Brauer Quotient of Interior Quandle $G$-Algebra

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Abstract

In this paper, we present the concept of Brauer morphism in the quandle $G$-algebra setting together with an illustration example of this construction. We study the relationship between the relative trace map and the Brauer morphism. We introduce the concept of the defect group of a primitive idempotent by using the relative trace map and the Brauer morphism. The object of an interior quandle $G$-algebra is studied, as well as, we define the homomorphism of interior quandle $G$-algebras.

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1 Introduction

Let $p$ be a prime number. Let $G$ be a finite group and $(k, O, K)$ be a $p$-modular system. This means that $O$ is a complete discrete valuation ring with residue field $K = O/J(O)$ of characteristic $p$ and field of fractions $k$ of characteristic 0, where $J(O)$ is the Jacobson radical of the ring $O$. In (1968) Green in his paper [3] initiated the notion of $G$-algebra as a convenient tool in the representation theory of finite group. The coefficients in Green work was from either $O$ or $K$. A special case of a $G$-algebra, which possesses some properties that a general $G$-algebra does not, is an interior $G$-algebra which is the heart of Puig’s work [5]. By the conjugation $G$-action, an interior $G$-algebra becomes a $G$-algebra. The concept of the Brauer homomorphism was introduced by Brauer for group
algebras only. Then Broué and Puig generalized such homomorphism for an arbitrary $G$-algebra [4]. We see that the work of Green, Puig and Broué was for an associative algebra.

The approach of the $G$-algebra motivates us to apply these tools on a non-associative algebraic system which is called Quandle system. In our paper [1], we created what we have called the Quandle $G$-algebra over the field $K$ and we denoted by $K[T]$, where $T$ is a finite quandle system and $G$ is a finite subgroup of the automorphism group of the quandle system $T$. Indeed, this type of $G$-algebras need not to be associative nor to be unitary. The theory is much similar to that for group algebra.

In the present paper, we interest to studying the ideal theory, Brauer quotient, Brauer morphism and the concept of the interior $G$-algebra in the quandle algebra environment.

This paper is organized as follows. In Section 1, we recall from [1] the structure of the quandle $G$-algebra over the field $K$. We also restate some related results. For any subgroup $H$ of $G$, we denote by $(K[T])^H$ the subalgebra consisting of the fixed points of the quandle $G$-algebra $K[T]$ under the restriction action of $H$. If $L \leq H$ then we write $(K[T])^H_L := Tr^H_L(K[T])^H$ for the image of the relative trace map from $L$ to $H$ on $K[T]$. The interesting thing is that this image is an ideal of $(K[T])^H$ (see Section 1). This permits us to construct quotient of quandle $G$-algebra $(K[T])^H / \sum_{L < H} (K[T])^H_L$ where $L$ runs over the set of all proper subgroups of $H$. This quotient is a Brauer quotient and is denoted by $K[T](H)$. In Section 2, we give a definition of the Brauer homomorphism (between quandle $G$-algebras) using the Brauer quotient. Then, we discuss the relation between Brauer morphism and the relative trace map. We also give a description of the basis elements of the Brauer quotient in the quandle $G$-algebra. We end this section with defining the concept of the defect group of a primitive idempotent in the quandle $G$-algebra. In the last section, we introduce the object of an interior quandle $G$-algebra. In order to define an interior quandle $G$-algebra, as the quandle $G$-algebra $K[T]$ without identity, we embed it in the unital ring $K^\circ[T] = K[T] \oplus K$ which we call it the extended quandle ring and it contains $K[T]$ as a subring. Actually, this ring is a $K$-algebra and has a quandle $G$-algebra structure. A maximal multiplicative subgroup of $K^\circ[T]$ has been defined, see [2], and is denoted by $U(K^\circ[T])$. We use it to define the structure map. We show that the quotient of the interior quandle $G$-algebra is an interior quandle $G$-algebra and the Brauer quotient is an interior $C_G(H)$-algebra where $H \leq G$. Finally, we define the homomorphism of interior quandle $G$-algebras and prove that this homomorphism induces a homomorphism between Brauer quotients of interior quandle $G$-algebras.
2 Quandle $G$-algebra over a field

**Definition 2.1.** Let $T$ be a finite set. One can consider a binary operation on $T$ such that the operation obeys the following conditions:

1. For all $t \in T$, $t \sqcup t = t$.
2. There exists a unique $t \in T$ such that $t = x \sqcap y$ for all $x, y \in T$.
3. For all $x, y, z \in T$, we have $(x \sqcap y) \sqcap z = (x \sqcap z) \sqcap (y \sqcap z)$.

The algebraic system $(T, \sqcup)$ is called Quandle system.

Now let $K$ be a field as in the introduction. Let $K[\![T]\!]$ be the set of all formal finite sums of linear combinations of elements from $T$ with coefficients from the field $K$. Then such set has the form:

$$K[\![T]\!] := \left\{ \sum_{i} \beta_i t_i : \beta_i \in K, t_i \in T \right\},$$

and it is a vector space over $K$. Define a multiplication in $K[\![T]\!]$ by

$$\left( \sum_{i} \alpha_i t_i \right) \left( \sum_{j} \beta_j s_j \right) = \sum_{i,j} (\alpha_i \beta_j)(t_i \sqcup s_j).$$

Then $K[\![T]\!]$ forms a ring. Moreover the ring $K[\![T]\!]$ is an algebra (not associative in general) over $K$.

Now we shall consider a finite subgroup $G$ of the automorphism group of the Quandle system $T$. This means that $G$ acts on the Quandle algebra $K[\![T]\!]$.

The algebra $K[\![T]\!]$ has the structure of a $G$-algebra over the field $K$ which is called Quandle $G$-algebra, see [1].

If $H$ is any subgroup of $G$, the set of all $H$-fixed points in $K[\![T]\!]$ is

$$(K[\![T]\!])^H = \left\{ \sum_i \alpha_i t_i \in K[\![T]\!] : \left( \sum_i \alpha_i t_i \right)^h = \sum_i \alpha_i t_i \text{ for all } h \in H \right\}.$$ 

Clearly, the set $(K[\![T]\!])^H$ is a subalgebra of $K[\![T]\!]$ and if $H \subseteq L$ are subgroups of $G$, then $(K[\![T]\!])^L \subseteq (K[\![T]\!])^H$.

We may define on $K[\![T]\!]$ the **relative trace map** as follows:

**Definition 2.2.** Consider a quandle $G$-algebra $K[\![T]\!]$ over $K$, where $G$ is any finite subgroup of $\text{Aut}(T)$. Let $L \subseteq H$ be subgroups of $G$. The **relative trace** is the map $Tr_L^H : (K[\![T]\!])^L \rightarrow (K[\![T]\!])^H$ given by

$$Tr_L^H \left( \sum_i \alpha_i t_i \right) = \sum_{v \in V} \left( \sum_{i} \alpha_i t_i \right)^v = \sum_{v \in V} \left( \sum_{i} \alpha_i t_i^v \right); \quad \forall \sum_i \alpha_i t_i \in (K[\![T]\!])^L,$$

where $V$ is a transversal for the right cosets $Lh$ in $H$ (i.e. $V$ picks out one element in each right coset of $L$ in $H$).
This relative trace map is a $K$-linear map and independent of the choice of $V$. The following lemmas exhibit some basic properties of this map.

**Lemma 2.3.** For a quandle $G$-algebra $K[T]$ over $K$, let $L \subseteq H$ be subgroups of $G$ and let $\sum_i \alpha_i t_i \in (K[T])^H$, $\sum_j \beta_j s_j \in (K[T])^L$. Then

(i) $\text{Tr}_L^H (\sum_i \alpha_i t_i) = [H : L] \sum_i \alpha_i t_i$, where $[H : L]$ is the index of $L$ in $H$.

(ii) $\text{Tr}_H^H = \text{Id}_{(K[T])^H}$.

(iii) For any element $g \in G$, we have $(K[T])^H g = (K[T])^{g^{-1}Hg}$.

(iv) For $g \in G$, we have $(\text{Tr}_L^H (\sum_j \beta_j s_j))^g = \text{Tr}_{g^{-1}Lg}^H (\sum_j \beta_j s_j)^g$.

**Lemma 2.4.** (The transitivity formula). For a quandle $G$-algebra $K[T]$ over $K$, let $D \subseteq L \subseteq H$ be subgroups of $G$. Then the following diagram commutes

$$
\begin{array}{ccc}
(K[T])^D & \xrightarrow{\text{Tr}_L^D} & (K[T])^L \\
\downarrow{\text{Tr}_L^D} & & \downarrow{\text{Tr}_L^H} \\
(K[T])^H & & \end{array}
$$

That is, $\text{Tr}_L^H \circ \text{Tr}_D^L = \text{Tr}_D^H$.

**Lemma 2.5.** (The Frobenius formula for a quandle algebra). For a quandle $G$-algebra $K[T]$ over $K$, let $L \subseteq H$ be subgroups of $G$. Then for any element $(\sum_i \alpha_i t_i) \in (K[T])^H$ and $\sum_j \beta_j s_j \in (K[T])^L$, we have

$$
\text{Tr}_L^H (\sum_i \alpha_i t_i \sum_j \beta_j s_j) = \sum_i \alpha_i t_i \text{Tr}_L^H (\sum_j \beta_j s_j)
$$

and

$$
\text{Tr}_L^H (\sum_j \beta_j s_j \sum_i \alpha_i t_i) = (\text{Tr}_L^H (\sum_j \beta_j s_j) \sum_i \alpha_i t_i).
$$

The Frobenius formula shows the following:

**Corollary 2.6.** For a quandle $G$-algebra $K[T]$ over $K$ and for any subgroup $L$ of $H$, the image of the trace map, $\text{Tr}_L^H ((K[T])^L)$, is an ideal of $(K[T])^H$.

**Lemma 2.7.** (Mackey formula for a quandle algebra). For a quandle $G$-algebra $K[T]$ over $K$, let $H$ be a subgroup of $G$ and let $L, D$ be subgroups of $H$. Then, for all $\sum_i \alpha_i t_i \in (K[T])^L$, we have

$$
\text{Tr}_L^H (\sum_i \alpha_i t_i) = \sum_{LhD \in L\backslash H/D} \text{Tr}_{D \cap h^{-1}Lh}^D (\sum_i \alpha_i t_i^h),
$$

where $L\backslash H/D$ is the set of double cosets.
By a permutation quandle $G$-algebra, we mean a quandle $G$-algebra over $K$ such that this algebra is a permutation $KG$-module. Observe that, $K[T]$ is a permutation quandle $G$-algebra over the field $K$ since $K[T]$ is a quandle $G$-algebra with the $K$-basis $T$ on which $G$ acts as a permutation group.

Let $T$ be a finite quandle, for any element $t \in T$ and any subgroup $H$ of $G$, where $G$ is a finite subgroup of $\text{Aut}(T)$. Recall that the $H$-orbit of $t$ and the stabilizer of $t$ in $H$, which are denoted by $t^H$ and $H_t$, respectively, are

$$t^H = \{ t^h : h \in H \} \quad \text{and} \quad H_t = \{ h \in H : t^h = t \},$$

with $|t^H| = [G : H]$ by the Orbit-Stablizer Theorem.

**Proposition 2.8.** Consider a permutation quandle $G$-algebra $K[T]$, where $T$ is a finite quandle $K$-basis of $K[T]$ that is permuted by the action of $G$. Let $L \subseteq H$ be subgroups of $G$. Then for any $t \in T$ we have

$$Tr^H_L \left( (\hat{t}L) \right) = [H_t : L_t](\hat{t}L)$$

where $(\hat{t}L) = \sum_{l \in L} t^l$ and $[H_t : L_t]$ is the index of $L_t$ in $H_t$.

The following theorems give a basis of the domain and the range of the relative trace map. The proofs can be seen in [1].

**Theorem 2.9.** Consider a permutation quandle $G$-algebra $K[T]$ where a quandle $T$ is its finite $K$-basis that is permuted by the action of $G$. Let $H$ be a subgroup of $G$. Then if $t_1, \cdots, t_n$ are all representatives for the $H$-orbits of $T$, then the set

$$\{ (\hat{t}^H_i) : 1 \leq i \leq n \}$$

is a $K$-basis of $(K[T])^H$, where $(\hat{t}^H_i) = \sum_{h \in H} t_i^h$. (In other words the subalgebra $(K[T])^H$ of $H$-fixed points has as a $K$-basis the set of $H$-orbits sums in $T$.)

**Theorem 2.10.** Consider a permutation quandle $G$-algebra $K[T]$, let $L \subseteq H$ be subgroups of $G$. If $t_1, \cdots, t_n$ are all representatives for the $H$-orbits of $T$ and $\lambda_i$ is the greatest common divisor of $[H_t : hLh^{-1} \cap H_t]$ with $h$ is running over $H$, then the nonzero elements of the set

$$\{ \lambda_i(\hat{t}^H_i) : 1 \leq i \leq n \}$$

form a $K$-basis of $Tr^H_L(K[T])^L$. 

3 Brauer Morphism in quandle setting

Let $K$ be a field as in the introduction; that is of prime characteristic $p$, let $T$ be a finite quandle and $K[T]$ denote the quandle algebra over $K$. Let $G$ be any finite group of $Aut(K[T])$. So, as we have seen in Section 1, $K[T]$ is a quandle $G$-algebra over $K$.

For any subgroups $L$, $H$ of $G$, if $L$ is a proper subgroup of $H$ then the Frobenius formula shows that $(K[T])_L^H := Tr^H_L(K[T])^L$ is an ideal of $(K[T])^H$, where $Tr^H_L : (K[T])^L \to (K[T])^H$ is the trace map from $L$ to $H$ on $K[T]$. Thus $(K[T])_L^H := \sum_{L<H}(K[T])_L^H$ is an ideal of $(K[T])^H$ where $L$ ranges over the set of all proper subgroups of $H$. So, we can define the quotient algebra

$$K[T](H) := (K[T])^H/(K[T])_L^H$$

which we call it the **Brauer quotient** of the quandle $G$-algebra $K[T]$.

**Definition 3.1.** Let $H$ be a $p$-subgroup of $G$ and $K[T](H)$ be the Brauer quotient. The natural surjective map $Br_H : (K[T])^H \to K[T](H)$ giving by $Br_H(\sum_t \alpha_t i_t) = \sum_t \alpha_t i_t + (K[T])_L^H$ is called the **Brauer homomorphism** on $K[T]$ with respect to $H$.

**Lemma 3.2.** Let $H$ be a $p$-subgroup of $G$, then:

(i) $(K[T])^H$ is an $N_G(H)/H$-algebra and $(K[T])_L^H$ is an $N_G(H)/H$-ideal of $(K[T])^H$.

(ii) The Brauer homomorphism is a homomorphism of $N_G(H)/H$-algebra.

Where $N_G(H) = \{ g \in G : g^{-1}Hg = H \}$ is the normalizer of $H$ in $G$.

**Proof.** (i) From Lemma 2.3 we have $((K[T])^H)^g = (K[T])^{g^{-1}Hg}$ for any element $g$ in $G$. Let $g \in N_G(H)$ so $g^{-1}Hg = H$ and then $(K[T])^H = (K[T])^H$ proving that $(K[T])^H$ is an $N_G(H)$-algebra over $K$. Since $H$ acts trivially on $(K[T])^H$, i.e. $a^h = a$ for all $h \in H$ and $a \in (K[T])^H$, this implies that $a^{-1}Hg = a^{-1}Hg$ for $g \in N_G(H)$ and $a \in (K[T])^H$. Thus, we can consider $(K[T])^H$ as an $N_G(H)/H$-algebra.

Now let $g$ be any element of $G$, then

$$((K[T])^H)_L^g = (\sum_{L<}\sum_{L<L} Tr^L_H(K[T])^L)^g$$

$$= \sum_{L<H}(Tr^L_H(K[T])^L)^g$$

$$= \sum_{L<H}Tr^g_{g^{-1}Lg}(K[T])^L$$

$$= \sum_{L<H}Tr^g_{g^{-1}Lg}(K[T])^{g^{-1}Lg}$$

$$= (K[T])^{g^{-1}Hg}_{<g^{-1}Hg}.$$
If \( g \in N_G(H) \) then \((K[T])^H_{<H}\)^g = \((K[T])^H_{<H}\). Therefore, \((K[T])^H_{<H}\) is an \(N_G(H)/H\)-ideal of \((K[T])^H_{<H}\).

(ii) This is a direct consequence of (i).

\[\text{Note that, if } L \text{ and } H \text{ are subgroups of } G \text{ and } L \text{ is a } p\text{-subgroup of } H, \text{ then the above lemma asserts that } (K[T])^L, (K[T])^L_{<L} \text{ are fixed under } N_H(L). \text{ So we can write } (K[T])^L = (K[T])^L_{<L} = (K[T])^L_{N_H(L)}. \text{ Moreover, since } L \text{ acts trivially on } K[T], \text{ we can write } K[T] = (K[T])^L_{N_H(L)}. \]

\text{Lemma 3.3. For a quandle } G\text{-algebra } K[T], \text{ let } H \text{ and } L \text{ be subgroups of } G \text{ and } L \text{ be a } p\text{-subgroup of } H. \text{ Then}

\[Br_L \circ Tr^H_L = Tr^{N_H(L)}_1 \circ Br_L\]

where \(N_H(L) = N_H(L)/L\) and \(Tr^{N_H(L)}_1 : K[T](L) \to (K[T](L))^{N_H(L)}\).

In other words, the following diagram commutes,

\[\begin{array}{ccc}
(K[T])^L & \xrightarrow{Tr^H_L} & (K[T])^H \\
\downarrow{Br_L} & & \downarrow{Br_L} \\
K[T](L) & \xrightarrow{Tr^{N_H(L)}_1} & K[T](L) = (K[T](L))^{N_H(L)}
\end{array}\]

\text{Proof. Let } \sum_i \alpha_i t_i \in (K[T])^L, \text{ then by Mackey formula, we have}

\[Tr^L_L(\sum_i \alpha_i t_i) = \sum_{LhL \subseteq L} Tr^L_{L \cap h^{-1} Lh}(\sum_i \alpha_i t_i)^h).\]

If \( h \notin N_H(L), \text{ that is, if } L \cap h^{-1} Lh \text{ is a proper subgroup of } L, \text{ then}

\[Tr^L_{L \cap h^{-1} Lh}(\sum_i \alpha_i t_i)^h) \in (K[T])^L_{L \cap h^{-1} Lh} \subseteq (K[T])^L_{<L} = Ker(Br_L).
\]

Therefore, in this case, we have

\[Br_L(Tr^L_{L \cap h^{-1} Lh}(\sum_i \alpha_i t_i)^h) = 0.\]
Otherwise, i.e. if \( h \in N_H(L) \), that is, if \( L \cap h^{-1}Lh = L \) then \( L = h^{-1}Lh \). Therefore, \( hL = Lh \), which implies that \( LhL = Lh \). Thus,

\[
Br_L(Tr_L^H(\sum_i \alpha_i t_i)) = Br_L(\sum_{Lh \in N_H(L)} (Tr_{L\cap h^{-1}Lh}^L(\sum_i \alpha_i t_i)^h))
\]

\[
= \sum_{Lh \in N_H(L)} Br_L(Tr_L^L((\sum_i \alpha_i t_i)^L)^h) (as \sum_i \alpha_i t_i \in (K[T]^L))
\]

\[
= \sum_{Lh \in N_H(L)} Br_L((\sum_i \alpha_i t_i)^{Lh}) (as Tr_L^L = Id_{(K[T]^L)}
\]

\[
= \sum_{Lh \in N_H(L)} (\sum_i \alpha_i t_i)^{Lh} (K[T])^{Lh}) (as \in N_H(L))
\]

\[
= \sum_{Lh \in N_H(L)} (\sum_i \alpha_i t_i + (K[T])^{Lh}\Lh (as \in Aut(K[T]))
\]

\[
= \sum_{Lh \in N_H(L)} (Br_L(\sum_i \alpha_i t_i))^{Lh} (by \ definition \ of \ Brauer \ map)
\]

\[
= Tr_L^{N_H(L)}(Br_L(\sum_i \alpha_i t_i)) (by \ definition \ of \ trace \ map)
\]

\[
= Tr_L^{N_H(L)}(Br_L(\sum_i \alpha_i t_i)).
\]

\[\square\]

**Corollary 3.4.** For a quandle \( G \)-algebra \( K[T] \) over the field \( K \). Let \( P \subseteq L \subseteq H \) be subgroups of \( G \) and let \( P \) be a \( p \)-subgroup of \( L \). Then

\[
Br_P \circ Tr_L^H \circ Tr_P^L = Tr_L^{N_H(P)} \circ Br_P \circ Res_P^L \circ Tr_P^L,
\]

where \( Res_P^L : (K[T])^P \rightarrow (K[T])^L \) is the restriction map from \( L \) to \( P \) on \( K[T] \).

**Proof.**

\[
Br_P \circ Tr_L^H \circ Tr_P^L = Br_P \circ Tr_P^H
\]

(By the trasivity formula)

\[
= Tr_L^{N_H(P)} \circ Br_P
\]

(By Lemma 3.3)

\[
= Tr_L^{N_H(P)} \circ Tr_L^{N_L(P)} \circ Br_P
\]

(by the transitivty formula)

\[
= Tr_L^{N_H(P)} \circ Tr_L^{N_L(P)} \circ Br_P \circ Res_P^L \circ Tr_P^L.
\]

\[\square\]
Note that, from Section 1, we see that a quandle algebra $K[T]$ is a permutation quandle $G$-algebra over $K$. Now let $H$ be any subgroup of $G$, then a quandle $G$-algebra $K[T]$ can be viewed as a quandle, and a permutation quandle, $H$-algebra by restriction.

The following result describes a $K$-basis for the Brauer quotient $K[T](H)$.

**Theorem 3.5.** Consider a permutation quandle $P$-algebra $K[T]$, where $P$ is a $p$-subgroup of $G$ and $T$ is a $K$-basis of $K[T]$ that is permuted by $P$. Let $T^P$ be the set of $P$-fixed elements in $T$. Then the set $\{Br_P(t) : t \in T^P\}$ forms a $K$-basis of $K[T](P)$.

**Proof.** Assume that $t_1, \ldots, t_n, t_{n+1}, \ldots, t_m$ are all representatives for the $P$-orbits of $T$ and $t_1, \ldots, t_n$ are all distinct elements in $T^P$. Let $Q_i$ be the stabilizer of $t_i$ in $P$, $1 \leq i \leq m$. Since $t_i \in T^P$, $1 \leq i \leq n$ this implies that $Q_i = P$ for all $1 \leq i \leq n$. The remaining $Q_i$, $n + 1 \leq i \leq m$ are all proper subgroups of $P$. By Theorem 2.9, the set $\{(t_i^P) : 1 \leq i \leq m\}$ is a $K$-basis for $(K[T])^P$. Therefore, the nonzero elements of the set $\{Br_P(t_i^P) : 1 \leq i \leq m\}$ form a $K$-basis for $K[T](P)$. Since, each $\hat{t_i^P} = Tr_{Q_i}(t_i)$ where $Q_i$ is the stabilizer of $t_i$ in $P$, it follows that, for $1 \leq i \leq n$,

$$Tr_{Q_i}(t_i) = Tr_P(t_i) = Id(t_i) = t_i. $$

Now for all $n + 1 \leq i \leq m$, where $Q_i$ is a proper subgroup of $P$, we have

$$Tr_{Q_i}(t_i) \in (K[T])^P_{Q_i} \subseteq (K[T])^P_{Q_i}.$$ 

So, we have $Br_P(\hat{t_i^P}) = Br_P(t_i)$ for $t_i \in T^P$, and $Br_P(\hat{t_i^P})$ is zero in $K[T](P)$ for $t_i \notin T^P$. Hence the set $\{Br_P(t) : t \in T^P\}$ is a $K$-basis of $K[T](P)$. 

The following is an illustrating example of Brauer morphism in quandle $G$-algebra.

**Example 3.6.** Let $G = S_4$, $T = \text{Conj}(G)$ be the conjugation quandle with the operation defined by $x \sqcup y = x^{-1}yx$ for $x, y \in T$. Let $H = D_4$ which is a dihedral Sylow 2-subgroup of $G$ of order 8 and $H$ acts on $T$ by conjugation. That is $t^h = h^{-1}th$ for $t \in T$ and $h \in H$.

**The $D_4$-orbits of the elements of $T$ are:**

- $(1)^{D_4} = \{(1)\}$ so, $\hat{(1)}^{D_4} = (1)$.
- $(12)^{D_4} = \{(12), (23), (14), (34)\}$ so, $\hat{(12)}^{D_4} = (12) + (23) + (14) + (34)$.
- $(13)^{D_4} = \{(13), (24)\}$ so, $\hat{(13)}^{D_4} = (13) + (24)$. 

• \(( (12)(34) )^{D_4} = \{ (12)(34), (14)(23) \} \), so, \(( (12)(34) )^{D_4} = (12)(34) + (14)(23) \).

• \(( (13)(24) )^{D_4} = \{ (13)(24) \} \), so, \(( (13)(24) )^{D_4} = (13)(24) \).

• \((123)^{D_4} = \{ (123), (132), (124), (142), (134), (143), (234), (243) \} \), so, \((123)^{D_4} = (123) + (132) + (124) + (142) + (134) + (143) + (234) + (243) \).

• \((1234)^{D_4} = \{ (1234), (1432) \} \), so, \((1234)^{D_4} = (1234) + (1432) \).

• \((1243)^{D_4} = \{ (1243), (1324), (1423), (1342) \} \), so, \((1243)^{D_4} = (1243) + (1324) + (1423) + (1342) \).

Therefore, the set
\[
\{ (1)^{D_4}, (12)^{D_4}, (13)^{D_4}, (12)(34)^{D_4}, (13)(24)^{D_4}, (123)^{D_4}, (1234)^{D_4} \}
\]

is a basis of \( (K[T])^{D_4} \).

The Stabilizer of the representatives for the \(D_4\)-orbits of \(T\) are:

• \((D_4)(1) = D_4\).

• \((D_4)(12) = \{ (1), (12)(34) \} \).

• \((D_4)(13) = \{ (1), (13), (24), (13)(24) \} \).

• \((D_4)(12)(34) = \{ (1), (12)(34), (13)(24), (14)(23) \} \).

• \((D_4)(13)(24) = D_4\).

• \((D_4)(123) = \{ (1) \} \).

• \((D_4)(1234) = \{ (1), (13)(24), (1234), (1423) \} \).

• \((D_4)(1243) = \{ (1), (14)(23) \} \).

Now we will find \( Br_{D_4}((K[T])^{D_4}) \).

• \( Br_{D_4}( (1)^{D_4} ) = Br_{D_4}( (1) ) = (1) + (K[T])^{D_4}_{<D_4} \).

Note that \((1)\) is fixed by \(D_4\) and \((1)^{D_4} = (1) \not\in (K[T])^{D_4}_{<D_4} \).

• \( Br_{D_4}( (12)^{D_4} ) = (12)^{D_4} + (K[T])^{D_4}_{<D_4} \).

Note that, \((12)^{D_4} = Tr_{(D_4)(12)}^{D_4} ( (12) ) = Tr_{(D_4)(12)}^{D_4} ( (12)^{(D_4)(12)} ) \in (K[T])^{D_4}_{<D_4} \).

So, \( Br_{D_4}( (12)^{D_4} ) = 0_{K[T](D_4)} \).
\begin{itemize}
  \item \( Br_{D_4}((13)^{D_4}) = (13)^{D_4} + (K[T])^{D_4}_{D_4} \).
  
  But \((13)^{D_4} = Tr_{D_4}^{D_4}_{D_4}((13)) \in (K[T])^{D_4}_{D_4} \).
  
  Therefore, \( Br_{D_4}((13)^{D_4}) = 0_{K[T]} \).

  \item \( Br_{D_4}((12)(34)^{D_4}) = ((12)(34)^{D_4} + (K[T])^{D_4}_{D_4} \).
  
  Since \((12)(34)^{D_4} = Tr_{D_4}^{D_4}_{D_4}((12)(34)) \in (K[T])^{D_4}_{D_4} \),
  
  this implies that, \( Br_{D_4}((12)(34)^{D_4}) = 0_{K[T]} \).

  \item \( Br_{D_4}((13)(24)^{D_4}) = ((13)(24)^{D_4} + (K[T])^{D_4}_{D_4} = (13)(24) + (K[T])^{D_4}_{D_4} \).
  
  Note that \((13)(24)\) is fixed by \( D_4 \) and \((13)(24)^{D_4} = (13)(24) \notin (K[T])^{D_4}_{D_4} \).

  \item \( Br_{D_4}((123)^{D_4}) = (123)^{D_4} + (K[T])^{D_4}_{D_4} \).
  
  But \((123)^{D_4} = Tr_{D_4}^{D_4}_{D_4}(123) \in (K[T])^{D_4}_{D_4} \). Therefore, \( Br_{D_4}((123)^{D_4}) = 0_{K[T]} \).

  \item \( Br_{D_4}((1234)^{D_4}) = (1234)^{D_4} + (K[T])^{D_4}_{D_4} \).
  
  But \((1234)^{D_4} = Tr_{D_4}^{D_4}_{D_4}(1234) \in (K[T])^{D_4}_{D_4} \).
  
  So, \( Br_{D_4}((1234)^{D_4}) = 0_{K[T]} \).

  \item \( Br_{D_4}((1243)^{D_4}) = (1243)^{D_4} + (K[T])^{D_4}_{D_4} \).
  
  But \((1243)^{D_4} = Tr_{D_4}^{D_4}_{D_4}(1243) \in (K[T])^{D_4}_{D_4} \).
  
  So, \( Br_{D_4}((1243)^{D_4}) = 0_{K[T]} \).
\end{itemize}

Therefore, the set \( \{ Br_{D_4}((1) ), Br_{D_4}((13)(24)) \} \) is a basis for \( K[T](D_4) \).

We have introduced the relative trace map and Brauer morphism. We can now introduce and study the defect group of a primitive idempotent in a quandle \( G \)-algebra.

**Definition 3.7.** Consider a quandle \( G \)-algebra \( K[T] \) over a field \( K \) of prime characteristic \( p \), and let \( e \in (K[T])^G \) be a primitive idempotent. A subgroup \( P \) of \( G \) is said to be a **defect subgroup** of \( e \) in a quandle \( G \)-algebra if \( P \) is a minimal subgroup of \( G \) with the property that \( e \in (K[T])^P \).

Note that the defect group always exists since
\[
e \in (K[T])^G = Id_{(K[T])^G}(((K[T])^G) = (K[T])^G.
\]

**Lemma 3.8.** Let \( P \) be a defect group of \( e \) in a quandle \( G \)-algebra \( K[T] \) over a field \( K \) of prime characteristic \( p \). Then:
(i) If $H$ is a subgroup of $G$ such that $e \in (K[T])^G_H$, then $P \subseteq g^{-1}Hg$ for $g \in G$.

(ii) $P$ is a $p$-subgroup of $G$.

**Proof.** (i) Let $H$ be a subgroup of $G$ such that $e \in (K[T])^G_H$. Since $e \in (K[T])^G_P$, we can write

$$e = Tr_H^G(a), \quad e = Tr_P^G(b) \quad \text{for} \quad a \in (K[T])^H, \quad b \in (K[T])^P.$$ 

Then,

$$e = e.e = (Tr_H^G(a))(Tr_P^G(b))$$

$$= Tr_P^G(Tr_H^G(a)b) \quad \text{(From Lemma 2.5)}$$

$$= Tr_P^G(\sum_{g \in H \cap G/P} Tr_{g^{-1}Hg \cap P}(a^g)b) \quad \text{(By Mackay formula)}$$

$$= \sum_{g} Tr_P^G(Tr_{g^{-1}Hg \cap P}(a^g)b) \quad \text{(From Lemma 2.5)}$$

$$= \sum_{g} Tr_{g^{-1}Hg \cap P}(a^g)b \quad \text{(by the transitivty formula).}$$

Therefore $e \in \sum_{g}(K[T])^{g^{-1}Hg \cap P}$ and, by Rosenberg’s lemma ([5], Proposition 4.9), we get $e \in (K[T])^{g^{-1}Hg \cap P}$. By minimality of $P$ we have $g^{-1}Hg \cap P = P$. Consequently $P \subseteq g^{-1}Hg$.

(ii) Let $S$ be a Sylow $p$-subgroup of $G$. Since $e \in (K[T])^G$, we have $Tr_S^G(e) = [G : S]e$, where $[G : S]$ is the index of $S$ in $G$. Since $p \nmid [G : S]$ this implies that $[G : S]$ is invertible in $K$. So, we have $Tr_S^G([G : S]^{-1}e) = e$ meaning that $e \in (K[T])^G_S$. Therefore $P \subseteq g^{-1}Sg$ for some $g \in G$. Thus $P$ is a $p$-subgroup of $G$.

The following results characterize the concept of the defect group of a primitive idempotent in term of Brauer morphism.

**Proposition 3.9.** Consider a quanle $G$-algebra $K[T]$ over the field $K$ of prime characteristic $p$. Let $P$ be a defect group of a primitive idempotent $e \in (K[T])^G$ such that $Br_Q(e) \neq 0$ for any $p$-subgroup $Q$ of $G$ then $Q \subseteq g^{-1}Pg$ for some $g \in G$. (In other words, if $P$ is a defect group of a primitive idempotent $e \in (K[T])^G$ then any other $p$-subgroup $Q$ of $G$ with $Br_Q(e) \neq 0$ is a $G$-conjugate to a subgroup of $P$.)

**Proof.** Assume that $Q$ is a $p$-subgroup of $G$ such that $Br_Q(e) \neq 0$. Since $P$ is a defect group of $e$, $P$ is minimal with $e \in (K[T])^G_P$. Let $e = Tr_P^G(a)$ for some $a \in (K[T])^P$. From Mackey formula we have,

$$e = Tr_P^G(a) = \sum_{g \in Q \cap G/P} Tr_{g^{-1}Pg \cap Q}(a^g).$$
Since \( Br_Q(e) \neq 0 \), it follows that \( \sum_{g \in Q \setminus G/P} Tr^Q_{g^{-1}Pg \cap Q}(a^g) \notin (K[T])^Q \). Meaning that \( g^{-1}Pg \cap Q = Q \). Thus, \( Q \subseteq g^{-1}Pg \).

**Theorem 3.10.** Consider a quandle \( G \)-algebra \( K[T] \) over the field \( K \) of prime characteristic \( p \). Let \( P \) be a subgroup of \( G \) and let \( e \) be a primitive idempotent in \((K[T])^G\). The following are equivalent:

(i) The group \( P \) is a defect group of \( e \).

(ii) The group \( P \) is a \( p \)-subgroup of \( G \) such that \( e \in (K[T])^G_P \) and \( Br_P(e) \neq 0 \).

(iii) \( P \) is maximal \( p \)-subgroup of \( G \) with the property that \( Br_P(e) \neq 0 \).

**Proof.** (i) \( \Rightarrow \) (ii) : Let \( P \) be a defect group of \( e \). So by Lemma 3.8, \( P \) is a \( p \)-subgroup of \( G \) and by Definition 3.7 of the defect group, \( P \) is minimal subgroup of \( G \) such that \( e \in (K[T])^G_P \). So we can write \( e = Tr^G_P(a) \) for some \( a \in (K[T])^P \). Now suppose that \( Br_P(e) = 0 \) that means \( e \in (K[T])^P_L \). Put \( e = \sum_{L \subset P} Tr^P_L(b) \) for some \( b \in (K[T])^L \). Therefore we have,

\[
e = e.e = (Tr^G_P(a)) (\sum_{L \subset P} Tr^P_L(b)) \quad (e \text{ is an idempotent})
\]

\[
= Tr^G_P(a \sum_{L \subset P} Tr^P_L(b)) \quad (\text{as } \sum_{L \subset P} Tr^P_L(b) = e \in (K[T])^G_P)
\]

\[
= Tr^G_P(\sum_{L \subset P} Tr^P_L(ab)) \quad \text{(From Lemma 2.5 since } a \in (K[T])^P)\)
\]

\[
= \sum_{L \subset P} Tr^G_P(ab) \quad \text{(by the transitivity formula)}.
\]

So, \( e \in \sum_{L \subset P}(K[T])^G_L \) and then, by Rosenberg’s Lemma ([5], Proposition 4.9), \( e \in (K[T])^G_P \) for a proper subgroup \( L \) of \( P \). But this contradicts the minimality of \( P \) such that \( e \in (K[T])^G_P \). Thus, \( Br_P(e) \neq 0 \).

(ii) \( \Rightarrow \) (iii) : Assume \( Br_P(e) \neq 0 \), and let \( Q \) be a \( p \)-subgroup of \( G \) such that \( Br_Q(e) \neq 0 \). Then By Proposition 3.9, since \( e \in (K[T])^Q_P \), we have \( Q \) is \( G \)-conjugate to a subgroup of \( P \) (in other words, \( P \) contains a \( G \)-conjugate of \( Q \)). Thus \( P \) is maximal \( p \)-subgroup of \( G \) with the property \( Br_P(e) \neq 0 \).

(iii) \( \Rightarrow \) (i) Assume that (iii) holds and suppose that \( L \) is a defect group of \( e \). Since \( P \) is a \( p \)-subgroup of \( G \) such that \( Br_P(e) \neq 0 \), Proposition 3.9 yields that \( P \subseteq g^{-1}Lg \) (which means, \( gPg^{-1} \subseteq L \)). So, we may assume that \( P \subseteq L \). Since \( L \) is a defect group of \( e \), it follows that from the implication (i) \( \Rightarrow \) (ii) \( Br_L(e) \neq 0 \). But \( P \) is maximal \( p \)-subgroup with the property \( Br_P(e) \neq e \). So \( L \subseteq P \) and then \( P = L \). Hence \( P \) is a defect group of \( e \). □
4 Interior quandle $G$-algebra

Throughout this section, we assume that $K$ is the field of prime characteristic $p$. Let $T$ be a finite quandle set and $K[T]$ denote the quandle $G$-algebra over $K$ where $G$ is a finite subgroup of $\text{Aut}(K[T])$.

As $K[T]$ is a ring without identity, it is convenient to embed it into a ring with identity. The ring $K^\circ[T] = K[T] \oplus K = \{(\sum \alpha_i t_i, k) : \sum \alpha_i t_i \in K[T], k \in K\}$ with the multiplication defined by

$$(\sum \alpha_i t_i, k_1) \square (\sum \beta_j s_j, k_2) = (\sum \alpha_i \beta_j t_i \square s_j + k_1 \sum \beta_j s_j + k_2 \sum \alpha_i t_i, k_1 k_2)$$

is a ring with identity $(0, 1)$ and we shall call it the extended quandle ring of $T$.

Note that $K^\circ[T]$ is a quandle $G$-algebra where the action of $G$ on $K^\circ[T]$ is given by

$$(\sum \alpha_i t_i, k) g = (\sum \alpha_i t_i g, k g) = (\sum \alpha_i t_i, k)$$

for $g \in G$, $\sum \alpha_i t_i \in K[T]$ and $k \in K$, where we regard $K$ as a trivial $G$-algebra over itself.

Now we will define the concept of interior quandle $G$-algebra over $K$.

**Definition 4.1.** A quandle algebra $K^\circ[T]$ over $K$ is said to be an interior quandle $G$-algebra if there exists a group homomorphism $\phi : G \to \mathcal{U}(K^\circ[T])$, where $\mathcal{U}(K^\circ[T])$ denotes a maximal multiplicative subgroup of the ring $K^\circ[T]$.

The following is an example of an interior quandle $G$-algebra.

**Example 4.2.** Let $G = \{1, g, g^2\}$ be a group of order 3, $T = \text{Core}(G)$ be the core quandle with the operation defined by $x \square y = yx^{-1}y$, [2, Section 2] and let $K = \mathbb{Z}_3 = \{-1, 0, 1\}$, so $\mathbb{Z}_3^\circ[T] = \mathbb{Z}_3[T] \oplus \mathbb{Z}_3$ is a quandle algebra with identity $(0, 1)$.

Note that, for $g \in G$, we have $(g - 1, 1) \in \mathcal{U}(\mathbb{Z}_3^\circ[T])$ where $(g - 1, 1)^{-1} = (g^2 - 1, 1)$, because

$$(g - 1, 1) \square (g^2 - 1, 1) = (1 - g^2 - g + 1 + g - 1 + g^2 - 1, 1) = (0, 1).$$

We can define a map, $f : G \to \mathcal{U}(\mathbb{Z}_3^\circ[T])$ by $f(g) = (g - 1, 1)$ The map $f$ is a group homomorphism, because

- $f(1) = (1 - 1, 1) = (0, 1)$, so $f(\text{Id}_G) = \text{Id}_{\mathbb{Z}_3^\circ[T]}$. 

• \( f(g^{-1}) = (g^{-1} - 1, 1) = (g^2 - 1, 1) \) and,
• \( (f(g))^{-1} = (g - 1, 1)^{-1} = (g^2 - 1, 1) \). So, \( f(g^{-1}) = (f(g))^{-1} \).

• For \( g^n, g^m \in G, 1 \leq n, m \leq 2 \), we have
  
  \[
  f(g^n \cdot g^m) = f(g^{n+m}) = \begin{cases} 
  (g^{2n} - 1, 1), & \text{if } n = m \\
  (0, 1), & \text{if } n \neq m.
  \end{cases}
  \]

  and,

  \[
  f(g^n) \square f(g^m) = (g^n - 1, 1) \square (g^m - 1, 1) \\
  = (g^m g^{2n} g^m - g^{2n} - g^{2m} + 1 + g^n + g^m - 1, 1) \\
  = (g^{2(n+m)} - g^{2n} - g^{2m} + g^n + g^m - 1, 1)
  \]

  So,

  \[
  f(g^n) \square f(g^m) = \begin{cases} 
  (g^n - 2g^{2n} + 2g^n - 1, 1) = (g^{2n} - 1, 1), & \text{if } n = m \\
  (1 - g^{2n} - g^{4n} + g^n + g^{2n} - 1, 1) = (0, 1), & \text{if } n \neq m.
  \end{cases}
  \]

  where \( g^m = (g^n)^{-1} = g^{2n} \) when \( n \neq m \). Therefore,

  \[
  f(g^n \cdot g^m) = f(g^n) \square f(g^m).
  \]

Thus, the quandle algebra \( \mathbb{Z}_3[T] \) is an interior quandle \( G \)-algebra.

**Lemma 4.3.** If \( K^o[T] \) is an interior quandle \( G \)-algebra over \( K \) then it is a quandle \( G \)-algebra over \( K \).

**Proof.** Let \( K^o[T] \) be an interior quandle \( G \)-algebra over \( K \), then there is a group homomorphism \( \phi : G \to \mathcal{U}(K^o[T]) \).

For each \( u \in \mathcal{U}(K^o[T]) \) we can define a group homomorphism \( f_u : K^o[T] \to K^o[T] \) by \( f_u(x) = \text{Inn}(x) = uxu^{-1} \). The group generated by \( f_u \) is called an inner automorphism group of \( K^o[T] \) and it is a subgroup of \( \text{Aut}(K^o[T]) \). Therefore, we have a group homomorphism \( \psi : \mathcal{U}(K^o[T]) \to \text{Aut}(K^o[T]) \); \( u \mapsto f_u = \text{inn}(x) \), for \( x \in K^o[T] \). Consequently, there is a group homomorphism \( \psi \circ \phi : G \to \text{Aut}(K^o[T]) \). In other words, if \( K^o[T] \) is an interior quandle \( G \)-algebra then \( G \) acts on \( K^o[T] \) via an inner automorphism of \( K^o[T] \). \[ \square \]

**Lemma 4.4.** Let \( K^o[T] \) be an interior quandle \( G \)-algebra over a field \( K \) with group homomorphism \( \phi : G \to \mathcal{U}(K^o[T]) \). Then, for any subgroup \( H \) of \( G \), we have

\[
(K^o[T])^H = C_{K^o[T]}(\phi(H))
\]

If, in particular, \( \phi \) is surjective, then \( (K^o[T])^G = C_{K^o[T]}(\mathcal{U}(K^o[T])) \).
Proof. Let \( a \in (K^\circ[T])^H \), so \( a^h = a \) for all \( h \in H \). Since \( K^\circ[T] \) is an interior quandle \( G \)-algebra, from Lemma 4.3, \( G \) acts on \( K^\circ[T] \) via inner automorphism, that is, \( x^g = (\phi(g))^{-1}x\phi(g) \) for all \( g \in G \) and \( x \in K^\circ[T] \). Therefore, for all \( h \in H \leq G \) and \( a \in (K^\circ[T])^H \subseteq K^\circ[T] \), \( a^h = a^b = (\phi(h))^{-1}a\phi(h) \), which implies that \( \phi(h)a = a\phi(h) \). So, \( a \in C_{(K^\circ[T])^H}(\phi(H)) \) and then \( a \in C_{K^\circ[T]}(\phi(H)) \).

Thus, \((K^\circ[T])^H \subseteq C_{K^\circ[T]}(\phi(H)) \).

Now, assume that \( b \in C_{K^\circ[T]}(\phi(H)) \), so \( b \in K^\circ[T] \) and \( \phi(h)b = b\phi(h) \) for all \( h \in H \). Then \( b = (\phi(h))^{-1}b\phi(h) \). Since \( K^\circ[T] \) is interior quandle \( G \)-algebras, we have \( b^h = (\phi(h))^{-1}b\phi(h) \) for all \( h \in H \leq G \) and \( b \in K^\circ[T] \). Then \( b = b^h \) which means \( b \in (K^\circ[T])^H \). Hence, \((K^\circ[T])^H \subseteq C_{K^\circ[T]}(\phi(H)) \).

If, in particular, \( \phi \) is surjective so \( \phi(G) = \mathcal{U}(K^\circ[T]) \), then \((K^\circ[T])^G = C_{K^\circ[T]}(\mathcal{U}(K^\circ[T])) \).

\( \square \)

**Remark 4.5.** Note that an interior quandle \( G \)-algebra \( K^\circ[T] \) can be regarded as a left \( KG \)-module where \( g^a = \phi(g)a \) and as a right \( KG \)-module as well where \( a^g = a\phi(g) \) for \( g \in G \) and \( a \in A \). Also, for \( g \in G \) we have

\[ g(0,1) = \phi(g)(0,1) = \phi(g), \quad \text{and} \quad (0,1)^g = (0,1)\phi(g) = \phi(g) \]

So, \( \phi(g) = g(0,1) = (0,1)^g \) where \((0,1)\) is the identity element of the interior quandle \( G \)-algebra \( K^\circ[T] \).

In the following we will define a homomorphism between interior quandle \( G \)-algebra.

Let \( T_1 \) and \( T_2 \) be finite quandle sets and let \( G_1 \) and \( G_2 \) be finite subgroups of \( \text{Aut}(K^\circ[T_1]) \) and \( \text{Aut}(K^\circ[T_2]) \), respectively. Let \( K^\circ[T_1] \) be the interior quandle \( G_1 \)-algebra and \( K^\circ[T_2] \) be the interior quandle \( G_2 \)-algebra. Assume that \( G_1 \) is isomorphic to \( G_2 \). Set \( G = G_1 \simeq G_2 \), so, \( K^\circ[T_1] \) and \( K^\circ[T_2] \) become interior quandle \( G \)-algebras.

Consider \( G \) as defined above, we introduce the following definition.

**Definition 4.6.** Let \( K^\circ[T_1] \) and \( K^\circ[T_2] \) be quandle \( G \)-algebras over a field \( K \). A quandle \( K \)-algebra homomorphism \( f : K^\circ[T_1] \rightarrow K^\circ[T_2] \) is a \( K \)-algebra homomorphism from \( K^\circ[T_1] \) to \( K^\circ[T_2] \) that preserves the algebraic structures of quandle algebra.

**Definition 4.7.** Let \( K^\circ[T_1] \) and \( K^\circ[T_2] \) be quandle \( G \)-algebras over a field \( K \). A homomorphism of quandle \( G \)-algebras is a quandle \( K \)-algebra homomorphism \( f : K^\circ[T_1] \rightarrow K^\circ[T_2] \) satisfying \( f(a^g) = (f(a))^g \) for all \( g \in G \) and \( a \in K^\circ[T_1] \). If, moreover, \( K^\circ[T_1] \) and \( K^\circ[T_2] \) are interior quandle \( G \)-algebras with group homomorphisms \( \phi_1 : G \rightarrow \mathcal{U}(K^\circ[T_1]) \) and \( \phi_2 : G \rightarrow \mathcal{U}(K^\circ[T_2]) \), then
a homomorphism of interior quandle $G$-algebras is a quandle $K$-algebra homomorphism $f : K^\circ[T_1] \to K^\circ[T_2]$ such that $f \circ \phi_1 = \phi_2$, i.e. $f(\phi_1(g)) = \phi_2(g)$ for all $g \in G$.

**Proposition 4.8.** Let $K^\circ[T_1]$ and $K^\circ[T_2]$ be interior quandle $G$-algebras, and let $f : K^\circ[T_1] \to K^\circ[T_2]$ be a quandle $K$-algebra homomorphism. Then the following are equivalent.

(i) $f$ is a homomorphism of interior quandle $G$-algebras.

(ii) $f(ga) = g(f(a))$ and $f(a^g) = (f(a))^g$ for all $g \in G$ and $a \in K^\circ[T_1]$.

(iii) $f(g(0,1)) = g(f(0,1))$ for all $g \in G$ and $a \in K^\circ[T_1]$.

**Proof.** (i) $\Rightarrow$ (ii) : Since $K^\circ[T_1]$ and $K^\circ[T_2]$ are interior quandle $G$-algebras, there are group homomorphisms

$$\phi_1 : G \to U(K^\circ[T_1]) \quad \text{and} \quad \phi_2 : G \to U(K^\circ[T_2]).$$

Let $f : K^\circ[T_1] \to K^\circ[T_2]$ be a homomorphism of interior quandle $G$-algebras, so by Definition 4.7 $f(\phi_1(g)) = \phi_2(g)$ for all $g \in G$. Therefore, for all $g \in G$ and $a \in K^\circ[T_1]$ we have,

$$f(ga) = f(\phi_1(g)a) \quad \text{(from Remark 4.5 since $a \in K^\circ[T_1]$)}$$

$$= f(\phi_1(g))f(a) \quad \text{(since $f$ is a quandle K-homo.)}$$

$$= \phi_2(g)f(a)$$

$$= g(f(a)) \quad \text{(from Remark 4.5 since $f(a) \in K^\circ[T_2]$).}$$

Also,

$$f(a^g) = f(a\phi_1(g)) \quad \text{(from Remark 4.5 since $a \in K^\circ[T_1]$)}$$

$$= f(a)f(\phi_1(g)) \quad \text{(since $f$ is a quandle K-homo.)}$$

$$= f(a)\phi_2(g)$$

$$= (f(a))^g \quad \text{(from Remark 4.5 since $f(a) \in K^\circ[T_2]$).}$$

(ii) $\Rightarrow$ (iii) : Since $(0,1) \in K^\circ[T_1]$, which is the identity element, by using (ii), we have

$$f(g(0,1)) = g(f(0,1)) \quad \text{and} \quad f((0,1)^g) = (f(0,1))^g.$$
So, $f \circ \phi_1 = \phi_2$ and then $f$ is a homomorphism of interior quandle $G$-algebra.

We can consider ideal theory in an interior quandle $G$-algebra.

**Lemma 4.9.** Let $K^\circ[T]$ be an interior quandle $G$-algebra over $K$ and let $I$ be an ideal of $K^\circ[T]$, then the quotient $K^\circ[T]/I$ is an interior quandle $G$-algebra over $K$.

**Proof.** Since $K^\circ[T]$ is an interior quandle $G$-algebra, there is a group homomorphism $\phi : G \to U(K^\circ[T])$. Since $I$ is a proper ideal of $K^\circ[T]$, it follows that for all $u \in U(K^\circ[T])$, $u \in U(K^\circ[T]) - I$ meaning that if $u \in U(K^\circ[T])$ then $u + I \in U(K^\circ[T]/I)$. Therefore we can define a group homomorphism $f : U(K^\circ[T]) \to U(K^\circ[T]/I)$ via $u \mapsto u + I$ and then the map $f \circ \phi : G \to U(K^\circ[T]/I)$ is a group homomorphism. Thus $K^\circ[T]/I$ is an interior quandle $G$-algebra over $K$.

**Remark 4.10.** Let $K^\circ[T]$ be an interior quandle $G$-algebra, $H$ be a subgroup of $G$ and let $\rho : H \to U(K^\circ[T])$ be the restriction of $\phi : G \to U(K^\circ[T])$. So $K^\circ[T]$ is an interior quandle $H$-algebra over $K$.

Note that $\phi(C_G(H)) \subseteq U((K^\circ[T])^H)$, where $U((K^\circ[T])^H) = U(K^\circ[T]) \cap (K^\circ[T])^H$. To show that, let $x \in \phi(C_G(H))$ so $x \in U(K^\circ[T])$ and $x = (\phi(h))^{-1}x\phi(h) = (\rho(h))^{-1}x\rho(h) = x^h$ therefore, $x \in U(K^\circ[T]) \cap (K^\circ[T])^H = U((K^\circ[T])^H)$.

**Lemma 4.11.** Let $K^\circ[T]$ be an interior quandle $G$-algebra over $K$, $H$ be a $p$-subgroup of $G$ then the Brauer quatient $K^\circ[T](H)$ is an interior quandle $C_G(H)$-algebra over $K$.

**Proof.** Since $K^\circ[T]$ is an interior quandle $G$-algebra, there is a group homomorphism $\phi : G \to U(K^\circ[T])$. Then $K^\circ[T]$ is an interior quandle $C_G(H)$-algebra. Since $\phi(C_G(H)) \subseteq U((K^\circ[T])^H) \subseteq U(K^\circ[T])$, the injective map $C_G(H) \to U((K^\circ[T])^H)$ is a group homomorphism. Thus $(K^\circ[T])^H$ is an interior quandle $C_G(H)$-algebra over $K$. By Lemma 4.9, the Brauer quotient $K^\circ[T](H)$ is an interior quandle $C_G(H)$-algebra.

**Proposition 4.12.** Let $K^\circ[T_1]$ and $K^\circ[T_2]$ be interior quandle $G$-algebras over $K$, where $G = G_1 \cong G_2$, $G_1 \leq \text{Aut}(K^\circ[T_1])$ and $G_2 \leq \text{Aut}(K^\circ[T_2])$. Let $f : K^\circ[T_1] \to K^\circ[T_2]$ be a homomorphism of interior quandle $G$-algebras over $K$. Then, for any $p$-subgroup $H$ of $G$, $f$ induces two homomorphisms of interior quandle $C_G(H)$-algebras,

$$f^H : (K^\circ[T_1])^H \to (K^\circ[T_2])^H \quad \text{and} \quad f(H) : K^\circ[T_1](H) \to K^\circ[T_2](H).$$
Proof. Let $H$ be a $p$-subgroup of $G$, then the restriction of $f$ to $(K^o[T_1])^H$ induces a homomorphism of interior quandle $G$-algebras $f^H : (K^o[T_1])^H \rightarrow K^o[T_2]$. Note that, $f^H((K^o[T_1])^H) = (K^o[T_2])^H$. To this end, let $a \in (K^o[T_1])^H$ so, $a^h = a$ for all $h \in H$. Therefore,

$$f^H(a) = f^H(a^h) = (f^H(a))^h \quad \text{(from Proposition 4.8, since } f \text{ is homo. of interior)}$$

$$\in (K^o[T_2])^H.$$ 

Then, $f^H((K^o[T_1])^H) \subseteq (K^o[T_2])^H$. Now, let $b \in (K^o[T_2])^H$, so $b^h = b$ for all $h \in H$ and we can find $a \in (K^o[T_1])^H$ such that $b = f^H(a)$. Thus,

$$b = b^h = (f^H(a))^h = f^H(a^h) = f^H(a) \quad \text{(because } a \in (K^o[T_1])^H)$$

$$\in f^H(K^o[T_1])^H.$$ 

Then $(K^o[T_2])^H \subseteq f^H((K^o[T_1])^H)$. Hence, we can consider $f^H : (K^o[T_1])^H \rightarrow (K^o[T_2])^H$. From Lemma 4.11, $(K^o[T_1])^H$, $(K^o[T_2])^H$, $K^o[T_1](H)$ and $K^o[T_2](H)$ are interior quandle $C_G(H)$-algebras. Therefore, the homomorphism $f^H : (K^o[T_1])^H \rightarrow (K^o[T_2])^H$ and the Brauer homomorphism $Br^K_{H[T_2]} : (K^o[T_2])^H \rightarrow K^o[T_2](H)$ are homomorphisms of interior quandle $C_G(H)$-algebras. Then, the composition map $Br^K_{H[T_2]} \circ f^H : (K^o[T_1])^H \rightarrow K^o[T_2](H)$ is a homomorphism of quandle interior $C_G(H)$-algebras. Moreover, for all proper subgroups $L$ of $H$ and $a \in (K^o[T_1])^L$, we have

$$f^H(Tr_L^H(a)) = f^H(\sum_{hL \subseteq H/L} a^h)$$

$$= \sum_{hL \subseteq H/L} f^H(a^h)$$

$$= \sum_{hL \subseteq H/L} (f^H(a))^h$$

$$= Tr_L^H(f^H(a)) \in (K^o[T_2])^L.$$ 

So, $f^H((K^o[T_1])^L \subseteq (K^o[T_2])^L$. Then,

$$Br^K_{H[T_2]}(f^H((K^o[T_1])^L)) \subseteq Br^K_{H[T_2]}((K^o[T_2])^L) = 0_{K^o[T_2](H)}.$$ 

Consequently, $(K^o[T_1])^L \subseteq Ker(Br^K_{H[T_2]} \circ f^H)$. Thus, by Factor Lemma, there is a unique homomorphism $f(H) : K^o[T_1](H) \rightarrow K^o[T_2](H)$ such that
$f(H) \circ Br_{H}^{K^{\circ}[T_1]} = Br_{H}^{K^{\circ}[T_2]} \circ f^H$, or equivalently, the following diagram commutes,

\[
\begin{array}{ccc}
(K^{\circ}[T_1])^H & \xrightarrow{Br_{H}^{K^{\circ}[T_2]} \circ f^H} & K^{\circ}[T_2](H) \\
\downarrow_{Br_{H}^{K^{\circ}[T_1]}} & & \\
K^{\circ}[T_1](H) & \xrightarrow{f(H)} & \\
\end{array}
\]

References


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