International Journal of Algebra, Vol. 15, 2021, no. 4, 171 - 179
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2021.91563

Classes of Numerical Semigroups with Embedding Dimension 3: An algorithm for Computing the Frobenius Number

Violeta Angjelkoska

Faculty of Information and Communication Technology
FON University, Skopje, Republic of North Macedonia

Dončo Dimovski

Macedonian Academy of Sciences and Arts
Skopje, Republic of North Macedonia

Irena Stojmenovska

Faculty of Computer Science and Information Technology
University American College, Skopje, Republic of North Macedonia

Abstract

In this paper we present an algorithm for computing the Frobenius number of a numerical semigroup G with embedding dimension equal to 3 such that

$$G = [n; \{i, j\}; \{b_i, b_j\}], GCD(n, i) = GCD(n, j) = 1, n < x < y,$$
$$x = b_i n + i \text{ and } y = b_j n + j.$$

As a consequence, we give an algorithm for determining the set $\mathcal{F}$ of all numerical semigroups with embedding dimension 3, given its multiplicity n and the corresponding remainders i and j of the generating elements x and y modulo n i.e., the set

$$\mathcal{F} = \{[n; \{i, j\}; \{b_i, b_j\}] | b_i, b_j \in \mathbb{N}, n < b_i n + i < b_j n + j\}.$$

Keywords: Frobenius problem, Frobenius number, numerical semigroups

1 Introduction

We start with the well-known Frobenius problem (F.G. Frobenius, 19th century) and a short review of the obtained results.

Given positive integers $a_1, \dots, a_m$ *with* $GCD(a_1, \dots, a_m) = 1$, *compute the largest integer* $F(a_1, \dots, a_m)$ *not representable as a non-negative integer linear combination of* $a_1, \dots, a_m$, *called Frobenius number, and compute how many positive integers do not have such representation.*

For $m = 2$, there have been proved by J. J. Sylvester that $F(a_1, a_2) = a_1 a_2 - a_1 - a_2$ (J. C. Rosales at al. [15]). The question of finding a general formula for computing the corresponding Frobenius number when $m \geq 3$, turned out to be much more difficult to answer and still remains open in general case.

It has been proved that Frobenius number cannot be given by "closed" formulas of a certain type when $m \geq 3$. (F. Curtis [7]).

S. M. Johnson had first developed an algorithm computing the Frobenius number when $m = 3$ ([12]). R. Brauer and J. Shockley in 1962 were the first who provided an algorithm for solving the Frobenius problem, by using the brute force algorithm (R. Owens [13]). They obtained an L-shaped region of lattice points in the plane, where each point is uniquely associated with one of the residue classes modulo a_1, where a_1 is the smallest element of three given positive integers. Then, the solution of the Frobenius problem corresponds to one of two so-called corner points. Later, E. S. Selmer and Ø. Beyer developed an algorithm using the continued fractions, which was simplified by Ø. Rødseth (J. L. R. Alfonsín [3]). H. Greenberg in 1988 developed an algorithm using the continued fraction and the Euclidean algorithm ([11]). J. L. Davison in 1994 proposed an algorithm using the Euclidean algorithm ([8]). The algorithms by J. L. Davison and by H. Greenberg are the fastest known algorithms for computing the Frobenius number when $m = 3$ (D. Beihoffer at al. [6]). Recently, minimal distance diagrams, from theory of graphs, are also used for computing the Frobenius number for $m = 3$ ([1, 2, 10]).

A variety of formulas for special cases, algorithms and bounds for the Frobenius number when $m \geq 3$, are systematically elaborated in [3].

The Frobenius problem is closely connected to the notion of a numerical semigroup introduced bellow.

Let $\mathbb{N}_0$ be the set of the nonnegative integers and $\mathbb{N}$ be the set of the positive integers. A proper nonempty set G of $\mathbb{N}_0$ is a **numerical semigroup** if G is closed under addition, $0 \in G$ and $\mathbb{N}_0 \backslash G$ is finite. In other words, a numerical semigroup is an additive semigroup of nonnegative integers such that the greatest common divisor of its elements is 1. A set $S = \{n_1, \dots, n_m\} \subseteq G$ is a set of generators for G, denoted by $G = < S >$ or $G = < n_1, \dots, n_m >$, if

$$G = \{\alpha_1 n_1 + \cdots + \alpha_m n_m | \alpha_1, \dots, \alpha_m \in \mathbb{N}_0\}.$$

Let S be a set of generators for G. We say that S is a **minimal set of generators** for G, if none of its proper subsets generates G. Every numerical semigroup has a unique minimal set of generators, which is finite (J. C. Rosales [15]).

The cardinality of the minimal set of generators for G is called **embedding dimension** of G, denoted by $ed(G)$. The smallest number in the minimal set of generators is called **multiplicity** of G, denoted by n. The largest number not belonging to G is called **Frobenius number** of G, denoted by $F(G)$. In fact, $F(G)$ is the largest integer that cannot be presented as a linear combination of its minimal set of generators, with non-negative integer coefficients. The set $\mathbb{N}_0 \backslash G$ is called the set of gaps of G. The cardinality of $\mathbb{N}_0 \backslash G$ is called **genus** of G, denoted by $g(G)$. Thus, the Frobenius problem indeed consists of finding a formula for calculating the Frobenius number and the genus of a numerical semigroup in terms of its minimal set of generators.

2 Preliminaries

The aim of this paper is to present an original algorithm for computing the Frobenius number of numerical semigroups with embedding dimension 3. Moreover, we have also developed an algorithm for finding the set of all numerical semigroups with embedding dimension 3 under certain conditions.

For that purpose, we recall the structure of additive semigroups of integers given in [9].

Theorem 2.1 (Theorem 1.2. in [9]). Let G be a semigroup consisting of positive integers. Let n be the smallest integer in G, d the greatest common divisor of the elements of G and $n = kd$. Let us denote by A_i the set of all the elements in G whose remainder after division by n is id i.e., $A_i = \{a | a \in G, a = nt + id, t \in \mathbb{N}\}$.

Then:

(i) $G = A_0 \cup A_1 \cup ... \cup A_{k-1}$, the union is disjoint.

(ii) There exist $a_0 = 1, a_1, \dots, a_{k-1}$, such that $A_i = \{tn + id | t \geq a_i\}$ and

$$a_i + a_j \geq \begin{cases} a_{i+j}, & i + j < k \\ a_{i+j-k} - 1, & i + j \geq k. \end{cases}$$

(iii) If $m_i = a_i n + id$, then $\{n = m_0, m_1, \dots, m_{k-1}\}$ is a set of generators for G.

(iv) Let $b = max\{a_0, a_1, \dots, a_{k-1}\}$, $s = max\{i | a_i = b\}$ and $c = (b-1)k + s + 1$. Then $(c-1)d \notin G$ and $\{td | t \geq c\} = G_* \subseteq G$. (We say that G_* is the *regular part* of G.) ∎

In [5] we have made a characterization of the embedding dimension of numerical semigroups in the same manner as additive semigroups of integers were characterized in [9], and moreover, we have obtained an explicit formula for computing $F(G)$ for $ed(G) = 3$. This is fundamental i.e., a starting point for the results obtained in this paper.

The addition of integers modulo n will be denoted by $\oplus$ and the additive group of integers modulo n will be denoted by $(\mathbb{Z}_n, \oplus)$. If $X \subseteq \mathbb{Z}_n$, the subgroup of $(\mathbb{Z}_n, \oplus)$ generated by X will be denoted by $< X >$. The multiplication of integers modulo n will be denoted by $\odot$. For a real number x, let $[x]$ be the integer part of x i.e., let $[x]$ be the biggest integer less than or equal to x, and let

$$\lceil x \rceil = \begin{cases} [x] + 1, & x \notin \mathbb{Z} \\ [x], & x \in \mathbb{Z}. \end{cases}$$

In continuation, we will use notations and results from [5].

For $i_1,\ldots,i_k \in \mathbb{Z}_n$ and $k \in \mathbb{N}$ the integer part $\left[\frac{i_1+\ldots+i_k}{n}\right]$ will be denoted by $[n; i_1,\ldots,i_k]$.

A numerical semigroup G will be denoted as $G = [n; T; B(T)]$, where n is multiplicity of G, $T \subseteq \mathbb{Z}_n \backslash \{0\}$ such that $< T > = \mathbb{Z}_n$ and $|T| = ed(G) - 1$, and $B(T) = \{b_s | s \in T\} \subseteq \mathbb{N}$ satisfies the following condition:

if $t \in T$ and $t = i_1 \oplus \ldots \oplus i_r$, for $i_1,\ldots,i_r \in T\backslash\{t\}$ and $r \in \mathbb{N}$, then

$$b_t < b_{i_1} + \ldots + b_{i_r} + [n; i_1,\ldots,i_r].$$

We recall the particular characterization of the Frobenius number of numerical semigroups with embedding dimension less than or equal to 3, in terms of its minimal set of generators. For deeper reading, see [5].

Let $G = [n; \{i,j\}; \{b_i, b_j\}]$ be a numerical semigroup with embedding dimension 3 and $GCD(n,i) = GCD(n,j) = 1$. Then

$$G = \langle n, x, y\rangle = \{m_k + a | k \in \mathbb{Z}_n, a \in \mathbb{N}_0,\},$$

where:

$$x = b_i n + i,\ y = b_j n + j \text{ and } \{m_k | k \in \mathbb{Z}_n\} = \mathcal{A} = \{a_k n + k | k \in \mathbb{Z}_n, a_0 = 1\}.$$

We have defined a homomorphism $\varphi: \mathbb{Z}^2 \to \mathbb{Z}_n$ by

$$\varphi(z_1, z_2) = t \text{ if and only if } z_1 i + z_2 j \equiv t \ (mod\, n).$$

It is easy to conclude that $H = ker\,\varphi$ is an additive subgroup of $\mathbb{Z}^k$ of rank k.

We say that a pair $(p, -q) \in H$, for $p, q \in \mathbb{Z}_n$, is a **minimal pair**, if there is no $(p', -q') \in H$, for $p', q' \in \mathbb{Z}_n$, such that $p' < p$ and $q' < q$. We say that two minimal pairs $(p,-q)$, $(u,-v)$ are **consecutive**, if

$$p > u,\ q < v \text{ and}$$
$$0 < c < p,\ 0 < d < v \Rightarrow (c, -d) \notin H.$$

Lemma 2.2 (Lemma 4.2. in [5]). Let $(p,-q), (u,-v)$ be two minimal consecutive pairs. Then

$$pv - qu = n \text{ and } \{s \odot i \oplus r \odot j | (s,r) \in A_L \cup A_R\} = \mathbb{Z}_n,$$

where

$$A_L = \{(s,r) | 0 \leq s < p, 0 \leq r < v - q\} \text{ and}$$
$$A_R = \{(s,r) | 0 \leq s < p - u, 0 \leq r < v\}. \blacksquare$$

Next, for $G = [n; \{i,j\}; \{b_i, b_j\}]$, $x = b_i n + i$ and $y = b_j n + j$, let i^{-1} be the inverse of i modulo n and:

- p be the smallest positive integer such that $px > (p \odot i \odot j^{-1})y$, and
- v be the smallest positive integer such that $vy > (v \odot j \odot i^{-1})x$.

A simple calculation implies that the pairs

$$(p, -p \odot i \odot j^{-1}) \text{ and } (v \odot j \odot i^{-1}, -v)$$

satisfy the condition of Lemma 2.2. Thus,

$$\mathcal{A} = \{sx + ry | (s,r) \in A_L \cup A_R\} \text{ and}$$
$$F(G) = (p-1)x + (v-1)y - min\{(v \odot j \odot i^{-1})x, (p \odot i \odot j^{-1})y\} - n.$$

To find all the minimal pairs, we start with the minimal pairs $(n, 0)$ and $(j \odot i^{-1}, -1)$. The next minimal pair is $\left(\left\lceil \frac{n}{j \odot i^{-1}} \right\rceil (j \odot i^{-1}) - n, -\left\lceil \frac{n}{j \odot i^{-1}} \right\rceil\right)$. If $(p,-q)$ and $(u,-v)$ are two consecutive minimal pairs such that $u \neq 0$, then the next minimal pair is

$$\left(\left\lceil\frac{p}{u}\right\rceil u - p, -\left(\left\lceil\frac{p}{u}\right\rceil v - q\right)\right).$$

Applying the results and discussion above, we have proved the following in [5]:

Theorem 2.3 (Theorem 4.3. in [5])**.** Let $G = \langle n, x, y\rangle$ be a numerical semigroup with $ed(G) = 3$. Then:

(i) There are unique $p, q, u, v \in \mathbb{N}$ obtained by the procedure given above, such that:

$$px \equiv qy (mod\, n),\ vy \equiv ux (mod\, n),$$
$$px > qy \text{ and } vy > ux;$$

(ii) The Frobenius number $F(G)$ of G is

$$px + vy - \frac{ux + qy - |ux - qy|}{2} - n - x - y. \blacksquare$$

As a consequence, we conclude that the numerical semigroup G in Theorem 2.3 is determined by the matrix $\begin{bmatrix} p & -q \\ -u & v \end{bmatrix}$. (For more details, see [4]).

The family of all numerical semigroups with embedding dimension 3, given its multiplicity n and remainders i and j of the generating elements x and y modulo n, respectively, where b_i and b_j are arbitrary positive integers such that $n < x = b_i n + i < y = b_j n + j$, will be denoted by $\mathcal{F}$. In other words,

$$\mathcal{F} = \{[n; \{i, j\}; \{b_i, b_j\}] \,|\, b_i, b_j \in \mathbb{N}, n < b_i n + i < b_j n + j\}.$$

3 An algorithm for computing the Frobenius number of a numerical semigroup with embedding dimension equal to 3

Applying the results in the previous section, bellow we create an algorithm for computing the Frobenius number of a numerical semigroup with embedding dimension equal to 3. Afterwards we illustrate it on an example.

Algorithm 1. An algorithm for computing the Frobenius number of a numerical semigroup $G = \langle n, x, y\rangle$ such that $ed(G) = 3, n < x < y,\ x \equiv i(mod\, n), y \equiv j(mod\, n)$ and $GCD(x, n) = GCD(y, n) = 1$.
Step 1. Let $$(p_0, -q_0) = (n, 0) \text{ и } (p_1, -q_1) = (j \odot i^{-1}, -1)$$ be the first consecutive minimal pairs and let $t = 0$. Step 2. If $(p_t, -q_t)$ and $(p_{t+1}, -q_{t+1})$ are two consecutive minimal pairs, then the next minimal pair is $$(p_{t+2}, -q_{t+2}) = (\left\lceil\frac{p_t}{p_{t+1}}\right\rceil p_{t+1} - p_t, -(\left\lceil\frac{p_t}{p_{t+1}}\right\rceil q_{t+1} - q_t)).$$ If $p_{t+1}x > q_{t+1}y$ and $p_{t+2}x < q_{t+2}y$, then $$F(G) = p_{t+1}x + q_{t+2}y - \frac{p_{t+2}x + q_{t+1}y - |p_{t+2}x - q_{t+1}y|}{2} - n - x - y,$$ and procedure ends. Otherwise, move to the next step. Step 3. Increase t by 1 and go to step 2.

The number of steps in Algorithm 1 is finite, since the sequence $p_0, p_1, p_2, \ldots$ is strictly decreasing. Namely, for two consecutive minimal pairs $(p_t, -q_t)$ and $(p_{t+1}, -q_{t+1})$ we have that $p_t > p_{t+1}$ and $\left\lceil \frac{p_t}{p_{t+1}} \right\rceil < \frac{p_t}{p_{t+1}} + 1$. Therefore,

$$\left\lceil \frac{p_t}{p_{t+1}} \right\rceil < \frac{p_t + p_{t+1}}{p_{t+1}}, \text{ i.e. } p_{t+2} = \left\lceil \frac{p_t}{p_{t+1}} \right\rceil p_{t+1} - p_t < p_{t+1}.$$

Example 3.1. We will compute the Frobenius number for the numerical semigroup $G = \langle 16783, 50802, 68132 \rangle$. We have that $n = 16783$, $i = 453$ and $j = 1000$.
According to Algorithm 1, we obtain the following steps:
Step 1. $(p_0, -q_0) = (16783, 0)$, $(p_1, -q_1) = (10524, -1)$ and $t = 0$.
Step 2. $(p_2, -q_2) = (4265, -2)$, $10525x > y$ and $4265x > 2y$.
Step 3. $t = 1$, $(p_3, -q_3) = (2271, -5)$, $4265x > 2y$ and $2271x > 5y$.
Step 4. $t = 2$, $(p_4, -q_4) = (277, -8)$, $2271x > 5y$ and $277x > 8y$.
Step 5. $t = 3$, $(p_5, -q_5) = (222, -67)$, $277x > 8y$ and $222x > 67y$.
Step 6. $t = 4$, $(p_6, -q_6) = (167, -126)$, $277x > 8y$ and $167x < 126y$.
Hence, we get that

$$F(G) = 222 \cdot 50802 + 126 \cdot 68132 - \frac{167 \cdot 50802 + 67 \cdot 68132 - |167 \cdot 50802 - 67 \cdot 68132|}{2} - 16783 - 50802 - 68132 = 15162115. \blacksquare$$

We note that using the algorithm above, we have computed the lengths of the L-shapes in a plane, that have been previously obtained by R. Brauer and J. Shockley by the brute force algorithm.

4 An algorithm for computing the set of numerical semigroups $\mathcal{F}$

In our research, we have come across several papers containing results for computing all numerical semigroups under certain conditions ([14, 16, 17]). In this paper we present an algorithm for determining the set $\mathcal{F}$ of all numerical semigroups with embedding dimension 3, for which the multiplicity n and remainders i and j of generating elements x and y modulo n, respectively, are given.

In fact, bellow we present an algorithm for determining the set

$$\mathcal{F} = \{[n; \{i, j\}; \{b_i, b_j\}] \,|\, b_i, b_j \in \mathbb{N}, n < b_i n + i < b_j n + j\}.$$

This algorithm is based on Algorithm 1 from the previous section.

Algorithm 2. An algorithm for computing the set of numerical semigroups

$$\mathcal{F} = \{[n; \{i, j\}; \{b_i, b_j\}] \,|\, b_i, b_j \in \mathbb{N}, n < b_i n + i < b_j n + j\}$$

with given multiplicity n, remainders i and j of generating elements x and y modulo n, respectively, for arbitrary positive integers b_i and b_j such that $n < b_i n + i < b_j n + j$.

Step 1. Let $(p_0, -q_0) = (n, 0)$ and $(p_1, -q_1) = (j \odot i^{-1}, -1)$ be the first consecutive minimal pairs and let $t = 0$.
Step 2. If $(p_t, -q_t)$ and $(p_{t+1}, -q_{t+1})$ are two consecutive minimal pairs, then the next minimal pair is

$$(p_{t+2}, -q_{t+2}) = \left(\left\lceil\frac{p_t}{p_{t+1}}\right\rceil p_{t+1} - p_t, -\left(\left\lceil\frac{p_t}{p_{t+1}}\right\rceil q_{t+1} - q_t\right)\right).$$

We obtain the numerical semigroup determined with the matrix $\begin{bmatrix} p_{t+1} & -q_{t+1} \\ -p_{t+2} & q_{t+2} \end{bmatrix}$.
If $p_{t+2} = 1$ the procedure ends. Otherwise, move to the next step.
Step 3. t increments by 1 and move to step 2.

The number of steps in Algorithm 2 is finite, for the same reasons as in Algorithm 1.
We illustrate the algorithm above on the following example:

Example 4.1. Let

$$\mathcal{F} = \{[2183; \{1,153\}; \{b_1, b_{153}\}] | b_1, b_{153} \in \mathbb{N}, 2183 < 2183b_1 + 1 < 2183b_{153} + 153\}.$$

Hence, $n = 2183$ and $j \odot i^{-1} = 153$. According to Algorithm 2, we obtain the following numerical semigroups:

Step 1. $(p_0, -q_0) = (2183, 0)$, $(p_1, -q_1) = (153, -1)$ and $t = 0$.
Step 2. $(p_2, -q_2) = (112, -15)$ and $\begin{bmatrix} 153 & -1 \\ -112 & 15 \end{bmatrix}$.
Step 3. $t = 1$, $(p_3, -q_3) = (71, -29)$ and $\begin{bmatrix} 112 & -15 \\ -71 & 29 \end{bmatrix}$.
Step 4. $t = 2$, $(p_4, -q_4) = (30, -43)$ and $\begin{bmatrix} 71 & -29 \\ -30 & 43 \end{bmatrix}$.
Step 5. $t = 3$, $(p_5, -q_5) = (19, -100)$ and $\begin{bmatrix} 30 & -43 \\ -19 & 100 \end{bmatrix}$.
Step 6. $t = 4$, $(p_6, -q_6) = (8, -157)$ and $\begin{bmatrix} 19 & -100 \\ -8 & 157 \end{bmatrix}$.
Step 7. $t = 5$, $(p_7, -q_7) = (5, -371)$ and $\begin{bmatrix} 8 & -157 \\ -5 & 371 \end{bmatrix}$.
Step 8. $t = 6$, $(p_8, -q_8) = (2, -585)$ and $\begin{bmatrix} 5 & -371 \\ -2 & 585 \end{bmatrix}$.
Step 9. $t = 7$, $(p_9, -q_9) = (1, -1384)$ and $\begin{bmatrix} 2 & -585 \\ -1 & 1384 \end{bmatrix}$.

Since $p_9 = 1$, the procedure ends.

Thus, we conclude that,

$$\mathcal{F} = \left\{\begin{bmatrix} 153 & -1 \\ -112 & 15 \end{bmatrix}, \begin{bmatrix} 112 & -15 \\ -71 & 29 \end{bmatrix}, \begin{bmatrix} 71 & -29 \\ -30 & 43 \end{bmatrix}, \begin{bmatrix} 30 & -43 \\ -19 & 100 \end{bmatrix}, \begin{bmatrix} 19 & -100 \\ -8 & 157 \end{bmatrix}, \begin{bmatrix} 8 & -157 \\ -5 & 371 \end{bmatrix}, \begin{bmatrix} 5 & -371 \\ -2 & 585 \end{bmatrix}, \begin{bmatrix} 2 & -585 \\ -1 & 1384 \end{bmatrix}\right\}. \blacksquare$$

References

[1] F. Aguiló, C. Marijuán, Classification of numerical 3-semigroups by means of L-shapes, *Semigroup Forum*, **88** (2014), 670–688. https://doi.org/10.1007/s00233-013-9560-1

[2] F. Aguiló - Gost, A. Miralles, M. Zaragozá, Using Double-Loop digraphs for solving Frobenius problems, *Electronic Notes in Discrete Math.*, **24** (2006), 17–24. https://doi.org/10.1016/j.endm.2006.06.004

[3] J. L. R. Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, Oxford, 2005. https://doi.org/10.1093/acprof:oso/9780198568209.001.0001

[4] V. Angjelkoska, *Additive semigroups of nonnegative integers. Embedding dimension and Frobenius number of numerical semigroups*, Doctoral thesis, Skopje, 2021.

[5] V. Angjelkoska, D. Dimovski, Additive semigroups of integers. Embedding dimension of numerical semigroups, *Contributions, Masa*, **41** (2020), 49–55.

[6] D. Beihoffer, J. Hendry, A. Nijenhuis, S. Wagon, Faster Algorithms for Frobenius numbers, *The Electronic Journal of Combinatorics*, **12** (2005). https://doi.org/10.37236/1924

[7] F. Curtis, On formulas for the Frobenius number of a numerical semigroup, *Mathematica Scandinavica*, **67** (1990), 190–192. https://doi.org/10.7146/math.scand.a-12330

[8] J. L. Davison, On the linear Diophantine problem of Frobenius, *Journal of Number Theory*, **48** (1994), 353–363. https://doi.org/10.1006/jnth.1994.1071

[9] Д. Димовски, Адитивни полугрупи на цели броеви, *МАНУ, Скопје, Прилози*, **IX, 2** (1977), 21–25.

[10] M. A. Fiol, J. L. A. Yebra, I. Alegre, M. Valero, Discrete Optimization Problem in Local Networks and Data alignment, *IEEE Transactions on Computers*, **36** (1987), 702–713. https://doi.org/10.1109/tc.1987.1676963

[11] H. Greenberg, Solution to a linear diophantine equation for nonnegative integers, *Journal of Algorithms*, **9** (1988), 343-353. https://doi.org/10.1016/0196-6774(88)90025-9

[12] S. M. Johnson, A linear Diophantine Problem, *Canadian Journal of Mathematics*, **12** (1960), 390–398. https://doi.org/10.4153/cjm-1960-033-6

[13] R.W. Owens, An Algorithm to Solve the Frobenius Problem, *Mathematics Magazine*, **76** (2003), 264–275. https://doi.org/10.2307/3219081

[14] J.C. Rosales, M.B. Branco, On the enumeration of the set of elementary numerical semigroups with fixed multiplicity, Frobenius number or genus, *Kragujevac Journal of Mathematics*, **46** (2020), 433–442.

[15] J. C. Rosales, P. A. García-Sánchez, *Numerical Semigroups*, Developments in mathematics, Springer, New York, 2009.

[16] J. C. Rosales, P. A. García-Sánchez, J. I. García-García, M. Branco, Numerical semigroups with maximal embedding dimension, *International journal of commutative rings*, **1** (2003).

[17] Y. Zhao, Constructing numerical semigroups of a given genus, *Semigroup Forum*, **80** (2010), 242–254. https://doi.org/10.1007/s00233-009-9190-9

Received: June 25, 2021; Published: July 15, 2021