Some Classes of Four-Dimensional Real Division Algebras with Few Automorphisms

Oumar Diankha
Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
Université Cheikh Anta Diop
5005 Dakar, Senegal

Amar Fall
Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
Université Cheikh Anta Diop
5005 Dakar, Senegal

Oussama Fayz
Département de Mathématiques et Informatique
Faculté des Sciences Ben M’Sik
Université Hassan II
7955 Casablanca, Morocco

Abdellatif Rochdi
Département de Mathématiques et Informatique
Faculté des Sciences Ben M’Sik
Université Hassan II
7955 Casablanca, Morocco

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Abstract

We give a variety of classes of 4-dimensional real division algebras with few automorphisms. We specify the group of automorphisms of these algebras. Also, the
group of automorphisms of the isotopes of algebra \( \mathbb{H} \) is discussed.

**Mathematics Subject Classification:** 17A35, 17A36

**Keywords:** Real division algebra, automorphisms, isotopy

1. **Introduction**

Four-dimensional real division algebras have been extensively studied [Di 35], [Br 44], [BO 81], [AHK 92, 94, 98, 00], [R 99], [F 09], [TV 17]. The group of automorphisms of some particular classes of algebras has been specified [BO 81, Remark], [AHK 92, 94, 98, 00], [F 09], [DDR 16]. However, the determination of groups of automorphisms of all known four-dimensional real division algebras remains still an open problem. It is the case for algebras considered in [AHK 94], [TV 17].

In this paper we are interested in the computation of the group of automorphisms of certain 4-dimensional real division algebras, namely groups with few elements.

We introduce some definitions and notations in Section 2.

We treat in Section 3 the case of algebras considered in [AHK 94]. We show that their groups of automorphisms contains only the identity and a reflection (Theorem 2). The same happen in Section 4 for those algebras cited in [TV 17] (Theorem 4).

The Section 5 is reserved to the isotopes of the quaternion algebra \( \mathbb{H} \). Let \( f, g \) be two linear bijections from space \( \mathbb{H} \) onto itself fixing \( 1 \in \mathbb{H} \). We show that the group \( \text{Aut}(\mathbb{H}_{f,g}) \) contains a reflection if and only if there exists a non-trivial automorphism of \( \mathbb{H}_{f,g} \) fixing 1 (Proposition 2). We deduce that for every linear bijection \( f \) from space \( \mathbb{H} \) onto itself fixing 1, the group \( \text{Aut}(\mathbb{H}_{f,f}) \) is either trivial or contains a reflection (Corollary 1).

Among the known four-dimensional real division algebras \( \mathcal{A} \), the specified groups \( \text{Aut}(\mathcal{A}) \) are either trivial or contain a reflection. It may be conjectured that every non-trivial group of automorphisms of a four-dimensional real division algebra contains a reflection.

2. **Definitions and preliminary results**

All the vector spaces considered in this paper are finite-dimensional over the field \( \mathbb{R} \) of real numbers. Let \( \mathcal{A} \) be an arbitrary non-associative algebra and let \( L_a, R_a : \mathcal{A} \to \mathcal{A} \) be the linear operators of left and right multiplication defined, respectively, by \( L_a(x) = ax, \ R_a(x) = xa \) for all \( x \in \mathcal{A} \). If \( L_a, R_a \) are bijective for every non-zero element \( a \) in \( \mathcal{A} \), then \( \mathcal{A} \) is said to be a real division algebra. As usual, \( I_\mathcal{A} \) and \( \text{Aut}(\mathcal{A}) \) stand for the identity operator of \( \mathcal{A} \) and the group of automorphisms of \( \mathcal{A} \). Also, we denote by \( \text{Lin}\{a_1, \ldots, a_n\} \) the lineal hull spanned by \( a_1, \ldots, a_n \in \mathcal{A} \).
Let $\Phi \in Aut(A)$. We denote by $E_\lambda(\Phi)$ an eigenspace of $\Phi$ associated with an eigenvalue $\lambda$. $\Phi$ is called a reflection of $A$ if $\Phi \neq I_A$ and $\Phi^2 = I_A$. If, in addition, $A$ is a division algebra then the spectrum of $\Phi$ coincides with $\{1, -1\}$ and then $A = E_1(\Phi) \oplus E_{-1}(\Phi)$.

Let $A$ be an algebra and let $f, g : A \to A$ be linear bijections. We denote by $A_{f,g}$ the algebra obtained by endowing the space $A$ with the product $x \circledast y = f(x)g(y)$. $A_{f,g}$ is said to be an isotope of $A$. It is clear that $A$ is a division algebra if and only if $A_{f,g}$ is a division algebra. The notion of isotopy was introduced first by Albert [A 42, p. 696].

3. One example from fused algebras

It was built in [AHK 94], through an appropriate deformation of the Cayley-Dickson process, four-dimensional real division algebras by merging two real division algebras of dimension two which satisfy some conditions. Also, the mapping $(x, y) \mapsto (x, -y)$ remains a reflection.

We consider the algebras $F_\alpha$ ($\alpha \in \mathbb{R}$) whose multiplication with respect to the basis $e_1, e_2, e_3, e_4$ is given by the table below

\[
\begin{array}{cccc}
  e_1 & e_2 & e_3 & e_4 \\
  e_1 & e_1 & \alpha e_2 & e_3 & \alpha e_4 \\
  e_2 & -e_2 & e_1 & -e_4 & e_3 \\
  e_3 & e_3 & -e_4 & -e_1 & e_2 \\
  e_4 & -e_4 & -e_3 & -e_4 & e_2 & e_1 + e_2 \\
\end{array}
\]

$F_\alpha$ is a fusion of algebras $A_\alpha$, $B$ whose multiplication tables are respectively given by

\[
\begin{array}{cc}
  e_1 & e_2 \\
  e_1 & e_1 & \alpha e_2 \\
  e_2 & -e_2 & e_1 \\
\end{array}
\quad
\begin{array}{cc}
  e_1 & e_2 \\
  e_1 & e_1 & -v \\
  e_2 & -e_2 & -e_1 - e_2 \\
\end{array}
\]

**Theorem 1.** $F_\alpha$ is a real division algebra if and only if $\alpha > 0$.

**Proof.** According to [AK 83, Th. 3] $B$ is a division algebra and $A_\alpha$ is a division algebra if and only if $\alpha > 0$. Now, $F_\alpha$ is a division algebra if and only if $\alpha > 0$ by [AHK 94, Th. 3]. \(\square\)

In what follows we assume that $\alpha > 0$.

**Lemma 1.** The only solutions of equations $R_x^2 = I_{F_\alpha}$, $x^2 = -e_1$ are $x = \pm e_1$ and $\pm e_3$, respectively.

**Proof.** Let $x = ae_1 + be_2 + ce_3 + de_4$ be arbitrary in $F_\alpha$. Then the matrices of operators $R_x, R_x^2$ with respect to the basis $e_1, e_2, e_3, e_4$ are respectively given
by
\[
\begin{pmatrix}
  a & b & -c & d \\
  b\alpha & -a & d & c + d \\
  c & d & a & -b \\
  d\alpha & -c & -b & -a - b
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
  \Delta \\
  (\alpha + 1)cd + \alpha d^2 \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  * \\
  *
\end{pmatrix}
\]
with \(\Delta = a^2 + \alpha b^2 - c^2 + \alpha d^2\). Now it is easy to see that the only solutions of equation \(R_x^2 = I_{F_\alpha}\) are \(x = \pm e_1\). The rest can be verified by a simple calculation.

\[\square\]

**Lemma 2.** Let \(\Phi \in \text{Aut}(F_\alpha)\). Then \(\Phi(e_1) = e_1\) and \(\Phi(e_3) = \pm e_3\). In particular, \(\Phi\) has a nonempty spectrum.

**Proof.** The result follows by taking into account Lemma 3 and the fact that \(-e_1\) is not an idempotent. \(\square\)

**Theorem 2.** \(\text{Aut}(F_\alpha) = \mathbb{Z}_2\).

**Proof.** Assume that \(\text{Aut}(F_\alpha)\) contains \(\Phi \neq I_{F_\alpha}\). We have \(\Phi(e_1) = e_1\), \(\Phi(e_3) = \varepsilon e_3\), with \(\varepsilon = \pm 1\). On the other hand, it is clear that \(\Phi(e_2)\) belongs to the eigenspace \(E_\alpha(L_{e_1}) = \mathbb{R}e_2 + \mathbb{R}e_4\) and we have \(\Phi(e_2)^2 = \Phi(e_2^2) = \Phi(e_1) = e_1\). So \(\Phi(e_2)\) belongs to \(\mathbb{R}e_2\) and equals to \(\varepsilon' e_2\) for some \(\varepsilon' = \pm 1\). Finally, \(\Phi(e_4) = \Phi(-e_2 e_3) = -\varepsilon\varepsilon' e_2 e_3 = \varepsilon\varepsilon' e_4\) and then \(\Phi^2 = Id_{F_\alpha}\). On the other hand, \(\Phi(e_2)\) cannot be equal to \(-e_2\). Otherwise, we would have
\[
e_1 - e_2 = \Phi(e_1 + e_2) \\
= \Phi(e_4^2) \\
= (\Phi(e_4))^2 \\
= e_4^2
\]
which is absurd. Thus there is only one reflection. \(\square\)

**4. An example from [TV 17]**

In [TV 17] the authors considered the four-dimensional real algebra \(A_\alpha\) \((\alpha \in \mathbb{R})\) whose multiplication with respect to a basis \(e_1, e_2, e_3, e_4\) is given by:

<table>
<thead>
<tr>
<th></th>
<th>(e_1)</th>
<th>(e_2)</th>
<th>(e_3)</th>
<th>(e_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1)</td>
<td>(\alpha e_1)</td>
<td>(\alpha e_2)</td>
<td>(\alpha e_3)</td>
<td>(\alpha e_4)</td>
</tr>
<tr>
<td>(e_2)</td>
<td>(-\alpha e_2)</td>
<td>(\alpha e_1)</td>
<td>(\alpha e_4)</td>
<td>(-\alpha e_3)</td>
</tr>
<tr>
<td>(e_3)</td>
<td>(-\alpha e_3)</td>
<td>(-\alpha e_4)</td>
<td>(\alpha e_1)</td>
<td>(\alpha e_2)</td>
</tr>
<tr>
<td>(e_4)</td>
<td>(-e_3 - e_4)</td>
<td>(e_3 - e_4)</td>
<td>(e_1 - e_2)</td>
<td>(e_1 + e_2)</td>
</tr>
</tbody>
</table>
We need the following preliminary result which is easy to check.

**Lemma 3.** \( E_{-\alpha}(R_{e_1}) = \mathbb{R}e_2 + \mathbb{R}e_3. \)

The following two results constitute an improvement of [TV 17]:

**Theorem 3.** \( A_\alpha \) is a division algebra if and only if \( \alpha \neq 0. \)

*Proof.* Assume that \( \alpha \neq 0 \) and let \( \varphi \) be a linear bijection from \( \mathbb{H} \) onto itself whose matrix with respect to the canonical basis \( 1, i, j, k \) is given by

\[
\begin{pmatrix}
\alpha^2 & -\alpha^2 & 0 & 0 \\
-\alpha^2 & \alpha & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & -\alpha
\end{pmatrix}
\]

The multiplication of the division algebra \( \mathbb{H}_{\varphi,i,j,k} \), which is an isotope of \( \mathbb{H} \), with respect to the basis \( e_1 = \alpha^{-1}1, e_2 = \alpha^{-1}i, e_3 = \alpha^{-1}j, e_4 = -\alpha^{-1}k \) is the same as Table \( (T_\alpha) \). So \( A_\alpha \) is a division algebra. \( \square \)

**Theorem 4.** If \( \alpha \neq 0, -1 \) then \( \text{Aut}(A_\alpha) = \mathbb{Z}_2. \)

*Proof.* Let \( \Phi \) be in \( \text{Aut}(A_\alpha) \). We have \( \Phi(\alpha^{-1}e_1) = \alpha^{-1}e_1 \) because \( \alpha^{-1}e_1 \) is a left-unit. So \( \Phi(e_1) = e_1 \). Now, as \( e_2, e_3 \in E_{-\alpha}(R_{e_1}) = \mathbb{R}e_2 + \mathbb{R}e_3 \) and \( \Phi(e_1) = e_1 \) we have \( \Phi(e_2), \Phi(e_3) \in \mathbb{R}e_2 + \mathbb{R}e_3 \). Write \( \Phi(e_2) = ae_2 + be_3, \Phi(e_3) = d'e_2 + b'e_3 \). The equalities \( \Phi(e_2)^2 = \Phi(e_3)^2 = \Phi(\alpha e_1) \) imply \( a^2 + b^2 = a'^2 + b'^2 = 1 \) since \( \alpha \neq 0 \).

Now, write \( \Phi(e_4) = a''e_1 + b''e_2 + c''e_3 + d''e_4 \). As the matrix of \( \Phi \) with respect to the basis \( e_1, e_2, e_3, e_4 \) is invertible we have \( d'' \neq 0 \). The equality \( \Phi(e_4)e_1 = \Phi(e_4)e_1 = \Phi(-e_3 - e_4) \) gives \( a'' = 0 \) since \( \alpha \neq -1 \). The equality \( \Phi(e_4)^2 = \Phi(e_1 + e_2) \) then gives \( b'' = 0 \), since \( d'' \neq 0 \), and, consequently, \( b = 0 \) and \( a \neq 0 \). Finally, \( \alpha \Phi(e_4) = \Phi(e_2e_3) = ae_2(a'e_2 + b'e_3) = aa'e_1 + ab'e_4 \). So \( \Phi(e_4) = aa'e_1 + ab'e_4 \) and then \( aa' = 0 \) that is \( a' = 0 \).

We obtain the matrix of \( \Phi \) with respect to the basis \( e_1, e_2, e_3, e_4 \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & b' & 0 \\
0 & 0 & 0 & ab'
\end{pmatrix}
\]

for some real numbers \( a, b' \) with \( a^2 = b'^2 = 1 \).

If \( a = -1 \), we consider the following two subcases.

1. If \( b' = 1 \) then \( E_1(\Phi) = \mathbb{R}e_1 + \mathbb{R}e_3 \) and we have \( e_4 \in E_{-1}(\Phi) \). So \( e_4^2 \in E_1(\Phi) \), a contradiction.

2. If \( b' = -1 \) then \( E_1(\Phi) = \mathbb{R}e_1 + \mathbb{R}e_4 \) and we have \( e_4^2 \in E_1(\Phi) \), which is impossible.

Consequently, \( a = 1 \) and then \( \Phi \) is either the identity \( \text{Id}_{A_\alpha} \) or the reflection \( \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \mapsto \lambda_1 e_1 + \lambda_2 e_2 - \lambda_3 e_3 - \lambda_4 e_4. \) \( \square \)
5. Automorphisms of some algebras $\mathbb{H}_{f,g}$

Let $f, g : \mathbb{H} \rightarrow \mathbb{H}$ be linear bijections such that $f(1) = g(1) = 1$ and let $\Phi \in \text{Aut}(\mathbb{H}_{f,g})$. Note that the spectrum of $\Phi$ is contained in $\{1, -1\}$. Let $1, i, j, k$ be the canonical basis of space $\mathbb{H}$.

We have the following preliminary results:

**Lemma 4.** If $\Phi(1) = 1$ then $\Phi$ is an isometry of the euclidian space $\mathbb{H}$ and commutes with both $f, g$. Consequently $E_1(\Phi)$ is a subalgebra of $\mathbb{H}_{f,g}$ invariant under both $f, g$.

**Proof.** $\Phi$ is an automorphism of algebra $\mathbb{H}$ and commutes with both $f$ and $g$ [CKMMRR 11, Lemma 6.2]. In particular $\Phi$ is an isometry of the euclidian space $\mathbb{H}$ [HKR 91, p. 215]. Now, the subalgebra $E_1(\Phi)$ of $\mathbb{H}$ is invariant under both $f, g$; therefore it is a subalgebra of $\mathbb{H}_{f,g}$. \hfill $\square$

The group $\text{Aut}(\mathbb{H}_{f,g})$ can be trivial as shows the following:

**Proposition 1.** If $f$ is diagonalizable whose matrix with respect to the basis $1, i, i + j, i + k$ is $\text{diag}\{1, 2, 3, 4\}$ then $\text{Aut}(\mathbb{H}_{f,f}) = \{I_{\mathbb{H}}\}$.

**Proof.** Note that 1 is the only non-zero central idempotent of algebra $\mathbb{H}_{f,f}$ fixed by any automorphism of $\mathbb{H}_{f,f}$. Let $\Phi$ be in $\text{Aut}(\mathbb{H}_{f,f})$ then $\Phi$ commutes with $f$. As the spectrum of $f$ is consisting of four distinct numbers, the automorphism $\Phi$ is diagonalizable with the same basis $1, i, i + j, i + k$ consisting of eigenvectors corresponding, respectively, to the eigenvalues $1, \varepsilon_1, \varepsilon_2, \varepsilon_3$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$. Now, assume that $\Phi \neq I_{\mathbb{H}}$ then $\Phi$ is a reflection and we have $\mathbb{H} = E_1(\Phi) \oplus E_{-1}(\Phi)$. As $\Phi(1) = 1$, one of three eigenvectors $i, i + j, i + k$ belongs to $E_1(\Phi)$ and the other two in $E_{-1}(\Phi)$. We distinguish the following three cases:

1. If $i \in E_1(\Phi)$ then $i + j, i + k \in E_{-1}(\Phi)$ which gives $(i + j)(i + k) \in E_1(\Phi) = \text{Lin}\{1, i\}$ absurd.
2. If $i + j \in E_1(\Phi)$ then $i, i + k \in E_{-1}(\Phi)$ which gives $i(i + k) \in E_1(\Phi) = \text{Lin}\{1, i + j\}$ absurd.
3. If $i + k \in E_1(\Phi)$ then $i, i + j \in E_{-1}(\Phi)$ which gives $i(i + j) \in E_1(\Phi) = \text{Lin}\{1, i + k\}$ absurd. \hfill $\square$

**Lemma 5.** If $\Phi \neq I_{\mathbb{H}}$ then the subalgebra $E_1(\Phi)$ has dimension two and its orthogonal space $E_1(\Phi)^\perp$ is in turn invariant under both $f, g$.

**Proof.** According to the above notations $E_1(\Phi)$ has dimension $\leq 2$ because $\Phi \neq I_{\mathbb{H}}$. On the other hand, there is norm-one $a \in \mathbb{H} \setminus \{1, -1\}$ such that $\Phi = L_a \circ R_\pi$ [HKR 91, p. 215] and it is easily seen that $E_1(\Phi)$ contains $\mathbb{R} + \mathbb{R}a$ and is equal to it. In fact there is a norm-one $u \in \text{Im}(\mathbb{H})$ such that $\mathbb{R} + \mathbb{R}a = \mathbb{R} + \mathbb{R}u$.

On the other hand, $\Phi$ is an isometry [HKR 91, p. 215] and leaves invariant $E_1(\Phi)^\perp$. Let $v$ be norm-one in $E_1(\Phi)^\perp$. Then $\{v, uv\}$ is an orthonormal basis of $E_1(\Phi)^\perp$ can be extended to an orthonormal basis $\mathcal{E} = \{1, u, v, uv\}$ of $\mathbb{H}$.
Now $\Phi(v)$ is written $\alpha v + \beta uv$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$. Hence $\Phi(uv) = \Phi(u)\Phi(v) = u(\alpha v + \beta uv) = -\beta v + \alpha uv$ and the restriction of $\Phi$ to $E_1(\Phi) \perp$ is a rotation. The matrix $M_\Phi$ of $\Phi$ with respect to the basis $E$ is then given by

$$M_\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$  

The matrix $M_f$ of $f$ with respect to the basis $E$ is given by:

$$M_f = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$$

for some scalars $a_{12}, a_{13}, a_{14}, a_{22}, a_{23}, a_{24}, a_{33}, a_{34}, a_{43}, a_{44}$.

If $\beta = 0$ then $\alpha = \pm 1$ and we easily see that $E_1(\Phi) \perp$ (which is equal to $E_{-1}(\Phi)$ if $\alpha = -1$) is invariant under $f$. So we can assume that $\beta \neq 0$. Note that $\Phi, f$ commute if and only if their associated matrices commute. We have that

$$M_\Phi M_f = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ \alpha a_{33} - \beta a_{43} & \alpha a_{34} - \beta a_{44} \\ \beta a_{33} + \alpha a_{43} & \beta a_{34} + \alpha a_{44} \end{pmatrix},$$  

$$M_f M_\Phi = \begin{pmatrix} 1 & a_{12} & a_{13} + \beta a_{14} & -\beta a_{13} + \alpha a_{14} \\ 0 & a_{22} & a_{23} + \beta a_{24} & -\beta a_{23} + \alpha a_{24} \\ \alpha a_{33} + \beta a_{34} & -\beta a_{33} + \alpha a_{34} \\ \alpha a_{43} + \beta a_{44} & -\beta a_{43} + \alpha a_{44} \end{pmatrix}.$$  

Now

$$M_\Phi M_f = M_f M_\Phi \Rightarrow \begin{cases} (\alpha - 1)a_{13} + \beta a_{14} = 0 \\ -\beta a_{13} + (\alpha - 1)a_{14} = 0 \\ (\alpha - 1)a_{23} + \beta a_{24} = 0 \\ -\beta a_{23} + (\alpha - 1)a_{24} = 0 \end{cases}$$

Two systems consisting of equations (1), (2) and equations (3), (4) have the same discriminant $\Delta$ given by

$$\Delta = \begin{vmatrix} \alpha - 1 & \beta \\ -\beta & \alpha - 1 \end{vmatrix} = (\alpha - 1)^2 + \beta^2 \neq 0.$$  

So $a_{13} = a_{14} = a_{23} = a_{24} = 0$ and $f$ leaves invariant $E_1(\Phi) \perp$. The same goes for $g$. \hfill \square

We now state the main result of this section:
Proposition 2. The following affirmations are equivalent

(1) There is a non-trivial automorphism $\Phi$ of $\mathbb{H}_{f,g}$ fixing $1$.

(2) $E_1(\Phi)$ is a 2-dimensional subalgebra of $\mathbb{H}_{f,g}$ and its orthogonal space $E_1(\Phi)^\perp$ is invariant, as well as $E_1(\Phi)$, under both $f,g$.

(3) $\text{Aut}(\mathbb{H}_{f,g})$ contains a reflection fixing $1$.

(4) $-1$ belongs in the spectrum of some automorphism of $\mathbb{H}_{f,g}$ fixing $1$.

Proof. Denote by $\odot$ the product of the algebra $\mathbb{H}_{f,g}$. The implication $(1) \Rightarrow (2)$ is given by Lemmas 4, 5.

$(2) \Rightarrow (3)$ Let $E_1(\Phi) := \mathcal{B}$. Clearly $\mathcal{B}$ is a subalgebra of $\mathbb{H}$ and we have $\mathcal{B}^\perp \mathcal{B}^\perp \subseteq \mathcal{B}$, $\mathcal{B}\mathcal{B}^\perp = \mathcal{B}^\perp \mathcal{B} = \mathcal{B}^\perp$ [CKMRR 11, Lemma 6.3]. As $\mathcal{B}^\perp$ is invariant under both $f,g$ we have $\mathcal{B} \odot \mathcal{B}^\perp = \mathcal{B}^\perp \odot \mathcal{B} = \mathcal{B}^\perp$ and $\mathcal{B}^\perp \odot \mathcal{B}^\perp \subseteq \mathcal{B}$. So the mapping $\mathbb{H} = \mathcal{B} \oplus \mathcal{B}^\perp \rightarrow \mathbb{H} \quad x + y \mapsto x - y$ is a reflection of algebra $\mathbb{H}_{f,g}$ fixing $1$.

The implications $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$ are clear. □

Corollary 1. The following affirmations are equivalent:

(1) $\text{Aut}(\mathbb{H}_{f,f})$ is not trivial.

(2) $E_1(\Phi)$ is a 2-dimensional subalgebra of $\mathbb{H}_{f,f}$ containing $1$ and both $E_1(\Phi), E_1(\Phi)^\perp$ are invariants under $f$.

(3) $\text{Aut}(\mathbb{H}_{f,f})$ contains a reflection.

(4) $-1$ belongs in the spectrum of some automorphism of $\mathbb{H}_{f,f}$.

Proof. The result follows from Proposition 2 taking into account that $1 \in \mathbb{H}$ is the only non-zero central idempotent of $\mathbb{H}_{f,f}$ fixed by any automorphism of algebra $\mathbb{H}_{f,f}$. □

Remark 1. All known four-dimensional real division algebras whose group of automorphisms is not trivial admit a reflection. It may be conjectured that for every four-dimensional real division algebra $\mathcal{A}$, the group $\text{Aut}(\mathcal{A})$ is either trivial or contains a reflection.

References


Four-dimensional real division algebras with few automorphisms


Received: December 1, 2020; Published: December 19, 2020