Gorenstein Category with Respect to Twin Cotorsion Pair

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Abstract

In this paper the definition of Gorenstein category related to twin cotorsion pair is introduced. Let $M$ is $R$ module,$G(s)$ is the Gorenstein category related to twin cotorsion pair. $0 \to L \to M \to N \to 0$ is exact sequence. It is shown that if $N \epsilon G(S)$ then $LeG(S)$ if and only if $MeG(S)$ and if $L, MeG(S)$ then when $N \epsilon G(S)$ if and only if $Ext^1_R(N,U \cap \Gamma) = 0$.

It is established that the results obtained by X.Y.Yang and W.J.Chen. And then we study the Gorenstein homological dimensions is related to twin cotorsion pairThis of $R$–modules.

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1 Introduction

Cotorsion pairs and their relation to structures have been the topic of much recent research. The concept of cotorsion pair was first proposed by Sale in ([1]), in 1979. Nakaoka in ([2]) introduced the notion of hearts of (twin)cotorsion pairs on triangulated categories and showed that they have structures of (semi-) abelian categories. Yu Lin, he studied a twin cotorsion pair $(S,T)$, $(U,V)$ on an exact category $\beta$ with enough projectives and injectives and introduce a notion of the heart and showed that its heart is preabelian in ([3]). Yang and Chen extended the classical Gorenstein projective module to the Gorenstein category
related to the cotorsion pair, and proved that Lemma 3.2 let $M$ be an object in $R$ modules, then $M \in G(x)$ if and only if $\text{Ext}^{i \geq 1}_R(M, X \cap Y) > 0$, and there exists a $\text{Hom}(-, X \cap Y)$ exact exact sequence $0 \rightarrow M \rightarrow X_{i-1} \rightarrow X_{i-2} \rightarrow \cdots$, with each $X_i \in X \cap Y$. Proposition 3.3 (1) let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $R$ modules, if $N \in G(X)$, $L \in G(X)$ if and only if $M \in G(X)$. (2) let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $R$ modules if $L, M \in G(X)$ then $N \in G(X)$ if and only if $\text{Ext}^1_R(N, X \cap Y) = 0$ in ([4]). Inspired by the result of the appeal, the natural question is whether there will be similar results when the cotorsion pair is generalized to the twin cotorsion pair. The main purpose of this paper is to try to give the answers to the above ideas.

2 Preliminary

In this section we will recall the definition of Ore-extension and some basic results. Throughout this paper, we let $M$ be a $R$ modules and $(S, \Gamma)$ and $(U \cap V)$ be a twin cotorsion pair and $S \leq U$.

Definition 2.1. [4] A cotorsion pair is a pair $(S, \Gamma)$ of classes of $R$ modules such that $S^\perp = \Gamma$, $S = \perp S$, and $S^\perp = \{M \in R-\text{Mod}|\text{Ext}^1_R(S, M) = 0, \forall s \in S\}$; $\Gamma = \{N \in R-\text{Mod}|\text{Ext}^1_R(N, T) = 0, \forall T \in \Gamma\}$

Definition 2.2. [4] A complete $S$-resolution is a $\text{Hom}_R(-, S \cap \Gamma)$-exact exact sequence $\cdots \rightarrow S_1 \rightarrow S_0 \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots$ with each $s_i \in S$. An object $M$ is in $G(S)$ if there exists a complete $S$-RESOLUTION AS ABOVE SUCH THAT $M \cong \text{Ker}(S_{-1} \rightarrow S_{-2})$.

Definition 2.3. [4] A cotorsion pair $(S, \Gamma)$ is said to be complete if for any $R$ module $M$, there are exact sequence $0 \rightarrow M \rightarrow T \rightarrow S \rightarrow 0$ and $0 \rightarrow T' \rightarrow S' \rightarrow M \rightarrow 0$. with $T, T' \in \Gamma$ and $s, s' \in S$.

Definition 2.4. [4] Let $(S, \Gamma)(U, V)$ be complete cotorsion pairs. The pair $(S, \Gamma)$ and $(U, V)$ is called a twin cotorsion pair if it satisfies $\text{Ext}^1_R(S, V) = 0$. This is equivalent to $S \leq U$ or $V \leq \Gamma$.

Remark [4] (1)If $(S, T) = (U, V) = (\rho(R), R-\text{Mod})$, then $G(S)$ is just the class of Gorenstein projective $R$-modules, where $\rho(R)$ the class of $R$-modules.
(2)If $(S, T) = (U, V) = (X, Y)$, then $G(S)$ in here is just the Gorenstein category $G(S)$ with respect to cotorsion pair $(X, Y)$.
(3)If $(S, T) = (U, V) = (F(R), G(S))$, then $G(S)$ in here is just the Gorenstein flat cotorsion.
(4)If $(S, T) = (\rho(R), R-\text{Mod}), (u, v) = (F(R), G(S))$, then $G(S)$ in here is just
the class of Ding projective $R$– modules. By completeness of cotorsion pair $(S, T)$, $R$– module $M$ in $G(S)$ if and only if $Ext^{i \geq 1}(M, U \cap T) = 0$, and there exists a $Hom_R(-, U \cap T)$ exact exact sequence $0 \rightarrow M \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow 0 \rightarrow \cdots$, with each $S_i \in S$

3 Main Results 1

Definition 3.1. Let $(S, T)$ and $(U, V)$ be twin cotorsion pairs, $R$ module $M \in G(S)$, there exists a $Hom_R(-, U \cap T)$ exact exact sequence $\cdots \rightarrow S_{1} \rightarrow S_{0} \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots$ with each $S_i \in S$, $M \simeq Ker(S_{-1} \rightarrow S_{-2})$.

Lemma 3.2. Let $(S, T), (U, V)$ be twin cotorsion pair, $S \leq U, U \leq T, R$– module $M, M \in G(S)$ if and only if $Ext^{i \geq 1}(M, U \cap T) = 0$ and there exists a $Hom_R(-, U \cap T)$ exact exact sequence $0 \rightarrow M \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots$ with $S_i \in U \cap T$.

Proof. ($\iff$) The part is clear by definition. We only prove ”Only if” part.

($\implies$) By assumption $M \in G(S)$, there exists a complete $S$– resolution

\[ \cdots \rightarrow S_{1} \rightarrow S_{0} \rightarrow M \rightarrow 0 \]

such that

\[ M \simeq Ker(S_{-1} \rightarrow S_{-2}) \text{ with } S_i \in U. \]

Since

\[ \cdots \rightarrow S_{1} \rightarrow S_{0} \rightarrow M \rightarrow 0 \]

is $Hom_R(-, U \cap T)$ exact and $Ext^1_R(S_i, U \cap \Gamma) = 0, \forall i \geq 1$,

Then $Ext^{i \geq 1}(M, U \cap \Gamma) = 0$.

Next proof there exists a $Hom(-, U \cap \Gamma)$– exact exact sequence

\[ 0 \rightarrow M \rightarrow \overline{S}_{-1} \rightarrow \overline{S}_{-2} \rightarrow \cdots \]

with $\overline{S}_i \in U \cup \Gamma$.

Let $M_{-1} = \text{Im}(S_{-1} \rightarrow M_{-1} \rightarrow 0)$ be $Hom_R(-, U \cap \Gamma)$ exact.
Since \((S, \Gamma), (U, V)\) is complete. There is an exact sequence

\[
0 \to S_{-1} \to L_{-1} \to 0
\]

with \(S_{-1} \in U \cap \Gamma \), \(L_{-1} \in U \).

Consider the following pushout diagram

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \to & M & \to & S_{-1} & \to & M_{-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M & \to & S_{-1} & \to & N_{-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_{-1} & & L_{-1} & & L_{-1} & & L_{-1} & & L_{-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]  

(*1)

The exactness of \(0 \to M_{-1} \to N_{-1} \to L_{-1} \to 0\) implies that 
\(\text{Ext}_R^{i \geq 1}(N_{-1}, U \cap \Gamma) = 0\). Let \(M_{-2} = \text{Im}(S_{-2} \to S_{-3})\).

Then

\[
0 \to M_{-1} \to S_{-2} \to M_{-2} \to 0
\]

is \(\text{Hom}_R(-, U \cap \Gamma)\) exact.

Thus we have the following commutative diagram

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \to & M_{-1} & \to & N_{-1} & \to & L_{-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S_{-1} & \to & Q_{-2} & \to & L_{-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{-2} & & M_{-2} & & M_{-2} & & M_{-2} & & M_{-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]  

(*2)

Since \(S_{-2}\) and \(L_{-1}\) are in \(U\), So \(Q_{-2}\) in \(U\), Note that \(\text{Ext}_R^{i \geq 1}(M_{-2}, U \cap \Gamma) = \)
Then

\[ 0 \rightarrow N_{-1} \rightarrow Q_{-2} \rightarrow M_{-2} \rightarrow 0 \] be \( \text{Hom}(-, U \cap \Gamma) \) exact.

Since

\[ 0 \rightarrow M_{-2} \rightarrow S_{-3} \rightarrow S_{-4} \rightarrow \cdots \]

be \( \text{Hom}(-, U \cap \Gamma) \) exact. We get a \( \text{Hom}(-, U \cap \Gamma) \) exact exact sequence

\[ 0 \rightarrow N_{-1} \rightarrow Q_{-2} \rightarrow S_{-3} \rightarrow S_{-4} \rightarrow \cdots \]

By this way, we get a \( \text{Hom}_R(-, U \cap \Gamma) \) exact exact sequence

\[ 0 \rightarrow M \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots \]

with \( S_i \epsilon U \cap \Gamma \).

\[ \square \]

**Theorem 3.3.** Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be a short exact sequence of \( R \) module. If \( N, L \epsilon G(S) \) If and only if \( M \epsilon G(S) \).

**Proof.** (\( \Rightarrow \)), Let \( N, L \epsilon G(S) \). By Lemma, there exist \( \text{Hom}(-U \cap \Gamma) \) exact exact sequence

\[ 0 \rightarrow L \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots \]

\[ 0 \rightarrow N \rightarrow S'_{-1} \rightarrow S'_{-2} \rightarrow \cdots \]

with all \( S_i, S'_i \epsilon U \cap \Gamma \). Note that the sequence

\[ 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \]

is \( \text{Hom}(-U \cap \Gamma) \) exact. We can construct a \( \text{Hom}(-U \cap \Gamma) \) exact exact sequence

\[ 0 \rightarrow M \rightarrow S_{-1} \oplus S'_{-1} \rightarrow S_{-2} \oplus S'_{-2} \rightarrow \cdots \]
Also

\[ Ext^i_R(L, U \cap \Gamma) = 0 = Ext^i_R(N, U \cap \Gamma). \]

This implies that \( Ext^i_R(M, U \cap \Gamma) = 0 \), and hence \( M \in G(S) \).

\((\Leftarrow=)\) Assume that \( M \in G(S) \), then \( Ext^i_R(M, U \cap \Gamma) = 0 \), there exists a \( \text{Hom}_R(\cdot, U \cap \Gamma) \) exact exact sequence

\[ 0 \to M \to S_{-1} \to S_{-2} \to \cdots \]

with all \( S_i \in U \cap \Gamma \).

Let \( M_{-1} = Im(S_{-1} \to S_{-2}) \to \cdots \), Since

\[ Ext^i(R)^{i \ge 1}(M, U \cap \Gamma) = 0, Ext^i(R)^{i \ge 1}(M_{-1}, U \cap \Gamma) = 0. \]

The sequence

\[ 0 \to M \to S_{-1} \to M_{-1} \to 0 \]

is \( \text{Hom}_R(\cdot, U \cap \Gamma) \) exact.

with all \( S_{-1} \in U, M_{-1} \in G(S) \). Consider the following pushout diagram

\[ \begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & L & \to & M & \to & N & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L & \to & S_{-1} & \to & Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{-1} & & M_{-1} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \] (∗3)

Since \( M, M_{-1}, N \in G(S) \). Then \( Q \in G(S) \), there exists a \( \text{Hom}_R(\cdot, U \cap \Gamma) \) exact exact sequence \( 0 \to Q \to S_{-2} \to S_{-3} \to \cdots \) with all \( S_i \in U \), We get a \( 0 \to L \to S_{-1} \to S_{-2} \to S_{-3} \to \cdots \), with all \( S_i \in U \).

**Theorem 3.4.** Let \( 0 \to L \to M \to N \to 0 \) a short exact sequence of \( R \) module. If \( L, M \in G(S), N \in G(S) \) if and only if \( Ext^1_R(N, U \cap \Gamma) = 0 \).
Proof. \((\Longleftrightarrow)\) The part is clear by definition. We only prove "if" part.
\((\implies)\) Since \(L \in G(S)\), there exists a short exact sequence
\[
0 \to L \to X \to L_0 \to 0.
\]

And \(X \in U \cap \Gamma, L_0 \in G(S)\). Consider the following pushout diagram
\[
\begin{array}{cccc}
0 & 0 & \downarrow & \\
0 & L & M & N & 0 \\
0 & X & Q & N & 0 \\
L_0 & L_0 & & & \\
0 & 0 & & & \\
\end{array}
\]

(*4)

Since \(L, L_0 \in G(S)\), Thus (1) \(Q \in G(S)\). Thus there exists a short exact sequence
\[
0 \to Q \to X' \to K \to 0
\]

with \(X' \in S, k \in G(S)\). Consider the following pushout diagram
\[
\begin{array}{cccc}
0 & 0 & \downarrow & \\
0 & X & Q & N & 0 \\
0 & X & X' & H & 0 \\
K & K & & & \\
0 & 0 & & & \\
\end{array}
\]

(*5)

Note that the sequence \(0 \to N \to H \to K \to 0\) exactness, then \(H \in G(S)\)

\[
\text{Ext}_{R}^{\geq 1}(H, U \cap \Gamma) = 0\), By Lemma \(N \in G(S)\)
4 Main Results

Lemma 4.1. Let $M$ be any $R$–module and consider two exact sequence, $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0, 0 \to \bar{K}_n \to \bar{G}_{n-1} \to \cdots \to \bar{G}_0 \to M \to 0$ where $G_0 \cdots G_{n-1}$ and $\bar{G}_0 \cdots \bar{G}_{n-1} \epsilon G(S)$. Then $K_n \epsilon G(S)$ if and only if $\bar{K}_n \epsilon G(S)$.

Theorem 4.2. Let $M$ be an $R$–module, $G(S)$ is the Gorenstein category related to twin cotorsion pair, $G(s) - dimM < \infty$, $M \epsilon G(S)$, and let $n$ be an integer, then the following condition are equivalent

1. $G(S) - dimM < n$.
2. $Ext_R^n(M,L) = 0$ for all $i > n$, and all $R$–modules $L$ with $G(S) - dimL < \infty$.
3. $Ext_R^n(M,Q) = 0$ for all $i > n$, and all projective $R$–modules $Q$.
4. For every exact sequence $0 \to K_n \to C_{n-1} \to \cdots \to G_0 \to M \to 0$. Where $G_0 \cdots G_{n-1} \epsilon C(A)$, Then also $K_n \epsilon G(S)$.

Proof. The proof is cyclic. Obviously (2) $\implies$ (3) and (4) $\implies$ (1), so we only have to prove the last two implications.

To prove (1) $\implies$ (2), we assume that $G(S) - dimM < n$. We have an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$. Where $G_0 \cdots G_n \epsilon C(A)$. Then $Ext^n_R(G_n,L) \cong Ext^n_R(M,L) = 0$, where $i > n$. And $R$–modules $L$ with $G(S) - dimL < \infty$. As desired. Where $i > n$, and $R$–modules $L$ with $G(S) - dimL < \infty$.

To prove (3) $\implies$ (4), We consider an exact sequence,

$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$,

where $G_0 \cdots G_{n-1} \epsilon G(S)$. Then $Ext^n_R(K_n,Q) \cong Ext^{i+n}_R(M,Q) = 0$ for every integer $i > 0$ and every projective module $Q$. Decomposing $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ into short exact sequence and let

$0 \to K \to G \to M \to 0$ be an exact sequence of $R$–modules where $G \epsilon G(S)$.

If $M \epsilon G(S)$, then so is $K$. Otherwise we get $Gpd_k K = Gpd_k M - 1 < \infty$.

Since $G(S) - dimM < \infty$, Hence there is an exact sequence

$0 \to G'_m \to \cdots \to G'_0 \to K_n \to 0$, where $G'_0 \cdots G'_m \epsilon G(S)$.

We decompose it into short exact sequence:

$0 \to C'_j \to \cdots \to G'_{j-1} \to C'_{j-1} \to 0$, for $j = 1 \cdots m$, where $C'_m = G'_m$ and $C'_0 = K_n$.

Now, $Ext^n_R(C'_{j-1},Q) \cong Ext^n_R(K_n,Q) = 0$, for all $j = 1, \cdots m$, and all projective modules $Q$. Thus Proposition 2.11 in ([11]) can be applied successively to conclude that $C'_m, \cdots C'_0 \epsilon G(S)$. In particular $K_n = C'_0$ is Gorenstein projective, as desired.
References


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