(\varphi_i(j) = a_i + b_ij\), where \(a_i, b_i\) are integral and at least one of the \(\varphi_i\) is not constant. \(\varphi_1(j), \ldots, \varphi_l(j)\) are said to form a quasi-period. If all \(c_i\)'s are all absent the continued fraction is called pure. The non constant linear functions are said to be the non constant part of the quasi-period; the constant part of the quasi-period is formed by the constant functions.

**Example 2.1 ([5, 1]).** Euler gave the continued fraction for \(e\):

\[
e = [2, 1, 2, 1, 4, 1, 1, 6, 1, \ldots] = [2, \overline{1, 2j, 1}]_{j \geq 1}.
\]

For \(k \geq 2\),

\[
e^{1/k} = [1, (2j + 1)k - 1, 1]_{j \geq 0} = [1, k - 1, 1, 3k - 1, 1, 1, 5k - 1, 1, \ldots].
\]

For \(n\) odd, \(n \geq 1\),

\[
e^{2/n} = \begin{cases} [3 + 3j, 18 + 12j, 5 + 3j, 1]_{j \geq 0} & \text{for } n = 1 \\ [1, \frac{n-1}{2} + 3nj, 6n + 12nj, \frac{5n-1}{2} + 3nj, 1]_{j \geq 0} & \text{for } n > 1. \end{cases}
\]
Other known examples are the following.

\[ ke^{1/k} = [k + 1, 2k - 1, 2j, 1]_{j \geq 1}, \quad k \geq 1. \]  
\[ \frac{1}{k} e^{1/k} = [0, k - 1, 2k, 1, 2j, 2k - 1]_{j \geq 1}, \quad k > 1. \]

We are mainly interested in the Hurwitz numbers which have pure continued fraction expansions.

2.3 Möbius transformations

A different way of looking Hurwitz numbers is through $2 \times 2$ integral matrices. Consider the Möbius transformation

\[ f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc \neq 0. \]

We have a map given by

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_A(z) = \frac{az + b}{cz + d}. \]

The product of two matrices is mapped to the composition of the corresponding Möbius transformations. Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then

\[ A_1 A_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} \mapsto \]

\[ \frac{(a_1 a_2 + b_1 c_2)z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2} = f_{A_1}(f_{A_2}(z)). \]

Therefore, $A_1 A_2 \mapsto f_{A_1 A_2} = f_{A_1} \circ f_{A_2}$.

Let

\[ B = \det(A) A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \]

The the entries of $B$ are integral and $\det(B) \neq 0$. Since $AB = \det(A) I = BA$, where $I$ is the two by two identity matrix, one has that

\[ f_A \left( f_B(z) \right) = \frac{\det(A) z}{\det(A)} = z = f_B \left( f_A(z) \right) \]

and $f_B$ is the inverse function of $f_A$, i.e., $f_A^{-1} = f_{\det(A) A^{-1}}$. Of course $f_A^{-1} = f_{\lambda \det(A) A^{-1}}$ for any $\lambda \neq 0$. Clearly, the map $A \mapsto f_A$ is not injective. Indeed,
Let \( f_A(z) = f_{\lambda A}(z) \) for \( \lambda \neq 0 \). If \( f_A(z) = z \) for all \( z \), then \( cz^2 + (d - a)z - b = 0 \) and therefore \( a = d \) and \( b = 0 = c \); thus \( A = aI \) for some nonzero integer \( a \). Therefore, if \( f_A(z) = f_B(z) \), then \( f_{\det(B)B^{-1}A}(z) = z \) and \( A = (\lambda / \det(B))B \).

We say that a matrix is \textit{primitive} if the greatest common divisor of its elements is one. If \( f_A(z) = f_B(z) \) and \( A \) and \( B \) are primitive, then \( A = \pm B \) (If \( A = \lambda B \), with \( r/s \in \mathbb{Q} \), \( (r, s) = 1 \) and \( s > 0 \), then \( a_{ij}s = rb_{ij} \). Let \( p \mid s \) with \( p \) a prime; then \( p \mid b_{ij} \) which is contrary to the hypothesis. Then \( s = 1 \). Now, if \( q \mid r \) with \( q \) a prime, then \( q \mid a_{ij} \) which is a contradiction. Then \( r = \pm 1 \).

A square matrix over the integers with determinant \( \pm 1 \) is said to be \textit{unimodular}. We say that the \( 2 \times 2 \) integral and primitive matrices \( A \) and \( B \) are \textit{equivalent} if there exists a unimodular matrix \( C \) such that \( A = CB \). Equivalently, \( A \) and \( B \) are equivalent if and only if \( AB^{-1} \) has integral entries and \( \det(AB^{-1}) = \pm 1 \).

In general, the product of primitive matrices is not primitive.

\textbf{Lemma 2.2.} If \( C \) is an integral matrix of determinant \( \pm 1 \) and \( A \) is an integral primitive matrix, then \( CA \) is primitive.

\textit{Proof.} Let \( C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) \( , \ A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \) and assume that \( p \) is a prime that divide all the entries of \( CA = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \). Then \( p \) divides \( d(ax + bz) - b(cx + dz) = x(ad - bc) \) and thus \( p \mid x \). Also, \( p \) divides \( -c(ax + bz) + a(cx + dz) = z(ad - bc) \) and thus \( p \mid z \). In a very similar way, we get that \( p \) divides \( y \) and \( w \). \( \Box \)

\textbf{Lemma 2.3.} Assume \( A \) and \( B \) are integral primitive matrices. The numbers \( f_A(z) \) and \( f_B(z) \) are equivalent if and only if \( A \) and \( B \) are equivalent.

\textit{Proof.} The numbers \( f_A(z) \) and \( f_B(z) \) are equivalent if and only if there exists a matrix \( C \) of determinant \( \pm 1 \) such that \( f_A(z) = f_C(f_B(z)) = f_{\lambda B}(z) \). Then \( \lambda A = CB \) for some \( \lambda \neq 0 \). Since \( CB \) is primitive, \( \lambda A \) is also primitive, so that \( \lambda = \pm 1 \). Therefore, \( A \) is equivalent to \( B \). Conversely, if \( A \) and \( B \) are equivalent, there exists an integral unimodular matrix \( C \) such that \( A = CB \). Then \( f_A(z) = f_{CB}(z) = f_C(f_B(z)) \). \( \Box \)

\textbf{Corollary 2.4.} Assume \( A \) and \( B \) are integral primitive matrices. The numbers \( f_A(z) \) and \( f_B(z) \) are equivalent if and only if the entries of the matrix \( AB^{-1} \) are integral and \( |\det(A)| = |\det(B)| \).

\textit{Proof.} The integral matrices \( A \) and \( B \) are equivalent if and only if \( AB^{-1} \) has integral entries and \( \det(AB^{-1}) = \pm 1 \). \( \Box \)
Theorem 2.5. Every integral primitive matrix $A$, with $\det A \neq 0$, is equivalent to an integral primitive matrix of the form
\[
\begin{pmatrix}
\alpha & \beta \\
0 & \delta
\end{pmatrix}
\]  
(4)
where $\alpha, \delta > 0$ and $0 \leq \beta < \delta$. The matrix (4) is uniquely determined by $A$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $D = ad - bc$ and let $e$ be the greatest common divisor of $a$ and $c$; then $e > 0$. Let $r, s$ be integers such that $e = ar + cs$. Let us assume $D > 0$ and let $k$ be the unique integer such that
\[-1 + \frac{e}{D}(br + ds) < k \leq \frac{e}{D}(br + ds),
\]
i.e., let $k$ be the floor of $e(br + ds)/D$ and put $x = r + kc/e$ and $y = s - ka/e$. Then $0 \leq bx + dy < D/e$ and
\[
\begin{pmatrix}
x & y \\
-c/e & a/e
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
\frac{e}{D} & bx + dy \\
0 & D/e
\end{pmatrix}.
\]
If $D < 0$, let $x$ and $y$ as before but with $k$ replaced by the ceil of $e(br + ds)/D$. Then
\[
\begin{pmatrix}
x & y \\
c/e & -a/e
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
\frac{e}{D} & bx + dy \\
0 & -D/e
\end{pmatrix}
\]
and $0 \leq bx + dy < -D/e$.

Now, suppose that $A$ is equivalent to $\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{pmatrix}$, where $\alpha_1, \delta_1 > 0$, $0 \leq \beta_1 < \delta_1$. Then $\alpha \beta = \alpha_1 \beta_1$. Without lost of generality, we may further assume that $\beta_1 \leq \beta$. It follows that the matrix
\[
C = \frac{1}{\alpha_1 \delta_1} \begin{pmatrix}
\alpha & \beta \\
0 & \delta
\end{pmatrix}
\begin{pmatrix}
\delta_1 & -\beta_1 \\
0 & \alpha_1
\end{pmatrix}
= \begin{pmatrix}
\frac{\alpha}{\alpha_1} & \frac{-\beta_1}{\delta_1} + \frac{\beta}{\delta} \\
0 & \frac{\delta}{\delta_1}
\end{pmatrix}
\]
is integral and has determinant 1. Then $(\alpha/\alpha_1)(\delta/\delta_1) = 1$ implies $\alpha = \alpha_1$ and $\delta = \delta_1$. Since $0 \leq -\beta_1/\delta + \beta/\delta_1 < 1$, the entry $(1, 2)$ of $C$ the is zero so that $\beta = \beta_1$. This completes the proof. \qed

See [5] for a similar result.

Next, we consider Hurwitz numbers of a special type.
Proposition 2.6. Let $s, D$ be integers with $D \geq 1$. Fix $n$. Let

$$H_s = \frac{se^{2/n} - (s + 1)}{-De^{2/n} + D}.$$  \hfill (5)

For $k \in \mathbb{Z}$, the Hurwitz numbers $H_s$ and $H_{s+kD}$ are equivalent. For $0 \leq s, t < D$ with $s \neq t$, $H_s$ and $H_t$ are not equivalent.

Proof. Let $A_s$ be the integral matrix

$$A_s = \begin{pmatrix} s & -(s + 1) \\ D & D \end{pmatrix}.$$  

Since $s$ and $-(s + 1)$ are coprime, it follows that $A_s$ is primitive. Now,

$$A_s A_{s+kD}^{-1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_s A_t^{-1} = \begin{pmatrix} 1 & (t-s)/D \\ 0 & D \end{pmatrix}.$$  

Since the matrix $A_s A_{s+kD}^{-1}$ has integral entries and $\det(A_s) = \det(A_{s+kD})$, the first assertion follows from Corollary 2.4. On the other hand, for $0 \leq s, t < D$ and $s \neq t$, we have $0 < |t - s| < D$ so that $D$ does not divide $t - s$ and $(t-s)/D$ is not integral. Therefore $A_s$ and $A_t$ are not equivalent.

Proposition 2.7. Let $p$ be a prime and $k$ a positive integer. Then, there are exactly $p^k + p^{k-1}$ non equivalent matrices of determinant $p^k$, where $p$ is a prime. The matrices

$$\begin{pmatrix} s & -(s + 1) \\ -p^k & p^k \end{pmatrix}, \quad s = 0, 1, \ldots, p^k - 1,$$  \hfill (6)

$$\begin{pmatrix} 1 & p(t + 1) - 1 \\ 0 & p^k \end{pmatrix}, \quad t = 0, 1, \ldots, p^{k-1} - 1.$$  \hfill (7)

are representatives of the distinct classes of matrices of determinant $p^k$.

Proof. Denote by $A_s$ and $B_t$ the matrices of the form (6) and (7), respectively. By Proposition 2.6, the matrices $A_s$, $s = 0, \ldots, p^k - 1$ are not equivalent among them. For $0 \leq s, t < p^{k-1} - 1$ and $s \neq t$, we have

$$B_s B_t^{-1} = \begin{pmatrix} 1 & \frac{1}{p^k} (s - t) \\ 0 & 1 \end{pmatrix}.$$  

It follows that the entry $(1, 2)$ of $B_s B_t^{-1}$ is not an integer. Therefore, $B_s$ and $B_t$ are not equivalent. Finally, we see that

$$A_s B_t^{-1} = \begin{pmatrix} s & \frac{1}{p^k} (-ps(t + 1) - 1) \\ -p^k & p + pt \end{pmatrix}.$$
Since $p$ does not divide $ps(t + 1) + 1$, the entry $(1, 2)$ is not integral. This finishes the proof. 

**Corollary 2.8.** Fix $n$. For $p$ prime and $k \geq 1$, the Hurwitz numbers

\[
\frac{se^{2/n} - (s + 1)}{-pe^{2/n} + p}, \quad s = 0, 1, \ldots, p^k - 1,
\]

\[
\frac{1}{p^k(e^{2/n} + p(t + 1) - 1)}, \quad t = 0, 1, \ldots, p^{k-1} - 1.
\]

are non equivalent among them.

### 3 Families of pure continued fractions

Let $[B_1, \ldots, B_t, \varphi_1(j), \ldots, \varphi_l(j)]_{j \geq 0}$ be the continued fraction of the Hurwitz number $H$, where $\varphi_i(j) = f_i(n) + g_i(n)j$ is an arithmetical progression in $j$ and $f_i(n)$ and $g_i(n)$ are linear in $n$.

Let $p$ a prime. According to Proposition 2.7, there exist an only a finite number of non equivalent matrices having determinant $p^k$, and indeed, we know the exact number of them.

Let $a, b, c, d$ integers such that $D = ad - bc$. The numerical evidence shows that there are at most $2D$ different types of continued fractions, depending on the remainder left by $n$ when divided by $2D$ with maybe only one exception, namely, when $D = 2$.

When $D \geq 5$ is a prime, the numerical evidence shows that the length of the non constant part of the quasi-period of the continued fraction of a Hurwitz number of the form (5) exhibit regularity.

#### 3.1 Hurwitz numbers of determinant one

If $(ae^{2/n} + b)/(ce^{2/n} + d)$ and $(a'e^{2/n} + b')/(c'e^{2/n} + d')$ are Hurwitz numbers which both have determinant $\pm 1$, then, by Corollary 2.4 they are equivalent.

The matrix \[
\begin{pmatrix}
0 & -1 \\
-1 & 1
\end{pmatrix}
\]
which has determinant $-1$ yields to pure continued fractions.

\[
\frac{1}{e^{2/n} - 1} = \begin{cases}
\left[\frac{n-1}{2} + nj, 1, 1\right]_{j \geq 0} & \text{if } n \equiv 0 \text{ mod } 2 \text{ and } n \geq 2,

\left[\varphi_1(j), \varphi_2(j), \varphi_3(j), 1, 1\right]_{j \geq 0}, & \text{if } n \equiv 1 \text{ mod } 2 \text{ and } n \geq 1,
\end{cases}
\]

where

\[
\varphi_1(j) = \frac{n - 1}{2} + 3nj, \quad \varphi_2(j) = 6n + 12nj, \quad \varphi_3(j) = 2n + \frac{n - 1}{2} + 3nj.
\]
Remark 3.1. Observe that the continued fraction (8) for \( n \) odd, which was discovered empirically, is equivalent to (1).

3.2 Hurwitz numbers of determinant two

There are only three classes of equivalence matrices of determinant \( \pm 2 \). The following non equivalent matrices produce pure continued fractions.

\[
\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Depending on the remainder of \( n \) when is divided by 4: For \( n \equiv 0 \mod 4 \), \( n \geq 4 \), we have

\[
\frac{1}{2e^{2/n} - 2} = \left\lfloor \frac{n - 4}{4} + \frac{n}{2}, 1, 3 \right\rfloor_{j \geq 0}.
\]

For \( n \geq 1 \), \( n \equiv 1 \mod 4 \),

\[
\frac{1}{2e^{2/n} - 2} = [\varphi_1(j), \varphi_2(j), \varphi_3(j), 3, 1, \varphi_4(j), \varphi_5(j), \varphi_6(j), 1, 3]_{j \geq 0},
\]

where

\[
\varphi_1(j) = (n - 1)/4 + 3nj, \\
\varphi_2(j) = 12n + 48nj, \\
\varphi_3(j) = n + n - 1/4 + 3nj, \\
\varphi_4(j) = 2n - ((n - 1)/4 + 1) + 3nj, \\
\varphi_5(j) = 36n + 48nj, \\
\varphi_6(j) = 3n - ((n - 1)/4 + 1) + 3nj.
\]

For \( n \equiv 2 \mod 4 \), \( n > 2 \), then

\[
\frac{1}{2e^{2/n} - 2} = [\varphi_1(j), 3, 1, \varphi_2(j), 1, 3]_{j \geq 0},
\]

where

\[
\varphi_1(j) = (n - 2)/4 + n j, \quad \varphi_2(j) = (n - 2)/4 + n/2 - 1 + n j.
\]

For \( n \geq 3 \), \( n \equiv 3 \mod 4 \),

\[
\frac{1}{2e^{2/n} - 2} = [\varphi_1(j), 1, 1, \varphi_2(j), 1, 1, \varphi_3(j), 3, 1, \varphi_4(j), 1, 1, \varphi_5(j), 1, 1, \varphi_6(j), 1, 3]_{j \geq 0},
\]
where
\[
\begin{align*}
\varphi_1(j) &= (n - 3)/4 + 3nj, \\
\varphi_2(j) &= 3n - 1 + 12nj, \\
\varphi_3(j) &= n + (n - 3)/4 + 3nj, \\
\varphi_4(j) &= 2n - ((n - 3)/4 + 2) + 3nj, \\
\varphi_5(j) &= 9n - 1 + 12nj, \\
\varphi_6(j) &= 3n - ((n - 3)/4 + 2) + 3nj.
\end{align*}
\]

Let us consider Hurwitz numbers corresponding to the matrix \( \begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix} \).

Let \( n \geq 2 \). For \( n \equiv 0 \mod 4 \), we have
\[
\frac{e^{2/n} - 2}{-2e^{2/n} + 2} = \left[ \frac{n - 4}{4} + \frac{nj}{2} \right]_{j \geq 0},
\]
where
\[
\begin{align*}
\varphi_1(j) &= \frac{n - 1}{4} - 1 + 3nj, \\
\varphi_2(j) &= 3n - 1 + 12nj, \\
\varphi_3(j) &= n + \frac{n - 1}{4} - 1 + 3nj, \\
\varphi_4(j) &= 2n - \frac{(n - 1)}{4} + 1 + 3nj, \\
\varphi_5(j) &= 9n - 1 + 12nj, \\
\varphi_6(j) &= 3n - \frac{(n - 1)}{4} + 1 + 3nj.
\end{align*}
\]

For \( n \equiv 1 \mod 4 \),
\[
\frac{e^{2/n} - 2}{-2e^{2/n} + 2} = [\varphi_1(j), 1, 1, \varphi_2(j), 1, 1, \varphi_3(j), 1, 3, \varphi_4(j), 1, 1, \varphi_5(j), 1, 1, \varphi_6(j), 3, 1],_{j \geq 0},
\]
where
\[
\begin{align*}
\varphi_1(j) &= (n - 1)/4 - 1 + 3nj, \\
\varphi_2(j) &= 3n - 1 + 12nj, \\
\varphi_3(j) &= n + (n - 1)/4 - 1 + 3nj, \\
\varphi_4(j) &= 2n - ((n - 1)/4 + 1) + 3nj, \\
\varphi_5(j) &= 9n - 1 + 12nj, \\
\varphi_6(j) &= 3n - ((n - 1)/4 + 1) + 3nj.
\end{align*}
\]

For \( n \equiv 2 \mod 4 \),
\[
\frac{e^{2/n} - 2}{-2e^{2/n} + 2} = [\varphi_1(j), 1, 3, \varphi_2(j), 3, 1],_{j \geq 0},
\]
where
\[
\begin{align*}
\varphi_1(j) &= n - 3((n - 2)/4 + 1) + nj, \\
\varphi_2(j) &= n - ((n - 2)/4 + 1) + nj.
\end{align*}
\]

For \( n \equiv 3 \mod 4 \),
\[
\frac{e^{2/n} - 2}{-2e^{2/n} + 2} = [\varphi_1(j), \varphi_2(j), \varphi_3(j), 1, 3, \varphi_4(j), \varphi_5(j), \varphi_6(j), 3, 1],_{j \geq 0},
\]
where
\[
\begin{align*}
\varphi_1(j) &= (n - 3)/4 + 3nj, \\
\varphi_2(j) &= 12n + 48nj, \\
\varphi_3(j) &= n + (n - 3)/4 + 3nj, \\
\varphi_4(j) &= 2n - ((n - 3)/4 + 1) + 3nj, \\
\varphi_5(j) &= 36n + 48nj, \\
\varphi_6(j) &= 3n - (n - 3)/4 - 1 + 3nj.
\end{align*}
\]

Finally,
\[
\frac{e^{2/n} + 1}{e^{2/n} - 1} = \left[ n + 2nj \right]_{j \ge 0}, \quad n = 1, 2, 3, \ldots
\]

References


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