Some Properties of Hom-Crossed Products

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Abstract

This paper mainly studies two problems: on one hand, the Maschke theorem of Hom-crossed products is obtained by using the integral on the Hom-Hopf algebras. On the other hand, let \((H, \alpha)\) be a Hom-Hopf algebra, and \((A, \beta)\) be a Hom-algebra. We will introduce the \(H\)-cleft extension is equivalent to the Hom-crossed product.

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1. Introduction

In 1985, Radford proved that smash products \(A\#H\) and smash coproducts \(A \times H\) (where \(A\) is a left \(H\)-module algebra and a left \(H\)-comodule algebra) formed a Hopf algebra if and only if \(A\) is a bialgebra in the Yetter-Drinfeld modules category \(\mathcal{H} \mathcal{Y}\mathcal{D}\). This theorem is called the Radford biproduct theorem. For Radford biproduct, there have been many forms of generalizations, such as Radford biproduct of quasi-Hopf algebras (see[1]), Radford biproduct of weak Hopf algebras (see[2]), Radford biproduct of multiplier Hopf algebras (see[3]), etc.

Blattner, Cohen, Montgomery, Doi, Takeuchi and others independently extended the theory of group crossed products to Hopf algebras, which called by the crossed products on the Hopf algebras, as a generalization of smash products. The definition of crossed product is proposed by the literature [4], and its definition is as follows: let \(H\) be a Hopf algebra and \(A\) be an algebra, and \(\sigma : H \times H \rightarrow A\) is a \(k\)-bilinear map, then the crossed products \(A\#_{\sigma}H\) refer to the vector space with the following multiplication operations \(A \otimes_k H\). For any \(a, b \in A, h, g \in H\), the calculation rules are as follows:

\[(a\#_{\sigma}h)(b\#_{\sigma}g) = a(h_1 \cdot b)\sigma(h_2, g_1)\#_{\sigma}h_3g_2.\]
Crossed products played an important role in the extension theory of Hopf algebras. The crossed products with invertible cocycle is cleft extensions, and by crossed products one can construct new Hopf algebras. Later, the research on crossed products has aroused the interest of scholars and has achieved certain research results, for example: Cotorsion dimension of weak crossed products (see[5]), frame-indifference of crossed products, rotations, and the permutation tensor (see[6]), separable extensions for crossed products over monoidal Hom-Hopf algebras (see[7]) etc.

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by Makhlouf and Silvestrov in (see[8]). Hom-associative algebras and their related structures have recently become rather popular, due to the prospect of having a general framework in which one can produce many types of natural deformations of algebras. Among them such structures as Hom-coassociative coalgebras, Hom-Hopf algebras, Hom-Leibniz algebras (see[9]), and so on. Further, some actions and coactions on these Hom-algebras structures such as Hom-modules, Hom-comodules, Hom-Hopf modules, and Hom-module algebras were considered, and the fundamental structure theorem of Hom-Hopf modules was investigated in [10].

The framework of this paper is as follows: in section 1, we will review the basic concepts and conclusions of Hom-structure and Hom-crossed products, such as the definition of Hom-algebras, Hom-coalgebras and Hom-crossed products, etc. In Section 2, the Maschke theorem of Hom-crossed products is obtained by using the integral on the Hom-Hopf algebras. In Section 3, Cleft extensions is not only a special case of Galois extension, but also extends the classical theorem of group Galois theory, that is, if \( F \subset E \) is a finite Galois extension of the field with Galois group \( G \), then \( E/F \) has a normal basis. we will introduce the definition of \( H \)-cleft extensions, and prove that the crossed products are actually cleft extensions, and vice versa.

2. Preliminaries

The vector spaces \( V \), tensor products, modules, and linear maps involved in this paper are all researched on a fixed field \( k \), unless otherwise specified. The text continues to use Sweedler’s notations about coalgebras and comodules. For a coalgebra \( C \), we write its comultiplication \( \Delta(c) = c_1 \otimes c_2 \) for all \( c \in C \), and for a right \( C \)-comodule \( M \), we denote its coaction by \( \rho(m) = m_0 \otimes m_1 \) for all \( m \in M \). \( S \) represents the antipode map of the Hom-Hopf algebras \( H \). Let \( I \) be the identity map of vector space \( V \). Also, assume that \( \alpha, \beta \) are invertible maps.

In this section, we will review some definitions and basic conclusions of Hom-structure and Hom-crossed products.
Definition 2.1. [11] A Hom-associative algebra is a triple \((V, \mu, \alpha)\) where \(V\) is a \(k\)-linear space, \(\mu : V \otimes V \to V\) is a bilinear multiplication and \(\alpha : V \to V\) is a \(k\)-linear space homomorphism satisfying the Hom-associativity condition
\[
\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)).
\]

The Hom-associative algebra is said to be unital if there exists a homomorphism \(\eta : k \to V\) such that
\[
\mu(\eta(1) \otimes I(x)) = \alpha(x) = \mu(I(x) \otimes \eta(1)).
\]

Let \((V, \mu, \alpha)\) and \((V', \mu', \alpha')\) be two Hom-associative algebras. A linear map \(f : V \to V'\) is said to be a morphism of Hom-associative algebras if
\[
f \circ \alpha = \alpha' \circ f, \quad \mu' \circ (f \otimes f) = f \circ \mu.
\]

Definition 2.2. [11] A Hom-coalgebra is a triple \((V, \Delta, \beta)\) where \(V\) is a \(k\)-linear space and \(\Delta : V \to V \otimes V, \beta : V \to V\) are linear maps. A Hom-coassociative coalgebra is a Hom-coalgebra satisfying
\[
(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta.
\]

A Hom-coassociative coalgebra is said to be counital if there exists a map \(\varepsilon : V \to k\) satisfying
\[
(I \otimes \varepsilon) \circ \Delta = \beta = (\varepsilon \otimes I) \circ \Delta.
\]

Let \((V, \Delta, \beta)\) and \((V', \Delta', \beta')\) be two Hom-coassociative coalgebras. A linear map \(f : V \to V'\) is a morphism of Hom-coalgebras map if
\[
f \circ \beta = \beta' \circ f, \quad \Delta' \circ f = (f \otimes f) \circ \Delta.
\]

Definition 2.3. [9] A Hom-bialgebra is a quintuple \((V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)\) where
1) \((V, \mu, \alpha, \eta)\) is a Hom-associative algebra with unit \(\eta\).
2) \((V, \Delta, \beta, \varepsilon)\) is a Hom-coassociative coalgebra with counit \(\varepsilon\).
3) The linear maps \(\Delta\) and \(\varepsilon\) are morphisms of algebras \((V, \mu, \alpha, \eta)\), that is
\[
\begin{align*}
\Delta(e_1) &= e_1 \otimes e_1, \text{ where } e_1 = \eta(1), \\
\Delta(\mu(x \otimes y)) &= \Delta(x) \Delta(y), \\
\varepsilon(e_1) &= 1, \\
\varepsilon(\mu(x \otimes y)) &= \varepsilon(x) \varepsilon(y).
\end{align*}
\]

One can consider a more restrictive definition where linear maps \(\Delta\) and \(\varepsilon\) are morphisms of Hom-associative algebras that is the definition 2.3 3) becomes equivalent to
\[
\begin{align*}
\Delta(e_1) &= e_1 \otimes e_1, \text{ where } e_1 = \eta(1), \\
\Delta(\mu(x \otimes y)) &= \Delta(x) \Delta(y), \\
\varepsilon(\mu(x \otimes y)) &= \varepsilon(x) \varepsilon(y), \\
\varepsilon(x) &= \varepsilon(x). \\
\end{align*}
\]

\]
**Definition 2.4.** [9] A Hom-Hopf algebra is a Hom-bialgebra with an antipode. Then a Hom-Hopf algebra over a $k$-vector space $V$ is given by $H = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ where the following homomorphisms

$$
\mu : V \otimes V \to V, \eta : k \to V, \alpha : V \to V
$$

$$
\Delta : V \to V \otimes V, \varepsilon : V \to k, \beta : V \to V
$$

$$
S : V \to V
$$

satisfy the following conditions:

1) $(V, \mu, \alpha, \eta)$ is a unital Hom-associative algebra.
2) $(V, \Delta, \beta, \varepsilon)$ is a counital Hom-coalgebra.
3) The linear maps $\Delta$ and $\varepsilon$ are morphisms of algebras $(V, \mu, \alpha, \eta)$.
4) $S$ is the antipode, so

$$
\mu \circ S \otimes I \circ \Delta = \eta \circ \varepsilon = \mu \circ I \otimes S \circ \Delta.
$$

We have the following properties of the antipode when it exists:

$$
S(hg) = S(g)S(h), \quad S(\eta(1)) = 1,
$$

$$
\Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon,
$$

for any $h, g \in H$.

**Definition 2.5.** Let $(A, \beta)$ be a Hom-algebra, and $M$ is a vector space together with a linear map $\alpha : M \to M$. A left $(A, \beta)$-Hom-module together with a linear map $\varphi : A \otimes M \to M$, $a \otimes m \mapsto a \cdot m$ such that

$$
\beta(a) \cdot (b \cdot m) = (ab) \cdot \alpha(m), \quad 1_A \cdot m = \alpha(m), \quad \alpha(a \cdot m) = \beta(a) \cdot \alpha(m),
$$

for all $a, b \in A$, $m \in M$.

Let $(M, \alpha_M)$ and $(N, \alpha_N)$ be two left $(A, \beta)$-Hom-modules, then a linear map $f : M \to N$ is a called left $A$-module map if

$$
f(am) = af(m), \quad f \circ \alpha_M = \alpha_N \circ f,
$$

for all $a \in A, m \in M$.

**Definition 2.6.** Let $(H, \alpha)$ be a Hom-bialgebra, and $(A, \beta)$ be a Hom-algebra. $(A, \beta)$ is called left $(H, \alpha)$-Hom-module algebra, if there exists a linear map $\rho : H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$, such that:

$$
(hg) \cdot \beta(a) = \alpha(h) \cdot (g \cdot a), \quad 1_H \cdot a = \beta(a),
$$

$$
\beta(h \cdot a) = \alpha(h \cdot \beta(a)),
$$

$$
\alpha^2(h) \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_A = \varepsilon_H(h)1_A,
$$

for all $h, g \in H$, $a \in A$. 
Definition 2.7. Let \((C, \beta)\) be a Hom-coalgebra, and \(M\) is a vector space together with a linear map \(\alpha : M \rightarrow M\). A right \((C, \beta)\)-Hom-comodule together with a linear map \(\rho_M : M \rightarrow M \otimes C, m \mapsto m_0 \otimes m_1\) such that
\[
\alpha(m_0) \otimes m_{11} \otimes m_{12} = m_{00} \otimes m_{01} \otimes \beta(m_1),
\]
\[
m_0 \varepsilon(m_1) = \alpha(m), \quad \rho_M(\alpha(m)) = \alpha(m_0) \otimes \beta(m_1),
\]
for all \(m \in M\).

Let \((M, \alpha_M)\) and \((N, \alpha_N)\) be two right \((C, \beta)\)-Hom-comodules, then a linear map \(g : M \rightarrow N\) is a called right \(C\)-comodule map if
\[
\rho_N(g(m)) = (g \otimes I)\rho_M(m), \quad g \circ \alpha_M = \alpha_N \circ g,
\]
for all \(m \in M\).

Definition 2.8. A right integral in \((H, \alpha)\) is an \(\alpha\)-invariant element \(t \in H\) (i.e. \(\alpha(t) = t\)) such that
\[
th = \varepsilon(h)t,
\]
for all \(h \in H\). \(\int_H^t\) denotes the space of right integrals.

Definition 2.9. \([13]\) Let \((H, \alpha)\) be a Hom-bialgebra and \((A, \beta)\) be a left \((H, \alpha)\)-Hom-module algebra. Assume that \(\sigma : H \otimes H \rightarrow A\) is convolution invertible together with \(\sigma(\alpha(h), \alpha(g)) = \beta(\sigma(h, g))\), where for all \(h, g \in H\), and \(\alpha, \beta\) are invertible maps. The Hom-crossed product \((A^*_{\sigma}H, \beta^*_{\sigma}\alpha)\) of \((A, \beta)\) and \((H, \alpha)\) is defined as follows:

1) As a \(k\)-space, \((A^*_{\sigma}H, \beta^*_{\sigma}\alpha) = (A \otimes H, \beta \otimes \alpha),\)
2) Hom-multiplication: for all \(a, b \in A, h, g \in H,\)
\[
(a^* h)(b^* g) = a((\alpha^{-3}(h_1) \cdot \beta^{-2}(b))\sigma(h_{21}, g_1)) \alpha^{-2}(h_{22}) \alpha^{-1}(g_2).
\]

Lemma 2.10. \([13]\) Let \((H, A)\) be a \((\alpha, \beta, \sigma)\)-compatible pair. Hom-crossed product \((A^*_{\sigma}H, \beta^*_{\sigma}\alpha)\) is a Hom-algebra if and only if the following conditions hold:
\[
\sigma(h, 1_A) = \varepsilon_H(h)1_A, \sigma(1_H, h) = \varepsilon_H(h)1_A, \quad (1)
\]
\[
(\alpha^{-1}(h_1) \cdot \sigma(\alpha^2(g_1), \alpha(k_1)))\sigma(\alpha^3(h_2), \alpha(g_2k_2))
\]
\[
= \sigma(\alpha^3(h_1), \alpha^2(g_1))\sigma(\alpha^2(h_2g_2), \alpha^3(k)), \quad (2)
\]
\[
(\alpha^{-1}(h_1) \cdot (\alpha^{-2}(g_1) \cdot \beta^{-1}(c)))\sigma(\alpha^3(h_2), \alpha^2(g_2))
\]
\[
= \sigma(\alpha^3(h_1), \alpha^2(g_1))(\alpha^{-2}(h_2g_2) \cdot c). \quad (3)
\]

3. The Maschke theorem of Hom-crossed product

In this section, we will use the integral on the Hom-Hopf algebra to obtain the Maschke theorem of the Hom-crossed product. Let \(A^*_{\sigma}H\) be a Hom-crossed product, we can define the map \(\gamma : H \rightarrow A^*_{\sigma}H, \ h \mapsto 1_A^\# h\) and \(\gamma \in Hom(H, A^*_{\sigma}H)\) is invertible.
Lemma 3.1. Let \((H, \alpha)\) be a finite dimensional Hom-Hopf algebra and \(0 \neq t \in \int_H^r\). Let \(A_{\sigma}^\# H\) be a Hom-crossed product and \((M, \mu), (N, \nu)\) be a left \(A_{\sigma}^\# H\)-Hom-module, assume that \(\alpha\) is involutive \((\alpha^2 = I)\). If \(\pi : (M, \mu) \to (N, \nu)\) is an \((A, \beta)\)-Hom-module map, then
\[
\tilde{\pi} : (M, \mu) \to (N, \nu), m \mapsto \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(m))
\]
is an \(A_{\sigma}^\# H\)-Hom-module map.

Proof. For ease of notation, we will regard \(A\) as embedded in \(A_{\sigma}^\# H\) and write \(a\) in place of \(a_{\sigma}^\# 1, a \in A\).

First, we check that \(\tilde{\pi}\) is an \((A, \beta)\)-Hom-module map. Since
\[
\gamma(h)a = (\alpha^{-2}(h_1) \cdot \beta^{-1}(a))\gamma(\alpha^{-1}(h_2)),
\]
for any \(h \in H\), and since \(\gamma\) is invertible, it follows that
\[
h \cdot a = (\gamma(\alpha^{-2}(h_1))\beta^{-1}(a))\gamma^{-1}(\alpha^{-1}(h_2)).
\]
Thus for all \(m \in M, a \in A\),
\[
\tilde{\pi}(a \cdot m) = \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(a \cdot m))
\]
\[
= \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot (\beta^{-2}(a) \cdot \mu^{-2}(m)))
\]
\[
= \gamma^{-1}(\alpha(t_1))\pi((\gamma(\alpha^{-1}(t_2))\beta^{-2}(a)) \cdot \mu^{-1}(m))
\]
\[
\overset{(4)}{=} \gamma^{-1}(\alpha(t_1))\pi(((\alpha^{-3}(t_{21}) \cdot \beta^{-3}(a))\gamma(\alpha^{-2}(t_{22}))) \cdot \mu^{-1}(m))
\]
since \(\pi\) is an \((A, \beta)\)-Hom-module map
\[
= \gamma^{-1}(\alpha(t_1))((\alpha^{-2}(t_{21}) \cdot \beta^{-2}(a))\pi(\gamma(\alpha^{-2}(t_{22}))) \cdot \mu^{-2}(m)))
\]
\[
= (\gamma^{-1}(t_1)\alpha^{-2}(t_{21}) \cdot \beta^{-2}(a))\pi(\gamma(\alpha^{-1}(t_{22}))) \cdot \mu^{-1}(m))
\]
\[
= (\gamma^{-1}(\alpha^{-1}(t_{11}))((\alpha^{-2}(t_{12}) \cdot \beta^{-2}(a)))\pi(\gamma(\alpha^{-2}(t_{22}))) \cdot \mu^{-1}(m))
\]
\[
\overset{(5)}{=} (\gamma^{-1}(\alpha^{-1}(t_{11})))((\gamma(\alpha^{-4}(t_{121}))\beta^{-3}(a))\gamma^{-1}(\alpha^{-3}(t_{122})))\pi(\gamma(t_2) \cdot \mu^{-1}(m))
\]
\[
= ((\gamma^{-1}(\alpha^{-2}(t_{11}))\gamma(\alpha^{-4}(t_{121}))\beta^{-3}(a))\gamma^{-1}(\alpha^{-2}(t_{122})))\pi(\gamma(t_2) \cdot \mu^{-1}(m))
\]
\[
= (((\gamma^{-1}(\alpha^{-3}(t_{111}))\gamma(\alpha^{-4}(t_{1211}))\beta^{-2}(a))\gamma^{-1}(\alpha^{-2}(t_{122})))))\pi(\gamma(t_2) \cdot \mu^{-1}(m))
\]
\[
= ((((\gamma^{-1}(\alpha^{-4}(t_{1111}))\gamma(\alpha^{-4}(t_{12111}))\beta^{-2}(a))\gamma^{-1}(\alpha^{-1}(t_{12})))))\pi(\gamma(t_2) \cdot \mu^{-1}(m))
\]
\[
= (\beta^{-1}(a)\gamma^{-1}(t_1))\pi(\gamma(t_2) \cdot \mu^{-1}(m))
\]
\[
= a(\gamma^{-1}(t_1)\pi(\gamma(a(t_2)) \cdot \mu^{-2}(m)))
\]
since \(\alpha(t) = t\)
\[
= a(\gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(m)))
\]
\[
= a \cdot \tilde{\pi}(m).
\]
Next, we prove that \( \tilde{\pi} \) is also \((H, \alpha)\)-Hom-module map. For all \( h \in H, m \in M \),
\[
\begin{align*}
\tilde{\pi}(\gamma(\alpha(h)) \cdot m) &= \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(\gamma(\alpha(h)) \cdot m)) \\
&= \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot (\gamma(\alpha^{-1}(h)) \cdot \mu^{-2}(m))) \\
&= \gamma^{-1}(\alpha(t_1))\pi((\gamma(\alpha^{-1}(t_2))\gamma(\alpha^{-1}(h))) \cdot \mu^{-1}(m)) \\
&= \gamma^{-1}(\alpha(t_1))\pi((\sigma(\alpha(t_{21}), h_1)\gamma(\alpha(t_{22})\alpha(h_2))) \cdot \mu^{-1}(m)) \\
&\quad \text{since } \pi \text{ is an } (A, \beta)\text{-Hom-module map} \\
&= \gamma^{-1}(\alpha(t_1))((\sigma(\alpha(t_{21}), h_1))\pi(\gamma(h_2) \cdot \mu^{-2}(m))) \\
&= \gamma^{-1}(\alpha(t_1))(\sigma(t_{12}, \alpha(h_1))\pi(\gamma(t_2) \cdot \mu^{-2}(m))) \\
&= (\gamma^{-1}(\alpha^{-1}(t_{11}))\sigma(t_{12}, \alpha(h_1))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= (\gamma^{-1}(\alpha^{-1}(t_{11}))(\gamma(t_{121})\gamma(h_{11}))\gamma^{-1}(t_{122}h_{12}))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= (((\gamma^{-1}(\alpha^{-2}(t_{11}))(\gamma(t_{121})\gamma(h_{11}))\gamma^{-1}(t_{122}h_{12}))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= (((\gamma^{-1}(\alpha^{-3}(t_{11}))\gamma(t_{121})\gamma(h_{11}))\gamma^{-1}(\alpha(t_{122}h_{12})))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= ((\gamma^2(h_{11}))\gamma^{-1}(\alpha(t_{11}))\gamma^{-1}(\alpha(t_{11}))\gamma^{-1}(t_{12}2\alpha(h_{12}))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= (\gamma^2(h_{11}))\gamma^{-1}(\alpha(t_{11}))\gamma^{-1}(t_{12}2\alpha(h_{12}))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&\quad \text{since } t_{11} \alpha(h_1) \otimes t_{22} \alpha(h_2) = \Delta(t \alpha(h)) = \varepsilon(h)t_1 \otimes t_2 \\
&= (\gamma(h)\gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-1}(m)) \\
&= \gamma(\alpha(h))(\gamma^{-1}(\alpha(t_1))\gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(m))) \\
&= \gamma(\alpha(h)) \cdot \tilde{\pi}(m).
\end{align*}
\]
Hence, \( \tilde{\pi} \) is an \( A^*_{\text{H}} \)-Hom-module map. \( \square \)

**Lemma 3.2.** Let \((H, \alpha)\) be a finite dimensional Hom-Hopf algebra and \( t \) be a right integral in \( H \). Assume that \( \alpha \) is involutive (\( \alpha^2 = I \)), \( A^*_{\text{H}} \) be a Hom-crossed product, \((M, \mu)\) be a left \( A^*_{\text{H}} \)-Hom-module and \((N, \nu)\) be an \( A^*_{\text{H}} \)-Hom-submodule of \((M, \mu)\). If \((N, \nu)\) is a direct summand of \((M, \mu)\) as \((A, \beta)\)-Hom-modules, then \((N, \nu)\) is also a direct summand of \((M, \mu)\) as \( A^*_{\text{H}} \)-Hom-modules.

**Proof.** Let \( \pi : (M, \mu) \to (N, \nu) \) be the canonical projection as \((A, \beta)\)-Hom-modules. Define
\[
\tilde{\pi} : (M, \mu) \to (N, \nu), m \mapsto \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \mu^{-2}(m)).
\]
By Lemma 3.1, \( \tilde{\pi} \) is a \( A^*_{\text{H}} \)-Hom-modules morphism. Next we claim that \( \tilde{\pi} \) is a projection of \( M \) onto \( N \). For any \( n \in N \), that is \( \tilde{\pi}(n) = n \).
\[
\tilde{\pi}(n) = \gamma^{-1}(\alpha(t_1))\pi(\gamma(t_2) \cdot \nu^{-2}(n)) = (\gamma^{-1}(t_1))\gamma(t_2) \cdot \nu^{-1}(n) = n.
\]
Thus \((N, \nu)\) is also a direct summand of \((M, \mu)\) as \(A^\#_\sigma H\)-Hom-modules.
This completes the proof. \(\square\)

With the above lemmas, we obtain the Hom-Maschke theorem.

**Theorem 3.3.** Let \((H, \alpha)\) be a finite dimensional Hom-Hopf algebra, assume that \(\alpha\) is involutive \((\alpha^2 = I)\), \(A^\#_\sigma H\) is Hom-crossed product. If \((A, \beta)\) is semisimple, then the Hom-crossed product \((A^\#_\sigma H, \beta^\#_\sigma \alpha)\) is semisimple.

As a special application of Hom-crossed product, we get the Hom-Maschke theorem of Hom-smash product.

**Corollary 3.4.** Let \((H, \alpha)\) be a finite dimensional Hom-Hopf algebra, and \((A, \beta)\) be an \((H, \alpha)\)-Hom-module algebra. If \((A, \beta)\) is semisimple, then the Hom-smash product \((A^\# H, \beta^\# \alpha)\) is semisimple.

### 4. Cleft extensions

In this section, we will prove that a Hom-crossed product is equivalent to a Hom-cleft extension. This result is a generalization of the theory in the usual Hopf algebras. In order to achieve the proof, we need the following definition.

**Definition 4.1.** Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \beta)\) a Hom-algebra. Assume that \(B\) is a right \(H\)-comodule algebra, and \(A = B^{coH} = \{b \in B \mid \rho(b) = \beta(b) \otimes 1_H\}\). Then \(A \subset B\) is called \(H\)-cleft extension if there exists a right \(H\)-comodule map \(\gamma : H \to B\) which is convolution inverse.

Note that we may always assume \(\gamma(1) = 1\).

**Lemma 4.2.** Assume that \((B, \beta)\) is a right \(H\)-Hom-comodule algebra, via \(\rho : B \to B \otimes H, b \mapsto b_0 \otimes b_1\), and that \(A \subset B\) is a \(H\)-cleft extension via \(\gamma\) with \(\gamma(1) = 1\). Then

1) \(\rho \circ \gamma^{-1} = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta\).
2) for all \(b \in B\), \(\sum b_0 \gamma^{-1}(b_1) \in A = B^{coH}\).

**Proof.** 1) Since \(\rho\) is an algebra map, \(\rho \circ \gamma^{-1}\) is the inverse of \(\rho \circ \gamma = (\gamma \otimes I)\Delta\). Let \(\lambda = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta\). Then for any \(h \in H\),

\[
((\rho \circ \gamma) \ast \lambda)(h) = ((\gamma \otimes I)\Delta(h_1))((\gamma^{-1} \otimes S) \circ \tau \circ \Delta(h_2)) = \gamma(h_{11}) \otimes h_{12} (\gamma^{-1}(h_{22}) \otimes S(h_{21})) = \gamma(h_{11})\gamma^{-1}(h_{22}) \otimes h_{12} S(h_{21}) = \gamma(\alpha(h_1))\gamma^{-1}(h_{22}) \otimes \varepsilon(h_{21})1 = \gamma(\alpha(h_1))\gamma^{-1}(\alpha(h_2)) \otimes 1 = \varepsilon(h)1 \otimes 1.
\]
Hence $\lambda$ is a right inverse of $\rho \circ \gamma$, and so $\lambda = \rho \circ \gamma^{-1}$ by the uniqueness of inverses.

2) For all $b \in B$,

$$
\rho(b_0\gamma^{-1}(b_1))
= \sum \rho(b_0)\rho(\gamma^{-1}(b_1))
= (b_{00} \otimes b_{01})(\gamma^{-1}(b_{12}) \otimes S(b_{11}))
= b_{00}\gamma^{-1}(b_{12}) \otimes b_{01}S(b_{11})
= \beta(b_0)\gamma^{-1}(\alpha^{-1}(b_{122})) \otimes b_{11}S(\alpha^{-1}(b_{121}))
= \beta(b_0)\gamma^{-1}(b_{12}) \otimes \alpha^{-1}(b_{111})S(\alpha^{-1}(b_{112}))
= \beta(b_0)\gamma^{-1}(\alpha(b_1)) \otimes 1
= \beta(b_0\gamma^{-1}(b_1)) \otimes 1.
$$

Thus for all $b \in B$, we have $\sum b_0\gamma^{-1}(b_1) \in A = B^{coH}$. \hfill \Box

**Lemma 4.3.** Let $A \subset B$ be a right $H$-coleft extension via $\gamma : H \to B$ such that $\gamma(1) = 1$. Then there is a crossed product action of $H$ on $A$ given by

$$
h \cdot a = (\gamma(\alpha^{-2}(h_1)))\beta^{-1}(a))\gamma^{-1}(\alpha^{-1}(h_2)),
$$

and a convolution invertible map $\sigma : H \otimes H \to A$ given by

$$
\sigma(h, k) = (\gamma(\alpha^{-6}(h_1))\gamma(\alpha^{-5}(k_1)))\gamma^{-1}(\alpha^{-6}(h_2)\alpha^{-5}(k_2)).
$$

Then we have the crossed product $A^\sharp_\sigma H$. Moreover $\Phi : A^\sharp_\sigma H \to B$, $a^\sharp_\sigma h \mapsto \beta^{-1}(a)\gamma(\alpha^{-1}(h))$ is a Hom-algebra map. Further, the algebra isomorphism $\Phi$ is both a left $A$-module and right $H$-comodule map, where $A^\sharp_\sigma H$ is a left $A$-module via $a \cdot (b^\sharp_\sigma h) = ab^\sharp_\sigma \alpha(h)$ and is a right $H$-comodule via $(a^\sharp_\sigma h)_0 \otimes (a^\sharp_\sigma h)_1 = \beta(a)\sharp_\sigma h_1 \otimes h_2$.

**Proof.** We first show that $h \cdot a \in A$, for any $a \in A$, $h \in H$. Now

$$
\rho(h \cdot a)
= \rho((\gamma(\alpha^{-2}(h_1)))\beta^{-1}(a))\gamma^{-1}(\alpha^{-1}(h_2))
= (\rho\gamma(\alpha^{-2}(h_1)))\rho(\beta^{-1}(a))\rho(\gamma^{-1}(\alpha^{-1}(h_2)))
= ((\gamma(\alpha^{-2}(h_{11})) \otimes \alpha^{-2}(h_{12}))(a \otimes 1_H))(\beta^{-1}(\gamma^{-1}(h_{20})) \otimes \alpha^{-1}(\gamma^{-1}(h_{21})))
= (\gamma(\alpha^{-2}(h_{11}))a \otimes \alpha^{-1}(h_{12}))(\beta^{-1}(\gamma^{-1}(h_{22})) \otimes \alpha^{-1}(S(h_{21})))
= (\gamma(\alpha^{-2}(h_{11}))a)\beta^{-1}(\gamma^{-1}(h_{22})) \otimes \alpha^{-1}(h_{12})\alpha^{-1}(S(h_{21}))
= (\gamma(\alpha^{-1}(h_1))a)\beta^{-1}(\gamma^{-1}(h_{22})) \otimes \varepsilon(h_{21})1
= (\gamma(\alpha^{-1}(h_1))a)\gamma^{-1}(h_2) \otimes 1
= \beta(h \cdot a) \otimes 1.
$$
Hence \( h \cdot a \in A = B^{coH} \). It is easy to see that \( H \) measures \( A \). Similarly we see that \( \sigma \) has values in \( A \). For any \( h, k \in H \), we have

\[
\rho(\sigma(h, k)) = \rho(\sum (\gamma(\alpha^{-6}(h_1))\gamma(\alpha^{-5}(k_1)))\gamma^{-1}(\alpha^{-6}(h_2)\alpha^{-5}(k_2)))
\]

\[
= (\rho\gamma(\alpha^{-6}(h_1))\rho\gamma(\alpha^{-5}(k_1)))\rho\gamma^{-1}(\alpha^{-6}(h_2)\alpha^{-5}(k_2))
\]

\[
= ((\gamma(\alpha^{-6}(h_{11})) \otimes \alpha^{-6}(h_{12}))(\gamma(\alpha^{-5}(k_{11})) \otimes \alpha^{-5}(k_{12})))
\]

\[
= (\gamma^{-1}(\alpha^{-6}(h_{22})\alpha^{-5}(k_{22})) \otimes S(\alpha^{-6}(h_{21})\alpha^{-5}(k_{21})))
\]

\[
= (\gamma(\alpha^{-6}(h_{11}))\gamma(\alpha^{-5}(k_{11})))\gamma^{-1}(\alpha^{-6}(h_{22})\alpha^{-5}(k_{22})) \otimes (\alpha^{-6}(h_{12})\alpha^{-5}(k_{12}))\gamma^{-1}(\alpha^{-6}(h_{21})\alpha^{-5}(k_{21}))
\]

\[
= (\gamma^{-1}(\alpha^{-5}(h_{1}))\gamma^{-1}(\alpha^{-4}(k_{1})))\gamma^{-1}(\alpha^{-6}(h_{22})\alpha^{-5}(k_{22})) \otimes \varepsilon(h_{21}k_{21})1
\]

\[
= (\gamma^{-1}(\alpha^{-5}(h_{1}))\gamma^{-1}(\alpha^{-4}(k_{1})))\gamma^{-1}(\alpha^{-5}(h_{2})\alpha^{-4}(k_{2})) \otimes 1
\]

\[
= \beta(\sigma(h, k)) \otimes 1.
\]

Hence \( \sigma(h, k) \in A = B^{coH} \).

Define \( \Psi : B \to A^*_{A^*H}, b \mapsto \beta^{-3}(b_{00})\gamma^{-1}(\alpha^{-1}(b_{01}))\varepsilon(\alpha(b_{11})) \). By Lemma 4.2 2), we know that it makes sense. Next we will show that \( \Phi \) and \( \Psi \) are mutual inverse. First, if \( b \in B \), then

\[
\Phi \Psi(b) = \Phi(\beta^{-3}(b_{00})\gamma^{-1}(\alpha^{-1}(b_{01}))\varepsilon(\alpha(b_{11})))
\]

\[
= (\beta^{-3}(b_{00})\gamma^{-1}(\alpha^{-2}(b_{01})))\gamma(b_{11})
\]

\[
= \beta^{-2}(b_{00})(\gamma^{-1}(\alpha^{-2}(b_{01})))\gamma(\alpha^{-1}(b_{11}))
\]

\[
= \beta^{-2}(b_{0})(\gamma^{-1}(\alpha^{-2}(b_{11})))\gamma(\alpha^{-2}(b_{12}))
\]

\[
= \beta^{-2}(b_{0})\varepsilon(b_{1})1 = b.
\]

Next, choose \( a^*_h \in A^*_{A^*H} \). Then

\[
\Psi \Phi(a^*_h) = \Psi(\beta^{-1}(a)\gamma(\alpha^{-1}(h)))
\]

\[
= (\beta^{-1}(a)\gamma(\alpha^{-4}(h_{00})))\gamma^{-1}(\alpha^{-2}(a_{01}))\gamma(\alpha^{-2}(h_{01}))\varepsilon(a_{1})\gamma(h_{1})
\]

\[
= (\beta^{-2}(a)\gamma(\alpha^{-4}(h_{00})))\gamma^{-1}(\alpha^{-3}(\gamma(h_{01})))\varepsilon(a_{1})\gamma(\alpha^{-1}(h_{1}))
\]

\[
= (\beta^{-2}(a)\gamma(\alpha^{-4}(h_{10})))\gamma^{-1}(\alpha^{-3}(\gamma(h_{11})))\varepsilon(a_{1})\gamma(\alpha^{-1}(h_{2}))
\]

\[
= (\beta^{-2}(a)\gamma(\alpha^{-4}(h_{11})))\gamma^{-1}(\alpha^{-3}(\gamma(h_{12})))\varepsilon(a_{1})\gamma(\alpha^{-1}(h_{2}))
\]

\[
= \beta^{-1}(a)(\gamma(\alpha^{-4}(h_{11})))\gamma^{-1}(\alpha^{-4}(h_{12}))\varepsilon(a_{1})\gamma(\alpha^{-1}(h_{2}))
\]

\[
= a^*_h.
\]
Hence $\Psi = \Phi^{-1}$. Moreover $\Phi$ is a Hom-algebra map. Indeed, firstly $\Phi \circ (\beta \otimes \alpha) = \beta \circ \Phi$, and

\[ \Phi((a^*_g h)(b^*_g)) = \Phi(a((\alpha^{-3}(h_1) \cdot \beta^{-2}(b))\sigma(h_{21}, g_1))\gamma^{-2} h_{22} \alpha^{-1}(g_2)) \]

\[ = (\beta^{-1}(a)((\alpha^{-4}(h_1) \cdot \beta^{-3}(b))\sigma(\alpha^{-1}(h_{21}), \alpha^{-1}(g_1)))\gamma(\alpha^{-3}(h_{22})\alpha^{-2}(g_2))) \]

\[ = (\beta^{-1}(a)(((\gamma(\alpha^{-6}(h_{11}))\beta^{-4}(b))\gamma^{-1}(\alpha^{-5}(h_{122}))(\gamma(\alpha^{-7}(h_{211})))\gamma(\alpha^{-6}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122})))\gamma(\alpha^{-3}(h_{222})\alpha^{-2}(g_2))) \]

\[ = (\beta^{-1}(a)(((\gamma(\alpha^{-6}(h_{11}))\beta^{-4}(b))\gamma^{-1}(\alpha^{-5}(h_{122})))\gamma(\alpha^{-6}(h_{2111}))\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122})))\gamma(\alpha^{-3}(h_{222})\alpha^{-2}(g_2))) \]

\[ = (\beta^{-1}(a)(((\gamma(\alpha^{-6}(h_{11}))\beta^{-4}(b))\gamma^{-1}(\alpha^{-5}(h_{122})))\gamma(\alpha^{-6}(h_{2111}))\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122})))\gamma(\alpha^{-3}(h_{222})\alpha^{-2}(g_2))) \]

\[ = (\beta^{-1}(a)(((\gamma(\alpha^{-6}(h_{11}))\beta^{-4}(b))\gamma^{-1}(\alpha^{-5}(h_{122})))\gamma(\alpha^{-6}(h_{2111}))\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122})))\gamma(\alpha^{-3}(h_{222})\alpha^{-2}(g_2))) \]

\[ = (\beta^{-1}(a)(((\gamma(\alpha^{-6}(h_{11}))\beta^{-4}(b))\gamma^{-1}(\alpha^{-5}(h_{122})))\gamma(\alpha^{-6}(h_{2111}))\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122})))\gamma(\alpha^{-3}(h_{222})\alpha^{-2}(g_2))) \]

\[ = ((\beta^{-2}(a)(\gamma(\alpha^{-4}(h_{11})))\beta^{-3}(b))\gamma(\alpha^{-4}(g_{111}))\gamma^{-1}(\alpha^{-5}(h_{211})\alpha^{-5}(g_{122}))) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-3}(h_{11})))\beta^{-2}(b))((\gamma(\alpha^{-4}(g_{111}))\gamma^{-1}(\alpha^{-5}(h_{211})\alpha^{-5}(g_{122}))) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-3}(h_{11})))\beta^{-2}(b))((\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122}))) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-5}(h_{11})))\beta^{-2}(b))((\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122}))) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-5}(h_{11})))\beta^{-2}(b))((\gamma(\alpha^{-5}(g_{111}))\gamma^{-1}(\alpha^{-7}(h_{222})\alpha^{-6}(g_{122}))) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-3}(h_{11})))\beta^{-2}(b))\gamma(g) \in (h_2) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-2}(h_{11})))\beta^{-2}(b))\gamma(g) \]

\[ = ((\beta^{-2}(a)(\gamma(\alpha^{-2}(h_{11})))\beta^{-1}(b)\gamma(g) \]

\[ = (\beta^{-1}(a)(\gamma(\alpha^{-1}(h_{11})))\beta^{-1}(b)\gamma(\alpha^{-1}(g))) \]

\[ = \Phi(a^*_g h)\Phi(b^*_g). \]

Hence $A^*_g H \cong B$ as Hom-algebra. The Lemma 2.10 (2) and (3) hold. Also, $\Phi$ is clearly a left $A$-module map. It is also a right $H$-comodule map,
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since

\[ \rho_B(\Phi(a^* h)) = \rho_B(\beta^{-1}(a)\gamma(\alpha^{-1}(h))) \]
\[ = \rho_B(\beta^{-1}(a))\rho_B(\gamma(\alpha^{-1}(h))) \]
\[ = (a \otimes 1_H)(\gamma(\alpha^{-1}(h_1)) \otimes \alpha^{-1}(h_2)) \]
\[ = \Phi(\beta(a)\sharp h_1) \otimes h_2 \]
\[ = (\Phi \otimes I) \circ \rho_{A^*_\sigma H}(a^* h). \]

This completes the proof. \[\square\]

Lemma 4.4. Let \( A^*_\sigma H \) be a crossed product, and define the map \( \gamma : H \to A^*_\sigma H \), by \( \gamma(h) = 1^*_\sigma \alpha(h) \). Then \( \gamma \) is convolution invertible with inverse

\[ \gamma^{-1}(h) = \sigma^{-1}(S(\alpha^2(h_{21})), \alpha(h_{22}))\sharp S(h_1). \]

Proof. Set \( \mu(h) = \sigma^{-1}(S(\alpha^2(h_{21})), \alpha(h_{22}))\sharp S(h_1) \). Then

\[ (\mu * \gamma)(h) = (\sigma^{-1}(S(\alpha^2(h_{121})), \alpha(h_{122}))\sharp S(h_{111}))(1^*_\sigma \alpha(h_2)) \]
\[ = \sigma^{-1}(S(\alpha^2(h_{121})), \alpha(h_{122}))(S(\alpha^{-3}(h_{111})) \circ \beta^{-2}(1))\sigma(S(h_{1121}), \alpha(h_{21}))) \]
\[ = \sharp S(\alpha^{-2}(h_{1122}))h_{22} \]
\[ = \sigma^{-1}(S(\alpha^2(h_{121})), \alpha(h_{122}))(S(\alpha^{-4}(h_{1122})) \circ \beta^{-2}(1))\sigma(S(h_{1121}), \alpha(h_{21}))) \]
\[ = \sharp S(\alpha^{-1}(h_{111}))h_{22} \]
\[ = \sigma^{-1}(S(\alpha^2(h_{212})), h_{2211})\sigma(S(\alpha^2(h_{211})), h_{2212})\sharp S(\alpha(h_1)\alpha^{-1}(h_{222})) \]
\[ = 1^*_\sigma \varepsilon(h_2) \varepsilon(h_{221})S(\alpha(h_1))\alpha^{-1}(h_{222}) \]
\[ = \varepsilon(h)1^*_\sigma 1. \]

Thus \( \mu \) is the left inverse of \( \gamma \). And it is straightforward to verify that \( \mu \) is the right inverse of \( \gamma \), as done in Proposition 7.2.7 in [12].

Obviously \( \gamma \) is a right \( H \)-comodule map. This completes the proof. \[\square\]

By lemma 4.3 and 4.4, we obtain the following theorem directly.

Theorem 4.5. \( A \subset B \) is a \( H \)-cleft extension if and only if \( B \cong A^*_\sigma H \).

REFERENCES


Some properties of Hom-crossed products


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