The Homotopical Proof of \( \prod_1(S, x_0) \) as a Fundamental Group with Respect to “\( \circ \)” in the Interval \([0, n]\)

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Abstract

In algebraic field, a homomorphism is a structure-preserving map between two algebraic structures of the same type (such as two groups, two rings, or two vector spaces). These maps are continuous functions from one topological space to another. In terms of two or more maps, one map can be continuously deformed into the other and such a deformation is called a homotopy between the two functions. It is however, interesting to know that a topological space is a geometric object, and the continuous stretching and bending of the object into a new shape is a homeomorphism. That is, homeomorphism are the mappings that preserve all the topological properties of the given space.

Any given pointed topological space that provides a way to determine when two or more paths, starting and ending at a fixed base point, can be continuously deformed into each other is a fundamental group. This fundamental group is a tool used for describing what a topological space looks like. It creates an algebraic image of the space using loops in the space and these loops forms a continuous deformation within a certain interval.

The goal of this study is to establish that, the equivalent class \( \prod_1(S, x_0) \) which is made up of homotopic loops is a fundamental group with respect to “\( \circ \)” in the interval \([0,n]\), \( \forall n \in \mathbb{Z} \).

This is of interest because it can be used as a great tool in mathematics to study many physical systems.

Keywords: Homotopy, Fundamental group, Homomorphism, Homeomorphism, Equivalent class
1 Introduction

The essential inkling behind algebraic topology is to associate a topological situation to an algebraic situation, and study the simpler algebraic system. To each topological space a group can be associated, such that homeomorphic spaces stretch rise to isomorphic groups.

Suppose we pick out a point $x_0$ of $X$ to serve as a ”base point” and restrict ourselves to those paths that begin and end at $x_0$ (just choose $e_{x_0} = e_{x_1}$). Then we can multiply any path with each other because one path will start where the last one ended, and we will automatically have a closed system. Associativity, the existence of an identity element $[e_{x_0}]$, and the existence of an inverse $[\bar{f}]$ for $[f]$ are immediate. Then all the group axioms are satisfied, and the set of these path homotopy classes form a group under “$\circ$”.

So, to a map of spaces we can associate a homomorphism of groups such that compositions of maps yield compositions of homomorphisms of groups. Interestingly, a topological situation actually gives information about the algebraic one. Thus the fundamental group is a tool used for describing what a topological space looks like. It creates an algebraic image of the space using loops in the space. However, the group does not tell us everything about a space.

In this paper, we defined homotopy and presented some group properties. We then described the fundamental group and its related properties such as group homomorphisms. We also looked at useful definitions and theorems that plays an important role in computing fundamental groups, and our main result was actually built upon these theorems. In the end, we established the proof that the equivalent class $\prod_1(S, x_o)$ is a fundamental group in the interval $[0, n]$ $\forall n \in Z$ using self explained diagrams.
Throughout this paper we assumed the knowledge of basic group theory and general topology.

2 Preliminary Notes

2.1 Homotopy

Definition 1.1 Let $X$ be a topological space. A path in $X$ from $x_0$ to $x_1$ is a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say that $x_0$ is the initial point and $x_1$ the final point.
2.2 Homotopic path Concatenation

Definition 2.2.1 Let \( f : I \rightarrow X \) be a path in \( X \) from \( x_0 \) to \( x_1 \) and \( g : I \rightarrow X \) be a path in \( X \) from \( x_1 \) to \( x_2 \). Then the product \( f \ast g \) is defined to be the path \( f \ast g : I \rightarrow X \) given by
\[
f \ast g(s) = \begin{cases} 
  f(2s) & \text{for } s \in [0, \frac{1}{2}], \\
  g(2s-1) & \text{for } s \in [\frac{1}{2}, 1],
\end{cases}
\]

Figure 1: Concatenation of \( f \) and \( g \).

Example 2.2.2 (Linear homotopies.) Let \( f \) and \( f' \) be any two paths in \( \mathbb{R}^n \) having the same endpoints \( x_0 \) and \( x_1 \). Then \( F(s, t) = (1-t)f(s) + tf'(s) \) is a homotopy between \( f \) and \( f' \). We verify this:

(i) \( F(s, 0) = f(s) \) and \( F(s, 1) = f'(s) \),

(ii) \( F(0, t) = (1-t)x_0 + tx_0 = x_0 \) and \( F(1, t) = (1-t)x_1 + tx_1 = x_1 \).

During the homotopy each point \( f(s) \) travels along a line segment to \( f'(s) \) at constant speed. It is called a straight line homotopy.

In particular, if \( U \subset \mathbb{R}^n \) is convex, then any two paths \( f, g : I \rightarrow U \) with same endpoints are homotopic.

Definition 2.2.3 A loop in \( X \) is a continuous map \( f : I \rightarrow X \) such that \( f(0) = f(1) \).

Then two loops can be combined together in an obvious way; first travel along the first loop, then along the second.

Definition 2.2.4 Let \( X \) be a topological space, and \( x_0 \) a point in \( X \). The fundamental group of \( X \) is the set of path homotopy classes \([f]\) of loops \( f : I \rightarrow X \) based at \( x_0 \), together with the operation \( \ast \). We denote it by \( \pi(X, x_0) \).

Definition 2.2.5 Given two loop classes \([f]\) and \([g]\) we define:
(i) \([f] \ast [g] = [f \ast g]\).

(ii) The inverse of \([f]\) is given by \([f^{-1}]\), that is \([f]^{-1}\), where \(f^{-1}(t) = \bar{f}(t) = f(1-t)\).

**Theorem 2.2.6** If \(X\) is a convex subset of \(\mathbb{R}^n\), and if \(a, b \in X\), \(\alpha_t(s) = (1-t)\alpha_0(s) + t\alpha_1(s)\). This defines a homotopy between \(\alpha_0\) and \(\alpha_1\). □

**Theorem 2.2.7** The relation \(f \simeq g\) is an equivalence relation on the set, \(\text{Hom}(X,Y)\), of continuous mappings from \(X\) to \(Y\).

*Proof.* Let \(f : X \to Y\) be a given mapping. The homotopy \(H(x,t) = f(x)\) is a continuous mapping \(H : X \times [0,1] \to Y\) and so \(f \simeq f\)

If \(f_0 \simeq f_1\) and \(H : X \times [0,1] \to Y\) is a homotopy between \(f_0\) and \(f_1\), then the mapping \(H' : X \times [0,1] \to Y\) given by \(H'(x,t) = H(x,1-t)\) is continuous and a homotopy between \(f_1\) and \(f_0\) that is \(f_1 \simeq f_0\)

Finally, for \(f_0 \simeq f_1\) and \(f_1 \simeq f_0\), suppose that \(H_1 : X \times [0,1] \to Y\) is a homotopy between \(f_0\) and \(f_1\), and \(H_2 : X \times [0,1] \to Y\) is a homotopy between \(f_1\) and \(f_2\). Define the homotopy \(H : X \times [0,1] \to Y\) by

\[
H(x,t) = \begin{cases} 
H_1(x,2t), & \text{if } 0 \leq t \leq 1/2 \\
H_2(x,2t-1), & \text{if } 1/2 \leq t \leq 1
\end{cases}
\]

Since \(H_1(x,1) = f_1(x) = H_2(x,0)\), the piecewise definition of \(H\) gives a continuous function. By definition, \(H(x,0) = f_0(x)\) and \(H(x,1) = f_2(x)\) and so \(f_0 \simeq f_2\) □

**Definition 2.2.8** The equivalence classes of maps from \(X\) to \(Y\) as in the Theorem 1.1.3 are called the homotopy classes ad denoted by \([f]\) as the homotopy class of the map \(f\). While we use the notation \(\langle \gamma \rangle\) when \(\gamma\) is a loop in \(X\) based at \(x_0 \in X\).

From now on we assume all spaces are topological spaces.

### 2.3 Homotopy Classes

**Definition 2.3.1** The equivalent classes \([f]\) determined by homotopy modulo \(x_0\) on the collection \(C(S,x_0)\) of all closed paths \(f\) on \(S\) based at \(x_0 \in S\) are called homotopy classes of \(C(S,x_0)\). The collection of these homotopy classes is denoted by \(\prod_1(s,x_0)\).
3 Main Results

Definition 3.1.0

1. If \( f, g \in C(S, x_0) \) we define the juxtaposition \( f \ast g \) of \( f \) and \( g \) as follows:

\[
(f \circ g)(s) = \begin{cases} 
  f(2s) & \text{if } 0 \leq s \leq \frac{n}{2} \\
  g(2s - n) & \text{if } \frac{n}{2} \leq s \leq n 
\end{cases}
\]

Thus \( f \circ g \in C(S, x_0) \) and “\( \circ \)” is a binary operation on \( C(S, x_0) \).

2. If \([f], [g] \in \prod_1(S, x_0)\), then let \([f] \circ [g] = [f \circ g]\)

Theorem 3.1.1 If \( X \) and \( Y \) are homotopy equivalent then their fundamental groups are isomorphic.

Proof: Let \( f \) and \( g \) be s.t \( fg \cong 1_X \) and \( gf \cong 1_Y \). In order to apply Theorem 3.1.1, we have \( gf, 1_X : X \longrightarrow X \) which are homotopic maps. Suppose that \( x_0 \in X, x_1 = gf(x_0) \) and \( \alpha(t) := F(x_0, t), 0 \leq t \leq 1 \), hence we have the following commutative diagram,

\[
\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{(1_X)_*} & \pi_1(X, x_0) \\
g \circ f \downarrow & & \alpha \downarrow \\
\pi_1(X, x_1) & & \pi_1(Y, f(x_0))
\end{array}
\]

Figure 2: homotopic maps which is commutative

since \((1_X)_*\) is an isomorphism then so is \( g_* \circ f_* \). Mimicking the way above to see that \( f_* \circ g_* \) is also an isomorphism. Therefore \( f_* \) is an isomorphism \( \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)) \).

\[\square\]

Definition 3.1.2 The group \( \pi_1(X, x_0) \) is called the fundamental group of \( X \) at the base point \( x_0 \).

Suppose \( x_1 \) is another choice of basepoint for \( X \). If \( X \) is path-connected, there is a path \( \gamma : [0, 1] \rightarrow X \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). This path induces
a mapping $u_\gamma : \pi_1(X,x_0) \to \pi_1(X,x_1)$ by $[\lambda] \mapsto [\gamma^{-1} \ast \lambda \ast \gamma]$, that is, follow $\gamma^{-1}$ from $x_1$ to $x_0$, then follow $\lambda$ around and back to $x_0$, then follow $\gamma$ back to $x_1$, all giving a loop based at $x_1$. Notice

$$u_\gamma([\lambda] \ast [\mu]) = u_\gamma([\lambda \ast \mu])
= [\gamma^{-1} \ast \lambda \ast \mu \ast \gamma]
= [\gamma^{-1} \ast \lambda \ast \gamma \ast \gamma^{-1} \ast \mu \ast \gamma]
= [\gamma^{-1} \ast \lambda \ast \gamma] \ast [\gamma^{-1} \ast \mu \ast \gamma] = u_\gamma([\lambda]) \ast u_\gamma([\mu])$$

Thus $u_\gamma$ is a homomorphism. The mapping $u_\gamma^{-1} : \pi_1(X,x_1) \to \pi_1(X,x_0)$ is an inverse, since

$$[\gamma^{-1} \ast (\gamma^{-1} \ast \lambda \ast \gamma) \ast \gamma^{-1}] = [\lambda].$$

Thus, $\pi_1(X,x_0)$ is isomorphic to $\pi_1(X,x_1)$ whenever $x_0$ is joined to $x_1$ by a path. Though it is a bit of a lie, we write $\pi_1(X)$ for a space $X$ that is path-connected since any choice of basepoint gives an isomorphic group. In this case, $\pi_1(X)$ denotes an isomorphism class of groups.

**Corollary 3.1.3** $\pi_1(R^n \setminus \{0\}) \cong \pi_1(S^{n-1})$.

**Theorem 3.1.4** $\Pi_1(S,x_0)$ is a fundamental group with respect to “$\circ$” in the interval $[0, n]$.

**Proof**

1. “$\circ$” is associative. We need to show that $(f \circ g) \circ k \sim x_0 f \circ (g \circ k)$ for $f, g, k \in C(S,x_0)$.

$$[(f \circ g) \circ k](s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{n}{4} \\ g(2s-n) & \text{if } \frac{n}{4} \leq s \leq \frac{n}{2} \\ k(2s-n) & \text{if } \frac{n}{2} \leq s \leq n \end{cases} \circ k = \begin{cases} f[2(2s)] \\ g[2(2s-n)] \\ k[2(2s-n)] \end{cases}$$

and

$$[f \circ (g \circ k)](s) = \begin{cases} f \circ \left( g(2s) \\ k(2s-n) \right) = \begin{cases} f(2s) \\ g(2s-n) \\ k[2(2s-n)-n] \end{cases} \end{cases}$$
The homotopical proof of $\prod_1(S,x_0)$ as a fundamental group

$$\begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{n}{2} \\ g(4s - 2n) & \text{if } \frac{n}{2} \leq s \leq \frac{3n}{4} \\ k(4s - 3n) & \text{if } \frac{3n}{4} \leq s \leq n \end{cases}$$

We define a homotopy between $(f \circ g) \circ k$ and $f \circ (g \circ k)$ as follows:

$$h(s,t) = \begin{cases} f\left(\frac{4s}{n+1}\right) & \text{if } \langle s,t \rangle \in I^2 \text{ and } t \geq 4s - n \\ g(4s - t - n) & \text{if } \langle s,t \rangle \in I^2 \text{ and } 4s - n \geq t \geq 4s - 2n \\ k\left(\frac{4s-t-2n}{2-t}\right) & \text{if } \langle s,t \rangle \in I^2 \text{ and } 4s - 2n \geq t \end{cases}$$

Then the following is true:

$$h(s,0) = \begin{cases} f(4s) & \text{if } 0 \geq 4s - n \quad [\text{i.e. } 0 \leq s \leq \frac{4}{n}] \\ g(4s - n) & \text{if } 4s - n \geq 0 \geq 4s - 2n \quad [\text{i.e. } \frac{4}{n} \leq s \leq \frac{4}{2}] \\ k(2s - n) & \text{if } 4s - 2 \geq 0 \quad [\text{i.e. } \frac{n}{2} \leq s \leq n] \end{cases}$$

$$h(s,n) = \begin{cases} f(2s) & \text{if } n \geq 4s - n \quad [\text{i.e. } 0 \leq s \leq \frac{3n}{4}] \\ g(4s - 2n) & \text{if } 4s - n \geq n \geq 4s - 2n \quad [\text{i.e. } \frac{n}{2} \leq s \leq \frac{3n}{4}] \\ k(4s - 3n) & \text{if } 4s - 2n \geq 1 \quad [\text{i.e. } \frac{3n}{4} \leq s \leq n] \end{cases}$$

Thus $h(s,0) = [(f \circ g) \circ k](s)$ and $h(s,n) = [f \circ (g \circ k)](s)$ for all $s \in I^n$

Also $h(0,t) = f(0) = x_0$ and $h(n,t) = k(n) = x_0$ for all $t \in I^n$

Hence $(f \circ g) \circ k \tilde{x}_0 f \circ (g \circ k)$

Figure 3: $(f \circ g) \circ k \tilde{x}_0 f \circ (g \circ k)$
2. We show that the constant mapping \( c : I^n \rightarrow \{x_0\} \) is such that \([c]\) is the identity element of \( \prod_1(S, x_0) \) with respect to “\( \circ \)”. Thus we must show that \( f \circ c \bar{x}_0 f \) for any \( f \in C(S, x_0) \)

Let \( h : I^2 \rightarrow S \) be defined as follows:

\[
h(s, t) = \begin{cases} 
  f(\frac{2s}{n+t}) & \text{if } (s, t) \in I^2 \text{ and } t \geq 2s - n \\
  x_0 & \text{if } (s, t) \in I^2 \text{ and } 2s - n \geq t
\end{cases}
\]

Then

\[
(f \circ c)(s) = h(s, 0) = \begin{cases} 
  f(2s) & \text{if } 0 \geq 2s - n \quad [\text{i.e. } 0 \leq s \leq \frac{n}{2}] \\
  x_0 & \text{if } 2s - n \geq 0 \quad [\text{i.e. } \frac{n}{2} \leq s \leq n]
\end{cases}
\]

and

\[
h(s, n) = f(s) \quad \text{if } 1 \geq 2s - n \quad [\text{i.e. } 0 \leq s \leq n]
\]

Thus \( h(s, 0) = (f \circ c)(s) \) and \( h(s, n) = f(s) \) for all \( s \in I^n \)

More so \( h(0, t) = f(0) = x_0 \) and \( h(n, t) = f(n) = x_0 \) for all \( t \in I^n \)

Hence \( f \circ c \bar{x}_0 f \)

3. Finally we want to show that each homotopy class \([f] \in \prod_1(S, x_0)\) has an inverse \([g] \in \prod_1(S, x_0)\) s.t \([f] \circ [g] = [c] \)

Thus we want to show that if \( f \in C(S, x_0) \) there exist a \( g \in C(S, x_0) \)

such that \( f \circ g \bar{x}_0 c \)

Let \( g(s) = f(n - s) \quad \forall s \in I^n \)

Since \( g(0) = f(n) = x_0 = f(0) = g(n) \), \( g \in C(S, x_0) \)
By definition we have

\[(f \circ g)(s) = \begin{cases} 
  f(2s) & \text{if } 0 \leq s \leq \frac{n}{2} \\
  g(2s - n) = f(2n - 2s) & \text{if } \frac{n}{2} \leq s \leq n
\end{cases}\]

We then define homotopy \(h\) between \(f \circ g\) and \(c\) as follows:

\[h(s, t) = \begin{cases} 
  x_0 & \text{if } 0 \leq s \leq \frac{nt}{2} \\
  f(2s - t) & \text{if } \frac{nt}{2} \leq s \leq \frac{n}{2} \\
  g(2s + t - n) & \text{if } \frac{n}{2} \leq s \leq n - \frac{nt}{2} \\
  x_0 & \text{if } n - \frac{nt}{2} \leq s \leq n
\end{cases}\]

Since \(f\) and \(g\) are continuous, \(h\) is continuous and we have

\[h(s, 0) = \begin{cases} 
  f(2s) & \text{if } 0 \leq s \leq \frac{n}{2} \\
  g(2s - n) & \text{if } \frac{n}{2} \leq s \leq n
\end{cases}\]

and

\[h(s, n) = x_0 \text{ if } 0 \leq s \leq n\]

Thus \(h(s, 0) = (f \circ g)(s)\) and \(h(s, n) = c(s)\) \(\forall s \in I^n\)

Also \(h(0, t) = h(n, t) = x_0\) for all \(t \in I^n\)

Hence \(f \circ g \tilde{\cong} x_0 c\)

4 Remark

This paper, in full accordance with the principles of homotopy, has been able to establish the proof that \(\prod_1(S, x_0)\) is a fundamental group in the general interval \([0, n]\), \(\forall n \in Z\)
In general, $\prod_1(S, x_0)$ depends upon $x_0$. However, in the case of an arc-wise-connected space $S$, we can show that $\prod_1(S, x_0)$ is independent of $x_0$.

References


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