Classification of Exceptional Train Algebras of Rank 3 and Type (4, 2)

Roseli Arbach
São Paulo State University (UNESP)
School of Engineering
Ilha Solteira - SP, Brazil

Luis Antônio Fernandes de Oliveira
São Paulo State University (UNESP)
School of Engineering
Ilha Solteira - SP, Brazil

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2020 Hikari Ltd.

Abstract

In this paper, we classify the exceptional train algebras of rank 3 and dimension 6 of type (4, 2). For that, some auxiliary results are proved.

Mathematics Subject Classification: 17A30, 17C27

Keywords: Peirce decomposition, P-monomials, exceptional t-algebras

1 Introduction

From 1939 to 1951, I.M. Etherington introduced some new concepts in non-associative algebras, in order to construct models for the studying of Genetic of Populations. In particular, he considered the non-associative t-algebras in which were defined a nonzero homomorphism in the scalar field, satisfying a polynomial equation involving this homomorphism. The classical material about the t-algebras can be found in [5].
2 Preliminary Notes

Let $F$ be a field with $\text{char}(F) \neq 2$ and $A$ be an $F$-algebra, not necessarily associative. When $\omega : A \rightarrow F$ is a nonzero homomorphism, the ordered pair $(A, \omega)$ is called a baric algebra over $F$ and $\omega$ its weight function. For each $x \in A$, $\omega(x)$ is called the weight of $x$. Let $(A, \omega)$ be a commutative baric algebra. If there exist $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in F$ such that, for all $x$ in $A$,

$$x^n + \gamma_1 \omega(x)x^{n-1} + \ldots + \gamma_{n-1} \omega(x)^{n-1}x = 0 \tag{1}$$

we say that $(A, \omega)$ is a (commutative) train algebra (in short, $t$-algebra). Moreover, if there is no similar relation involving $x^{n-1}, \ldots, x$, we say that $n$ is the rank of $(A, \omega)$ and (1) is its train equation (in short, $t$-equation). In this case, it is easy to see that $1 + \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} = 0$. Moreover, $x^n = 0$ for all $x \in \text{Ker}(\omega)$. If $(A, \omega)$ is a commutative $t$-algebra of rank $n$, then its $t$-equation is unique. In this paper, we will consider only $t$-algebras of rank 3, that is, those satisfying an equation

$$x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)^2x = 0 \tag{2}$$

where $\gamma \in F$ is a fixed element. For details about $t$-algebras of rank 3, the reader is referred to [1], [3] and [6]. In this paper we assume that $2\gamma \neq 1$ and, as a consequence, there is (at least) an idempotent $e \in A$ and relative to this element, $A$ has a Peirce decomposition $A = Fe \oplus U_e \oplus V_e$ in which, denoting $\text{Ker}(\omega)$ by $N$,

$$U_e = \{u \in N : 2eu = u\} \tag{3}$$
$$V_e = \{v \in N : ev = \gamma v\} \tag{4}$$
$$N = U_e \oplus V_e \tag{5}$$

The decomposition $A = Fe \oplus U_e \oplus V_e$ depends on the choice of the idempotent in $A$, but it can be proved that the dimensions of $U_e$ and $V_e$ are invariants, see [1]. Then we can define the invariant type of $A$ as the ordered pair of integers $(1 + r, s)$, where $r = \text{dim}(U_e)$ and $s = \text{dim}(V_e)$. The subspaces $U_e$ and $V_e$ satisfy the following relations:

$$U_e^2 \subseteq V_e \quad ; \quad U_eV_e \subseteq U_e \quad ; \quad V_e^2 = 0; \tag{6}$$
$$U_e^{2n+1} \subseteq U_e \quad (n \geq 0) \quad ; \quad U_e^{2n} \subseteq V_e \quad (n \geq 1) \tag{7}$$

Moreover, by the second linearization of (2), for all elements $x, y, z \in N$, we obtain the Jacobi’s identity

$$J(x, y, z) = x(yz) + y(xz) + z(xy) = 0 \tag{8}$$
We denote by \( m(U_e, V_e) \) all subspaces of \( A \) that are obtained by multiplication, in a non associative way, of the subspaces \( U_e \) and \( V_e \). These subspaces will be called Peirce’s monomials (or \( P \)-monomials). As example, we have the \( P \)-monomials \( U_e, V_e, U^2_e, U_eV_e, U^3_e, (U_eV_e)V_e \). It is easy to see that each \( P \)-monomial is contained in \( U_e \) or in \( V_e \) (but not in both, unless it is the null space). Moreover as \( N = \ker(\omega) \) is nil of index 3 (as \( x^3 = 0 \) for all \( x \in N \)) it is also nilpotent so that every \( P \)-monomial of sufficiently high degree is zero. The basic purpose of this paper is to give the table of multiplications of the exceptional \( t \)-algebras of rank 3 and type (4, 2). In such algebras, \( U_e(U_eV_e) = (U_eV_e)V_e = U^4_e = U^3_eV_e = U_e[U_e(U_eV_e)] = U_e[(U_eV_e)V_e] = (U_eV_e)^2 = [(U_eV_e)V_e]V_e = 0 \), since if one of these subspaces is not null, then \( \dim A \geq 6 \) (for details, see [1]).

For the Theorem 4.1, we will use the following result. Consider \( A = Fe \oplus U_e \oplus V_e \) a \( t \)-algebra of rank 3. To each fixed \( \gamma \in F, 2\gamma \neq 1 \), there is a class of \( t \)-algebras of rank 3 that satisfy the equation \( x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)x = 0 \). If \( A \) satisfy the equation \( x^3 - \omega(x)x^2 = 0 \) (that is, if \( \gamma = 0 \)), it is possible to pass to other class in which \( \gamma \neq 0 \) if we define over the \( F \)-linear space \( A \) new multiplication as follows:

\[
x \circ y = (1 - 2\gamma)xy + \gamma[x\omega(y) + y\omega(x)]
\]

with \( \gamma \in F \) different from \( \frac{1}{2} \). For details, see [2].

Let \( A = Fe \oplus U_e \oplus V_e \) be a \( t \)-algebra of rank 3 and dimension 6. The possible types of \( A \) are: (5, 1), (4, 2), (3, 3), (2, 4) and (1, 5). The \( t \)-algebras of rank 3 and types (\( n, 1 \)), (3, \( n - 2 \)), (2, \( n - 1 \)) and (1, \( n \)) had already been classified (see [3] and [4]). Therefore, in order to complete the classification of the \( t \)-algebra of rank 3 and dimension 6, we have to analyse such algebras of type (4, 2). In this paper, we begin this classification.

Finally, a \( t \)-algebra \((A, \omega)\) of rank 3 satisfying (2) is called a exceptional \( t \)-algebra of rank 3 if \( U^2 = 0 \), for some Peirce decomposition of \( A \) and a normal \( t \)-algebra of rank 3 if \( UV = 0 \), for some Peirce decomposition of \( A \). For details about these algebras, see [2].

## 3 Auxiliary results

We begin with the statement of some results that we need to obtain the complete classification of the exceptional train algebras of rank 3 and dimension 6 of type (4, 2). For that, we consider \( A = Fe \oplus U_e \oplus V_e \) a \( t \)-algebra of rank 3 and dimension 6, of type (\( 1 + 3, 2 \)). A simple observation of the relations (6) and (7) shows that \( 0 \leq \dim(U^2) \leq 2, 0 \leq \dim(U^3) \leq 3 \) and \( 0 \leq \dim(UV) \leq 3 \). In the following lemmas, we ameliorate these valuations. In order to simplify
notation, we use $U$ and $V$ in place of $U_e$ and $V_e$, respectively. In the following, we use $< w_1, w_2, \ldots, w_n >$ to denote the linear subspace generated by the vectors $w_1, w_2, \ldots, w_n$ in some linear space.

**Lemma 3.1** In the conditions above fixed, we have $\dim(UV) < 3$ and $\dim(U^3) < 3$. Moreover, if $\dim(U^3) \geq 1$, then $U^2 = V$.

**Proof:** If $\dim(UV) = 3$, then $UV = U$ and $(UV)V = UV \neq 0$, and this implies $\dim(A) \geq 7$ (see Lemma 4.8, [1]). If $\dim(U^3) = 3$, then $A$ is not a exceptional $t$-algebra. Moreover, $U^3 = U$ and $U^4 = U^3U = U^2 \neq 0$, and this implies $\dim(A) \geq 11$ (see Lemma 4.9, [1]). Finally, $\dim(U^3) \geq 1$ implies that there are $u_1, u_2 \in U$ such that $u_1^2 u_2 \neq 0$. If $\alpha u_1^2 + \beta u_1 u_2 = 0$, then $\beta u_1 (u_1 u_2) = 0$ and so, by (8) (for $x = y = u_1, z = u_2$), $\beta u_1^2 u_2 = 0$, which implies $\beta = 0$ and $\alpha = 0$. That means that the set $\{u_1^2, u_1 u_2\}$ is free and $\dim(U^2) = 2$; that is, $U^2 = V$.

**Lemma 3.2** If $\dim(UV) = 2$, then $UV = < uv, w \bar{v} >$, $U = < u, uv, w \bar{v} >$ and $A$ is a exceptional $t$-algebra.

**Proof:** Suppose that $UV = < uv, w \bar{v} >$. If $\alpha u + \beta uv + \sigma w \bar{v} = 0$, by $(UV)V = 0$ we have $\alpha uv = 0$ and so $\alpha = 0$. But $\{uv, w \bar{v}\}$ is a free set and so $\beta = \sigma = 0$. It follows that $U = < u, uv, w \bar{v} >$ and so $\bar{v} = \alpha_1 u + \alpha_2 uv + \alpha_3 w \bar{v}$. Then, again by $(UV)V = 0$, it follows that $0 \neq \bar{v} = \alpha_1 u \bar{v}$ and so $UV = < uw, w \bar{v} >$ and $U = < u, uv, w \bar{v} >$, which implies $U^2 = < u^2 >$. We observe now that, as $\dim(UV) = 2$, the set $\{v, \bar{v}\}$ is free and so, by $u^2 \in V$, it follows $u^2 = \beta_1 v + \beta_2 \bar{v}$. Then $0 = \beta_1 uv + \beta_2 u \bar{v}$ and so we have $\beta_1 = \beta_2 = 0$, which implies $u^2 = 0$ and, consequently, $U^2 = 0$; that is, $A$ is a exceptional $t$-algebra.

**Lemma 3.3** In the conditions above fixed, we have:

(i) If $\dim(U^2) = 2$, then $U^3 = UV$;

(ii) If $\dim(U^2) = 2$, then $\dim(U^3) = \dim(UV) \leq 1$.

**Proof:** (i) $\dim(U^2) = 2 = \dim(V)$. Then $U^2 = V$ and so $U^3 = U^2 U = UV$, as we said.

(ii) By (i) and Lemma 3.1, $U^3 = UV$. Suppose $\dim(UV) = 2$; that is, there is a free set $\{u_1 v_1, u_2 v_2\} \subset UV$ such that $UV = U^3 = < u_1 v_1, u_2 v_2 >$. Consider the set $B = \{u_1, u_1 v_1, u_2 v_2\}$. If $\alpha u_1 + \beta u_1 v_1 + \sigma u_2 v_2 = 0$, then $\alpha u_1 v_1 = 0$ and so $\alpha = 0$, which implies $\beta = \sigma = 0$. In this way, $B$ is a basis for $U$ and so $u_2 = \lambda u_1 + \alpha_1 u_1 v_1 + \alpha_2 u_2 v_2$. It follows that $0 \neq u_2 v_2 = \lambda u_1 v_2$ and then $\lambda \neq 0$ and $\{u_1 v_1, u_1 v_2\}$ is a free set; that is, $UV = U^3 = < u_1 v_1, u_1 v_2 >$ and $U = < u_1, u_1 v_1, u_1 v_2 >$. Thus $U^2 = < u_1^2 >$, which is a contradiction, since $\dim(U^2) = 2$. Thus $\dim(UV) \leq 1$. 

* Roseli Arbach and Luis Antônio Fernandes de Oliveira
Lemma 3.4 If \( \dim(U^2) = 1 \) and \( \dim(U^3) = 0 \), then \( \dim(UV) \leq 1 \).

Proof: By Lemma 3.1, we have that \( \dim(UV) < 3 \). Suppose \( \dim(UV) = 2 \) and consider a free set \( \{ u_1v_1, u_2v_2 \} \subset UV \) such that \( UV = < u_1v_1, u_2v_2 > \). By \( \dim(U^2) = 1 \), it follows that \( U^2 = < u^2 > \), for \( u \in U \). We shall prove that \( V = < u^2, v_1 > \) and \( U = < u_1v_1, u_2v_2, u > \). In fact, if \( \alpha u^2 + \beta v_1 = 0 \), as \( u^3 = 0 \) then \( \beta u_1v_1 = 0 \) and so \( \alpha = \beta = 0 \), which implies \( V = < u^2, v_1 > \). Otherwise, by \( U(UV) = 0 \), \( \alpha_1u_1v_1 + \alpha_2u_2v_2 + \sigma u = 0 \) implies \( \sigma u^2 = 0 \) and so \( \sigma = 0 \) and \( \alpha_1 = \alpha_2 = 0 \). This means that \( U = < u_1v_1, u_2v_2, u > \). Consequently, \( UV = < u_1v_1 > \), and this is a contradiction. Hence, \( \{ u_1v_1, u_2v_2 \} \) is not a free set, which implies that \( \dim(UV) \leq 1 \).

Since \( 0 \leq \dim(U^2) \leq 2 \), \( 0 \leq \dim(U^3) \leq 3 \) and \( 0 \leq \dim(UV) \leq 3 \), there are \( 3 \times 4 = 48 \) possible combinations of these dimensions. The above Lemmas reduce these possibilities to seven cases, which are described in the table given in the following.

Proposition 3.5 Let \( A = Fe \oplus U \oplus V \) be a \( t \)-algebra of rank 3 and dimension 6, of type \((4, 2)\). The table below describes the possibilities for \( \dim(U^2) \), \( \dim(U^3) \) and \( \dim(UV) \):

<table>
<thead>
<tr>
<th>CASE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dim(U^2) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \dim(U^3) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \dim(UV) )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

4 The classification of the exceptional \( t \)-algebras
of rank 3 and type \((4, 2)\)

In the main result of this paper (Theorem 4.1), we will classify the first, second and third cases of the Table 1. After that, we need to prove that there is no isomorphism among all cases described in Theorem 4.1. For that, as we said in section 1, we use only \( t \)-algebras of rank 3 in which \( \gamma = 0 \).

Theorem 4.1 Let \( A = Fe \oplus U \oplus V \) be an exceptional \( t \)-algebra of rank 3 and dimension 6, of type \((4, 2)\).
(i) if \( A \) is a normal \( t \)-algebra, then \( A \) has a basis \( \{ e, u_1, u_2, u_3, v_1, v_2 \} \) which multiplication table is

\[
\begin{align*}
e^2 &= e, \\
2eu_i &= u_i \ (i = 1, 2, 3), \\
e_iv_i &= \gamma v_i \ (i = 1, 2)
\end{align*}
\]

and the other products are zero.
(ii) if $\dim(\mathfrak{U}V) = 1$, then $A$ has a basis $\{e, u_1, u_2, u_3, v_1, v_2\}$ which multiplication table is

$$ e^2 = e, \quad 2e u_i = u_i \ (i = 1, 2, 3), \quad e_i v_i = \gamma v_i \ (i = 1, 2), $$

Case 1: $u_1 v_1 = u_3$, and the other products are zero.

or

Case 2: $e^2 = e, \quad 2e u_i = u_i \ (i = 1, 2, 3), \quad e_i v_i = \gamma v_i \ (i = 1, 2),$

$u_1 v_1 = u_3, \ u_2 v_2 = u_3$, and the other products are zero.

(iii) if $\dim(\mathfrak{U}V) = 2$, then $A$ has a basis $\{e, u_1, u_2, v_1, v_2\}$ which multiplication table is

$$ e^2 = e, \quad 2e u_i = u_i \ (i = 1, 2, 3), \quad e_i v_i = \gamma v_i \ (i = 1, 2), $$

$u_1 v_1 = u_2, \ u_1 v_2 = u_3$, and the other products are zero.

\textbf{Proof:} (i) If $A$ is a normal $t$-algebra, then $UV = 0$ and the enunciated result is obvious.

(ii) Since $\dim(\mathfrak{U}V) = 1$, there are $u_1 \in \mathfrak{U}, v_1 \in V$ such that $0 \neq u_1 v_1 \in \mathfrak{U}V$ and so $\mathfrak{U}V = \langle u_1 v_1 \rangle$. It is easy to see that $\{u_1, u_1 v_1\}$ is a free set, which implies that there are $u_2 \in \mathfrak{U}$ and $v_2 \in V$ such that the set $\{e, u_1, u_2, u_3 = -u_1 v_1, v_1, v_2\}$ is a basis for $A$. In relation to this basis, the table of multiplication of $A$ is:

$$ e^2 = e, \quad 2e u_i = u_i \ (i = 1, 2, 3), \quad e_i v_i = \gamma v_i \ (i = 1, 2), $$

$u_1 v_1 = u_3, \ u_1 v_2 = au_3, \ u_2 v_1 = bu_3, \ u_2 v_2 = cu_3,$

and the other products are zero.

where $a, b, c \in F$.

Each of the elements $a, b, c \in F$ may be or not zero. Then we have the eight possibilities, which we will analyse here.

P1: If $a = b = c = 0$, we obtain the table described in Case 1.

P2: If $a = b = 0$ and $c \neq 0$, we will obtain the table described in Case 2, by considering the change of coordinates $u_2 = c^{-1}u_2$.

P3: If $a = c = 0$ and $b \neq 0$, we consider the change of coordinates

$$ u'_2 = -u_1 + b^{-1}u_2, $$

$$ v'_1 = v_1 + v_2, $$

to obtain the table described in Case 1.

P4: If $b = c = 0$ and $a \neq 0$, we consider the change of coordinates

$$ v'_2 = v_1 - a^{-1}v_2, $$

to obtain the table described in Case 1.

P5: If $b, c \neq 0$ and $a = 0$, we consider the change of coordinates

$$ u'_1 = 2^{-1}u_1 + (2b)^{-1}u_2, $$

$$ u'_2 = 2^{-1}u_1 - (2b)^{-1}u_2, $$

$$ v'_2 = v_1 - 2bc^{-1}v_2, $$

to obtain the table described in Case 2.

P6: If $a, c \neq 0$ and $b = 0$, we consider the change of coordinates

$$ u'_2 = -ac^{-1}u_2, $$

$$ v'_2 = v_1 - a^{-1}v_2, $$

to obtain the table described in Case 2.
Classification of exceptional train algebras of rank 3 and type (4, 2)

P7: If \(a \cdot b \neq 0\) and \(c = 0\), we consider the change of coordinates
\[
\begin{align*}
\psi'_1 &= 2^{-1}u_1 \\
\psi'_2 &= -2^{-1}u_1 + (2b)^{-1}u_2 \\
\psi'_3 &= 2^{-1}u_3 \\
v'_2 &= v_1 - a^{-1}v_2
\end{align*}
\]
, to obtain the table described in Case 2.

P8: If \(a \cdot b \cdot c \neq 0\), we consider the change of coordinates \(v'_2 = av_1 - v_2\)

Here, if \(ab = c\), we fall back to P3 and if \(ab \neq c\), we fall back to P5.

(iii) By Lemma 3.2, we have \(UV = <uv, wv>, U = <u, uv, wv>\). Moreover, \(V = <v, v>\). Then, if we call \(u_1 = u, u_2 = uv, u_3 = wv, v_1 = v\) and \(v_2 = v\), we obtain the table of multiplication above and this completes the proof.

Proposition 4.2 In the conditions of Theorem 4.1, there is no isomorphism of \(F\)-algebras between any \(t\)-algebras there obtained.

Proof: As example, we will prove the result for the \(t\)-algebras described in (ii), Theorem 4.1. The other cases are completely analogous. For that, let \(A = Fe \oplus U_e \oplus V_e\) and \(A' = Ff \oplus U_f \oplus V_f\) be two exceptional \(t\)-algebras of rank 3 and dimension 6, of type (4, 2), such that \(A = <e, u_1, u_2, u_3, v_1, v_2>\) and \(A' = <f, u'_1, u'_2, u'_3, v'_1, v'_2>\). Suppose that the two tables of multiplication are, respectively,

\[
A : \begin{cases}
eq e, 2eu_i = u_i \ (i = 1, 2, 3), \\
u_1v_1 = u_3, u_2v_2 = u_3 and the other products are zero.
\end{cases}
\]

and

\[
A' : \begin{cases}
f^2 = f, 2fu_i' = u_i' \ (i = 1, 2, 3), \\
u_1'v_1' = u_3' and the other products are zero.
\end{cases}
\]

We now will prove that then there is no isomorphism of \(F\)-algebras between \(A\) and \(A'\). Suppose that \(\varphi : A \to A'\) is a isomorphism of \(F\)-algebras. Then:

1. \(\varphi(e) = f + au'_1 + bu'_2 + cu'_3\)

Suppose that \(\varphi(e) = kf + au'_1 + bu'_2 + cu'_3 + mu'_1 + nu'_2\). Then:

\[
kf + au'_1 + bu'_2 + cu'_3 + mu'_1 + nu'_2 = \varphi(e) = \varphi(e.e) = \varphi(e)^2 = k^2 f + kau'_1 + + kbu'_2 + (kc + 2am)u'_3\]

and so \(k = k^2, a = ka, b = kb, c = kc + 2am\) and \(m = n = 0\). Since \(e\) is a idempotent of weight 1, it follows that \(e \neq 0\) and so \(k = 1\) and \(\varphi(e) = f + au'_1 + bu'_2 + cu'_3\).

2. \(\varphi(u_i) = \lambda_1u'_1 + \lambda_2u'_2 + \lambda_3u'_3 \ (i = 1, 2, 3)\).

Suppose that \(\varphi(u_i) = k_i f + \lambda_1u'_1 + \lambda_2u'_2 + \lambda_3u'_3 + \theta_1u'_1 + \theta_2u'_2\). Then:

\[
k_i f + \lambda_1u'_1 + \lambda_2u'_2 + \lambda_3u'_3 + \theta_1u'_1 + \theta_2u'_2 = \varphi(u_i) = \varphi(2c.u_i) = 2\varphi(c.u_i) = 2(\varphi(e.u_i)) = 2\varphi(e)\varphi(u_i) = 2(f + au'_1 + bu'_2 + cu'_3)(k_i + \lambda_1u'_1 +
\]

\]
Now we will finally prove that $k_i = \theta_{i1} = \theta_{i2} = 0$ and so $\varphi(u_i) = \lambda_{i1}u'_1 + \lambda_{i2}u'_2 + \lambda_{i3}u'_3$, $i = 1, 2, 3$.

3. $\varphi(v_i) = -2a\sigma_{i1}u'_3 + \sigma_{i1}v'_1 + \sigma_{i2}v'_2$ ($i = 1, 2$).

Suppose that $\varphi(v_i) = k_if + \phi_{i1}u'_1 + \phi_{i2}u'_2 + \phi_{i3}u'_3 + \sigma_{i1}v'_1 + \sigma_{i2}v'_2$. Then: 

$$0 = \varphi(2ev_i) = 2\varphi(e).\varphi(v_i) = 2(f + au'_1 + bu'_2 + cu'_3).\phi_{i1}u'_1 + \phi_{i2}u'_2 + \phi_{i3}u'_3 + \sigma_{i1}v'_1 + \sigma_{i2}v'_2 + 2k_if + (\phi_{i1} + k_i)u'_1 + (\phi_{i2} + k_i)b)u'_2 + (\phi_{i3} + k_ic + 2a\sigma_{i1})u'_3.$$

It follows that $k_i = \phi_{i1} = \phi_{i2} = 0$ and $\phi_{i3} = -2a\sigma_{i1}$ and consequently $\varphi(v_i) = -2a\sigma_{i1}u'_3 + \sigma_{i1}v'_1 + \sigma_{i2}$, $i = 1, 2$.

Now we will finally prove that $\varphi$ cannot be a $F$-isomorphism of algebras:

(i) $\lambda_{31}u'_1 + \lambda_{32}u'_2 + \lambda_{33}u'_3 = \varphi(u_3) = \varphi(u_1v_1) = \varphi(u_1)\varphi(v_1) = (\lambda_{11}u'_1 + \lambda_{12}u'_2 + \lambda_{13}u'_3)(-2a\sigma_{11}u'_3 + \sigma_{11}v'_1 + \sigma_{12}v'_2) = \lambda_{11}\sigma_{11}u'_3.$

It follows that $\lambda_{31} = \lambda_{32} = 0$ and $\lambda_{33} = \lambda_{11}\sigma_{11}$. In this way, we have $\varphi(u_3) = \lambda_{33}u'_3$ and, as $\varphi$ is an isomorphism, $\lambda_{33} \neq 0$, and so $\lambda_{11} \neq 0$ and $\sigma_{11} \neq 0$.

(ii) $\lambda_{33}u'_3 = \varphi(u_3) = \varphi(u_2).\varphi(v_2) = (\lambda_{21}u'_1 + \lambda_{22}u'_2 + \lambda_{23}u'_3)(\sigma_{22}u'_2 - 2a\sigma_{21}u'_3 + \sigma_{21}v'_1 + \sigma_{22}v'_2) = \lambda_{21}\sigma_{21}u'_3.$

It follows that $\lambda_{33} = \lambda_{21}\sigma_{21}$ and, since $\varphi$ is an isomorphism, $\lambda_{33} \neq 0$, and so $\lambda_{21} \neq 0$ and $\sigma_{21} \neq 0$.

(iii) $0 = \varphi(u_2v_1) = \varphi(u_2)\varphi(v_1) = (\lambda_{21}u'_1 + \lambda_{22}u'_2 + \lambda_{23}u'_3)(-2a\sigma_{11}u'_3 + \sigma_{11}v'_1 + \sigma_{12}v'_2) = \lambda_{21}\sigma_{11}u'_3.$

Then $\lambda_{21} \sigma_{11} = 0$ and as $\sigma_{11} \neq 0$, we have $\lambda_{21} = 0$ and so $\varphi(u_2) = \lambda_{22}u'_2 + \lambda_{23}u'_3.$

It follows that $\lambda_{21}\sigma_{11} = 0$, and that is not possible since $\lambda_{21} \neq 0$ and $\sigma_{11} \neq 0$, which means that $\varphi$ is not a isomorphism of $F$-algebras.

References


Received: February 19, 2020; Published: March 11, 2020