A Brief Note on Absolute Valued Algebras with Involution

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Abstract

We give some necessary and sufficient conditions giving rise to the finite dimensionality of any absolute-valued algebra with involution.

Mathematics Subject Classification: 17A35, 17A80

Keywords: Absolute-valued algebra, involution, central idempotent

1. Introduction

An absolute-valued algebra is a non-zero real algebra $A$ endowed with a norm $||.||$ such that $||xy|| = ||x|| ||y||$ for all $x, y \in A$. From their appearance in [Os 18], the absolute-valued algebras have attracted the attention of many mathematicians [A 47], [UW 60], [CR 14], and physicists [Ok 78]. This is due to their beauty and diversity. Subsequently, work on absolute-valued algebras has succeeded one another and continues to do so until today.

Since the first paper on absolute-valued algebras with involution [U 61], several works in the same theme have emerged and continue to do so until today [G 63], [Elm 88, 90], [Rod 04], [BR 05], [EERR 06], [RR 09], [CR 14].

Let $A$ be an absolute-valued algebra $A$ with involution $x \mapsto \overline{x}$. By putting $x \diamond y = \overline{xy}$ on the normed space $A$ we get a new absolute-valued algebra $(A, \diamond)$ called the cracovian of $A$ [G 63]. This apparently provides the first example of infinite-dimensional absolute-valued algebras satisfying non-trivial identities, like $(x^2, y, x^2) = 0$ [EE 04], [CRR 12, Proposition 4.6], [DDFR].
Also, any absolute-valued algebras with involution satisfying an identity of the form \((x^p, x^q, x^r) = 0\), with \(p, q, r \in \{1, 2\}\), is finite-dimensional [Ch-R 08].

The existence of infinite-dimensional absolute-valued algebras is ensured even if the algebra is endowed with an involution [U 61] and having a non-zero central idempotent [Elm 88, Lemma 3.2]. Also, the infinite-dimensionality happen even for left-division absolute-valued algebras with left-unit [Cu 92], [Rod 92, 04].

Motivated by these facts, we became interested in conditions ensuring the finite dimensionality of any absolute-valued algebra with non-trivial involution. In section 2 we introduce the basic tools for the study of absolute-valued algebras \(A\) with a non-trivial involution. We show that if \(A\) contains a non-zero central element not collinear with a certain idempotent, then \(A\) is finite-dimensional and isomorphic to either \(\mathbb{C}\) or \(\mathbb{C}^\ast\) (Proposition 1).

The paper ends, in section 3, with the main result.

\[\text{2. Definitions and Notations}\]

Let \(A\) be a non-associative real algebra. We denote by \(L_x\) (resp \(R_x\)) the operator of left-multiplication (resp. right-multiplication) by \(x \in A\).

We also denote by \([x, y]\) (resp. \((x, y, z)\)) the commutator of \(x, y \in A\) (resp. the associator of \(x, y, z \in A\)). An element \(x \in A\) is said to be central if \([x, A] = 0\). It is said to left-invertible (resp. right-invertible) if \(L_x\) (resp. \(R_x\)) is bijective. It is said to invertible if both \(L_x, R_x\) are bijective. The algebra \(A\) is said to be left-division algebra (resp. right-division algebra) if \(L_x\) (resp. \(R_x\)) is bijective for all non-zero \(x \in A\). It is said to be a division algebra if both \(L_x, R_x\) are bijective for all non-zero \(x \in A\).

Let \(a_1, \ldots, a_n\) be in \(A\). We denote by \(\text{Lin}\{a_1, \ldots, a_n\}\) the lineal hull spanned by \(\{a_1, \ldots, a_n\}\).

An absolute-valued algebra is a real vector space \(A\) is endowed with a norm \(||.||\) such that \(||xy|| = ||x|| \cdot ||y||\) for all \(x, y \in A\). An involution over \((A, ||.||)\) is a linear mapping \(\sigma : A \to A\) \(x \mapsto \overline{x}\) satisfying the following conditions [U 61], [Rod 08]:

\[
\begin{align*}
(1) & \quad \overline{\overline{x}} = x \\
(2) & \quad x\overline{x} = \overline{x}x \\
(3) & \quad \overline{xy} = \overline{y} \overline{x}
\end{align*}
\]

for all \(x, y \in A\).

According to [U 61], there exists a distinguished element \(e \in A\) satisfying \(x\overline{x} = ||x||^2 e\) for every \(x \in A\), the absolute value of \(A\) derives from an inner product, \(A_\alpha := \{x \in A : \overline{x} = x\}\) is orthogonal to \(A_\beta := \{x \in A : \overline{x} = -x\}\), and elements of \(A_\alpha\) commute with those of \(A_\beta\). Clearly, the element \(e\) above is the unique nonzero self-adjoint idempotent of \(A\). We put \(B := \Re e \oplus A_\alpha\),
and we note that $B$ is a subalgebra of $A$ [Elm 88, Lemma 3.1] and that the idempotent $e$ is central in $B$.

We state the following result keeping the above notation:

**Proposition 1.** If $A_s \neq \{0\}$ and $A$ contains a non-zero central element $a$ not collinear with $e$ then $A$ is finite-dimensional and isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$.  

**Proof.** For every $x \in A_s$ we have $[x, e] = 0$ [U 61, Lemma 1]. On the other hand $[x, a] = 0$ because $a$ is a central element. As $e, a$ are linearly independent, we deduce that $x \in \text{Lin}\{e, a\}$ [Elm 83, Lemme 1.1]. So the underlying space of the subalgebra $B = \mathbb{R}e \oplus A_s$ coincides with $\text{Lin}\{e, a\}$. Therefore $B$ is a 2-dimensional commutative algebra, isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$. Now [Elm 88, Lemma 1.2] concludes. \hfill \Box

### 3. The main result

Let $A$ be an absolute-valued algebra with an involution.

We need the following preliminary result:

**Lemma 1.** Let $a \in A$. The following two statements are equivalents:

1. $a$ is left-invertible.
2. $\sigma(a)$ is right-invertible.

**Proof.** For every $x, y \in A$ we have $\sigma(xy) = \sigma(y)\sigma(x)$ which can be expressed by the following equality:

\begin{equation}
\sigma \circ L_x = R_{\sigma(x)} \circ \sigma.
\end{equation}

Taking into account that $\sigma^2 = I_A$ the equality (9) gives $R_{\sigma(x)} = \sigma \circ L_x \circ \sigma$ and shows the implication (1) $\Rightarrow$ (2). Also equality (9) gives $L_x = \sigma \circ R_{\sigma(x)} \circ \sigma$ and shows the implication (2) $\Rightarrow$ (1). \hfill \Box

**Theorem 1.** Let $A$ be an absolute-valued algebra with involution $\sigma$. The following seven statements are equivalents:

1. $L_a$ is bijective for some $a \in A$.
2. $R_b$ is bijective for some $b \in A$.
3. $A$ is left-division algebra,
4. $A$ is right-division algebra,
5. $A$ is division algebra,
6. $A$ is finite-dimensional,
7. The subalgebra $\mathbb{R}e \oplus A_s$ of $A$ is a finite-dimensional.

**Proof.** Since $\sigma$ is surjective, the equivalences (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) follow from Lemma 1. The implications (6) $\Rightarrow$ (3) $\Rightarrow$ (1), (6) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) are clear. The implication (5) $\Rightarrow$ (6) follows from [W 53]. The implication (7) $\Rightarrow$ (6) is proved in [Elm 88]. Finally, the implication (1) $\Rightarrow$ (6) is an immediate consequence of the equivalence (1) $\Leftrightarrow$ (2) and [Rod 04, Theorem 2.2]. \hfill \Box
Remark 1. The implications (1) ⇒ (3), (2) ⇒ (4) in Theorem 1 does not require that \( \mathcal{A} \) be an absolute-valued algebra [CR 14, Proposition 2.7.19].

A non-trivial identity in a real algebra is an identity which is not satisfied in arbitrary real algebras.

Taking into account [Rod 92, 04], [Cu 92] and the existence of infinite-dimensional absolute-valued algebras satisfying \((x^2, y, x^2) = 0\) [EE 04], [CRR 12, Prop. 4.6], it is relevant to know if there are infinite-dimensional absolute-valued algebras which satisfy some non-trivial identities, namely \((x^2, y, x^2) = 0\), and having left invertible elements.

References

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Received: October 21, 2020; Published: November 30, 2020