A Construction of Semigroups
Whose Elements Are Middle Units

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Abstract
An element $a$ of a semigroup $S$ is said to be a middle unit of $S$ if $xay = xy$ is satisfied for every $x, y \in S$. In this paper we show how to construct semigroups in which every element is a middle unit.

Mathematics Subject Classification: 20M10

Keywords: semigroup; left equalizer simple semigroup; middle unit of a semigroup; rectangular band

1 Introduction and motivation

In [6], P.M. Cohn gave necessary and sufficient conditions for a semigroup to be embeddable into a left simple semigroup. The conditions differ essentially according to whether or not the semigroup contains an idempotent element (i.e. an element $e$ satisfying $e^2 = e$). In both cases a necessary condition is the following: for every $a, b \in S$, if $xa = xb$ is satisfied for some $x \in S$, then $sa = sb$ is satisfied for all $s \in S$. By [11, Definition 2], a semigroup $S$ satisfying this condition is called a left equalizer simple semigroup. In [11], a complete description of left equalizer simple semigroups is given. It is proved in [11, Theorem 2.2] that a semigroup is left equalizer simple if and only if it is isomorphic to a semigroup defined in [11, Construction 1].

$^1$This work was supported by the National Research, Development and Innovation Office NKFIH, 115288.
An element $a$ of a semigroup $S$ is called a middle unit of $S$ if $xay = xy$ is satisfied for all $x, y \in S$. Semigroups with middle units are examined in many papers (see, for example, [1], [2], [3], [8]). In our present paper we concentrate our attention on semigroups in which every element is a middle unit. The reason for this attention is that these semigroups are left equalizer simple. This fact motivates us to try to construct them. We give a special case of [11, Construction 1], and prove that a semigroup $S$ has the property that every element of $S$ is a middle unit of $S$ if and only if $S$ is isomorphic to a semigroup defined in this construction.

A semigroup in which every element is an idempotent element is called a band. By [7, Proposition 3.2], every semigroup satisfying the identity $axa = a$ is a band; a semigroup with this property is called a rectangular band. It is known (see, for example, [12, II.1.3. Lemma]) that a band $B$ is a rectangular band if and only if every element of $B$ is a middle unit of $B$. We prove that the rectangular bands are exactly the left equalizer simple bands. Moreover, we show how to obtain them by using a special case of [11, Construction 1].

2 Preliminaries

By a semigroup we shall mean a multiplicative semigroup, that is, a non-empty set together with an associative multiplication. Let $S$ be a semigroup and $G^0$ be a semigroup arising from a one-element group $G = \{1\}$ by adjunction of a zero element 0. By an $S \times S$ matrix over $G^0$ we mean a mapping of $S \times S$ into $G^0$. Let $A$ be an $S \times S$ matrix over $G^0$. For an element $s \in S$, the set \{A(s, x) : x \in S\} is called the $s$−row of $A$. An $S \times S$ matrix $A$ over $G^0$ is called strictly row-monomial if each row of $A$ contains exactly one non-zero element of $G^0$. It is clear that, for every element $s \in S$, the $S \times S$ matrix $R(s)$ over $G^0$ defined by

$$R(s) : (a, b) \mapsto \begin{cases} 1, & \text{if } as = b \\ 0, & \text{otherwise} \end{cases}$$

is strictly row-monomial, and $s \mapsto R(s)$ ($s \in S$) is a representation of the semigroup $S$ by strictly row-monomial $S \times S$ matrices over $G^0$ (see [4, Exercise 4(b) for §35]). This representation is also called the right regular (matrix) representation of $S$.

Let $\theta$ denote the kernel of the homomorphism $s \mapsto R(s)$ ($s \in S$). It is clear that

$$\theta = \{(a, b) \in S \times S : xa = xb \text{ for all } x \in S\}.$$ 

A non-empty subset $L$ of a semigroup $S$ is called a left ideal of $S$ if $sa \in L$ for every $s \in S$ and $a \in L$. A semigroup is called a left simple semigroup if it does not properly contain any left ideal. It is known (see [4, pp. 6]) that a semigroup $S$ is left simple if and only if $Sa = S$ for every $a \in S$. 

A construction of semigroups whose elements are middle units

The investigation of left equalizer simple semigroups is based on the result of [11, Theorem 2.1]: a semigroup $S$ is left equalizer simple if and only if the factor semigroup $S/\theta$ is left cancellative (a semigroup $A$ is left cancellative if $xa = xb$ implies $a = b$ for every $x, a, b \in A$). Starting from a left cancellative semigroup, a special type of semigroups are defined in [11, Construction 1], and it is proved in [11, Theorem 2.2] that a semigroup is left equalizer simple if and only if it is isomorphic to a semigroup defined in [11, Construction 1].

This construction also plays an important role in our present investigation. Thus we cite it here.

**Construction 1** ([11, Construction 1]) Let $T$ be a left cancellative semigroup. For each $t \in T$, associate a nonempty set $S_t$ such that $S_t \cap S_r = \emptyset$ for every $t, r \in T$ with $t \neq r$. As $T$ is left cancellative, $x \mapsto tx$ is an injective mapping of $T$ onto $tT$. For arbitrary couple $(t, r) \in T \times T$ with $r \in tT$, let $\varphi_{t,r}$ be a mapping of $S_t$ into $S_r$. For all $t \in T$, $r \in tT$, $q \in rT \subseteq tT$ and $a \in S_t$, assume

\[(a) \varphi_{t,r} \circ \varphi_{r,q} = (a)\varphi_{t,q}. \quad (1)\]

On the set $S = \bigcup_{t \in T} S_t$ define an operation $\star$ as follows: for arbitrary $a \in S_t$ and $b \in S_x$, let

\[a \star b = (a)\varphi_{t,tx}. \quad (2)\]

If $a \in S_t$, $b \in S_x$, $c \in S_y$ are arbitrary elements then

\[a \star (b \star c) = a \star (b)\varphi_{x,xy} = (a)\varphi_{t,t(xy)} =\]

\[= (a) \varphi_{t,tx} \circ \varphi_{tx,t(xy)} = (a)\varphi_{t,tx} \star c = (a \star b) \star c.\]

Thus the operation $\star$ is associative, and so $S = \bigcup_{t \in T} S_t$ is a semigroup under the operation $\star$. This semigroup will be denoted by $(S; T, S_t, \varphi_{t,r}, \star)$.

**Proposition 2** ([11, Theorem 2.2]) A semigroup is left equalizer simple if and only if it is isomorphic to a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 1. \[\square\]

The following lemma will be used in our investigation several times.

**Lemma 3** In a semigroup $(S; T, S_t, \varphi_{t,r}, \star)$ defined in Construction 1, the $\theta$-classes are the sets $S_t$, and the factor semigroup $(S; T, S_t, \varphi_{t,r}, \star)/\theta$ is isomorphic to $T$.

**Proof.** As the semigroup $T$ is left cancellative, it is left reductive, that is, for arbitrary $a, b \in T$ if $xa = xb$ for all $x \in T$, then $a = b$. Thus, by [10, Theorem 1], the $\theta$-classes of $(S; T, S_t, \varphi_{t,r}, \star)$ are the sets $S_t$. As $S_t \star S_r \subseteq S_{tr}$ for every $t, r \in T$, the factor semigroup $(S; T, S_t, \varphi_{t,r}, \star)/\theta$ is isomorphic to $T$. \[\square\]

For notations and notions not defined in this paper, we refer to the books [4], [5], [9] and [12].
3 Semigroups whose elements are middle units

A semigroup satisfying the identity \(ab = a\) \([ab = b]\) is called a left [right] zero semigroup.

**Proposition 4** Every element of a semigroup \(S\) is a middle unit of \(S\) if and only if the factor semigroup \(S/\theta\) is a right zero semigroup.

**Proof.** The equation \(xay = xy\) is satisfied for all \(x, a, y \in S\) if and only if \((ay, y) \in \theta\) for all \(a, y \in S\), that is, the factor semigroup \(S/\theta\) is a right zero semigroup. \(\Box\)

It is clear that every right zero semigroup is left cancellative. Thus the semigroups \((S; T, S_t, \varphi_{t,r}, \ast)\) can be defined in that case when \(T\) is a right zero semigroup.

**Theorem 5** A semigroup has the property that its every element is a middle unit if and only if it is isomorphic to a semigroup \((S; T, S_t, \varphi_{t,r}, \ast)\) defined in Construction 1, where \(T\) is a right zero semigroup.

**Proof.** Let \((S; T, S_t, \varphi_{t,r}, \ast)\) be a semigroup defined in Construction 1, where \(T\) is a right zero semigroup. By Lemma 3, the factor semigroup \((S; T, S_t, \varphi_{t,r}, \ast)/\theta\) is isomorphic to \(T\). From Proposition 4 it follows that every element of the semigroup \((S; T, S_t, \varphi_{t,r}, \ast)\) is a middle unit.

Conversely, let \(S\) be a semigroup in which every element is a middle unit. By Proposition 4, \(S/\theta\) is a right zero semigroup. As a right zero semigroup is left cancellative, the semigroup \(S\) is left equalizer simple by [11, Theorem 2.1]. Then, by Proposition 2, \(S\) is isomorphic to a semigroup \((S; T, S_t, \varphi_{t,r}, \ast)\) defined in Construction 1. By Lemma 3, \(T\) is isomorphic to the factor semigroup \((S; T, S_t, \varphi_{t,r}, \ast)/\theta\). Thus \(T\) is a right zero semigroup. \(\Box\)

In the next example, we give a semigroup whose elements are middle units.

**Example 6** Let \(T\) be a right zero semigroup. Let \(S_t\) \((t \in T)\) be pairwise disjoint nonempty sets. Fix an element \(s_t^*\) in \(S_t\) for every \(t \in T\). For every \(t, r \in T\), let \(\varphi_{t,r}\) be the mapping of \(S_t\) into \(S_r = S_{tr}\) defined by \((a)\varphi_{t,r} = s_t^*\). It is easy to see that this system of mappings satisfies condition (1) of Construction 1. Let \(S = \cup_{t \in T} S_t\), and define an operation \(\ast\) on \(S\) as in (2) of Construction 1. Then the semigroup \((S; T, S_t, \varphi_{t,r}, \ast)\) defined by Construction 1 is a semigroup in which every element is a middle unit. For example, if \(T = \{x, y\}\), \(S_x = \{a, b\}\), \(s_x^* = a\), \(S_y = \{c\}\), \(s_y^* = c\), then the Cayley multiplicative table of \((S; T, S_t, \varphi_{t,r}, \ast)\) is the following:

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<td>(c)</td>
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</table>
It is easy to see that every element of \((S; T, S_t, \varphi_{t, r}, \ast)\) is a middle unit, indeed.

### 4 Left equalizer simple bands

**Lemma 7** ([12, II.1.3. Lemma, II.1.5. Lemma]) On an arbitrary semigroup \(S\), the following conditions are equivalent.

1. \(S\) is a rectangular band.
2. \(S\) is a band which satisfies the identity \(abc = ac\) (that is, every element of \(S\) is a middle unit).
3. \(S\) is a direct product of a left zero semigroup and a right zero semigroup.

**Theorem 8** A band is left equalizer simple if and only if it is a rectangular band.

**Proof.** Let \(S\) be a left equalizer simple band. Then, for every \(b, c \in S\), we have \(b(bc) = bc\) and so \(abc = ac\) is satisfied for all \(a \in S\). Thus \(S\) satisfies the identity \(abc = ac\). By Lemma 7, \(S\) is a rectangular band.

Conversely, assume that \(S\) is a rectangular band. Let \(a, b \in S\) be arbitrary elements. Assume that \(xa = xb\) is satisfied for some \(x \in S\). Then, for every \(s \in S\),

\[
\begin{align*}
sa &= sxa = sxb = sb.
\end{align*}
\]

Hence \(S\) is left equalizer simple. 

**Proposition 9** A band is embeddable into a left simple semigroup if and only if it is a left zero semigroup.

**Proof.** Assume that a band \(B\) is embeddable into a left simple semigroup \(S\). By Lemma 3, \(Se = S\) for every idempotent element \(e\) of \(S\). Thus the idempotent elements of \(S\) are right identity elements of \(S\) and so \(ef = e\) is satisfied for every \(e, f \in B\). Hence \(B\) is a left zero semigroup. As a left zero semigroup is left simple, the proposition is proved.

In the next we give a construction which is a special case of Construction 1.

**Construction 10** Let \(R\) be a right zero semigroup and \(L\) a non-empty set. For every \(e \in R\), let \(L_e\) denote a set such that there is a bijective mapping \(\tau_e\) of \(L\) onto \(L_e\), moreover \(L_e \cap L_f = \emptyset\) for all element \(e\) and \(f\) of \(R\) with \(e \neq f\). For every \(a \in L\) and \(e \in R\), let \(a_e\) denote the element \(\tau_e(a)\). For every couple
(e, f) ∈ R × R, let \( \varphi_{e,f} \) be the following mapping of \( L_e \) onto \( L_f \): for every \( a_e \in L_e \),
\[
(a_e)\varphi_{e,f} = a_f. \tag{3}
\]
It is easy to see that this system of mappings satisfies condition (1) of Construction 1. Let \( B = \bigcup_{e \in R} L_e \), and define an operation \( * \) on \( S \) as in (2) of Construction 1: for every \( a_e \in L_e \) and \( b_f \in L_f \), let
\[
a_e * b_f = (a_e)\varphi_{e,f} = a_f. \tag{4}
\]
By Construction 1, we can consider the semigroup \((B; R, L_e, \varphi_{e,f}, *)\).

**Theorem 11** A semigroup \((B; R; L_e, \varphi_{e,f}, *)\) defined in Construction 10 is a rectangular band. Moreover, every rectangular band is isomorphic to a semigroup defined in Construction 10.

**Proof.** Let \( a_e \in L_e \subseteq B \) be an arbitrary element. Then
\[
a_e * a_e = (a_e)\varphi_{e,e} = a_e.
\]
Thus \( B \) is a band. Let \( a_e, b_f \in B \) be arbitrary elements with \( a_e \in L_e \), \( b_f \in L_f \). Then
\[
a_e * b_f = (a_e)\varphi_{e,f} = a_f * a_e = a_e.
\]
Hence \((B; R; L_e, \varphi_{e,f}, *)\) is a rectangular band.

Conversely, let \( B \) be a rectangular band. By Lemma 7, \( B \) is a direct product of a left zero semigroup \( L \) and a right zero semigroup \( R \). Let
\[
L_e = L \times \{e\} = \{(a, e) : a \in L\} \quad (e \in R).
\]
It is clear that, for every \( e \in R \),
\[
\tau_e : a \mapsto (a, e) ; \quad a \in L
\]
is a bijection of \( L \) onto \( L_e \). Let the element \((a, e) \in L_e \) denoted by \( a_e \). For every \( e, f \in R \), let \( \varphi_{e,f} \) be a mapping of \( L_e \) into \( L_f \) defined by (3) of Construction 10: for every \( a_e \in L_e \),
\[
(a_e)\varphi_{e,f} = a_f.
\]
Consider the semigroup \((B; R, L_e, \varphi_{e,f}, *)\) in which the operation is defined by (4) of Construction 10: for every \( a_e \in L_e \) and \( b_f \in L_f \),
\[
a_e * b_f = (a_e)\varphi_{e,f} = a_f.
\]
As
\[
a_e * b_f = a_f = (a, f) = (a, e)(b, f) = a_e b_f
\]
for every \( e, f \in R \) and \( a_e \in L_e \), \( b_f \in L_f \), the semigroup \((B; R, L_e, \varphi_{e,f}, *)\) is isomorphic to the band \( B \). 
\( \square \)
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References


Received: January 14, 2020; Published: February 24, 2020