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Algebraic Logic and Topoi; a Philosophical Holistic Approach

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Abstract

We take a magical tour in algebraic logic and its most novel applications. In algebraic logic we start from classical results on neat embeddings due to Andreka, Henkin, Németi, Monk and Tarski, all the way to recent results in algebraic logic using so–called rainbow constructions. Highlighting the connections with graph theory, model theory, finite combinatorics, and in the last decade with the theory of general relativity and hypercomputation, this article aspires to present topics of broad interest in a way that is hopefully accessible to a large audience. Other topics deallt with include the interaction of algebraic and modal logic, the so–called (central still active) finitizability problem, Gödels's incompleteness Theorem in guarded fragments, counting the number of subvarieties of RCA_ω which is reminiscent of Shelah's stability theory and the interaction of algebraic logic and descriptive set theory as means to approach Vaught's conjecture in model theory. The interconections between algebraic geometry and cylindric algebra theory is surveyed and elaborated upon as a Sheaf theoretc duality. This article is not purely expository; far from it. It contains new results and new approaches to old paradigms. Furthermore, various scattered results in the literature are presented from a holistic perspective highlighting similarities between seemingly remote areas in the literature. For example topoi and category theory are approached as means to unify apparently scattered results in the literature.

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1 Introduction

1.1 Holism vs atomism

Holism as a philosophical concept is diametrically opposed to atomism. Where the atomist believes that any whole can be broken down or reduced into separate parts and to the interaction between such local smaller parts, the holist maintains that the whole is primary and more often than not reveals more than the sum of its parts. The atomist divides things up in order to know them more accurately; the holist looks at things or systems in aggregate and argues that we can know more about them and better understand their nature and their purpose if they are viewed in this global, so to speak, manner. The early Greek atomists Leucippus and Democritus (fifth century B.C.) were ancestors of new physics. There view is a forerunner to the intricate philosophy underlying quantum physics. Everything in the universe consists of indivisible, atoms of various kinds. Such atoms are indistructible; we cannot go further by splitting them into smaller 'subatoms'. The visible change on the macro level is nothing more than merely a re-arrangement of these atoms when viewed on the microlevel. This view was a reaction to the earlier holistic philosophy of the Greek philosopher Parmenides, who argued that at some primary level the world is a changeless static unity. Like Plato, change for Parmenides was disturbing and thus illusionary. In his theory of Forms and Ideals, Plato postulates that things around us are nothing but a copy of their perfect images.These images, he claims, are invariant, stable - transcending the world of senses - they lie beyond space and time and can be only visualized through the mind. Plato living in a period of great social disturbance was trying to arrest all forms of change by discovering unchanging rules in a 'world of ideals' to a world of flux based on the concept of relativism. He further suggests that his world of ideals is the real world and that what we percieve in our everyday life is actually an illusion, an imperfect copy of this perfect reality. Plato was trying to make a projection of the world of senses (which, being only appearance, is deceptive) onto the world of Forms (which he claimed to be the world of things as they are). On the other hand, Paramenidus said "All is One. Nor is it divisible, wherefore it is wholly continuous. It is complete on every side like the mass of a rounded sphere." In the seventeenth century, at the same time that classical physics ushered a renewed boost to atomism and reductionism, Spinoza developed a holistic philosophy, oriented more to ethics and morality, reminiscent of Parmenides. According to Spinoza, all the differences and apparent divisions we see in the world are really only aspects of an underlying single substance, which he called God or nature. All things existing are therefore only ripples on a universal ponde. This ardent emphasis on an underlying unity is reflected in the mystical thinking of most major spiritual traditions. Hegel, too, had mystical visions of the unity of all things, on which he based his own holistic philosophy of nature and the state. In his holistic philosophy of society he argued using dialectical logic or simply dialectics; progress occurs as a result of the unity of thesis and antithesis, the outcome is their synthesis. Progress is achieved due to, not inspite of, contradictions. The main consensus among dialecticians is that dialectics do not violate the law of contradiction of formal logic, although attempts have been made to create a 'paraconsistent logic' which is 'inconsistency tolerant'. Quoting Hegel: 'When the difference of reality is taken into account, it develops from difference into opposition, and from this into contradiction, so that in the end the sum total of all realities simply becomes absolute contradiction within itself.' Hegel's state is a quasi-mystical collective, an 'invisible and higher reality,' from which participating individuals derive their authentic identity, and to which they owe their loyalty and obedience. Karl Marx 'turned Hegel on his head' by turning the idealistic dialectic into a materialistic one, in proposing that material circumstances shape ideas, instead of the other way around. In this, Marx was following the lead of Feuerbach. All modern collectivist political thinkers - including, of course, Marx - stress some higher collective reality, the unity, the whole, the group, though nearly always at the cost of minimizing the importance of difference, the part, the individual. Against individualism, all emphasize that the social forces can be viewed as one social entity that somehow possess a holistic character and have a will of their own, not only over and above the characters, but in a way transcending the wills of individual members. Where atomism was apparently legitimized by the stupendous sucesses of quantum physics, holism found no such solace in the basic sciences, like mathematics and physics. It remained a change of emphasis rather than a new philosophical position. There were attempts to found holsim in other basic sciences. For example the idea of organism in biology, the emergence of biological form and the cooperative relation between biological and ecological systems; but these, too, were ultimately reducible to simpler parts, their properties, and the relation between them. Even systems theory, although it highlights the complexity of aggregates, it does so in terms of causal feedback loops between various constituent parts. It is only with quantum theory and the holistic approach to general relativiity, based on a (logistic positivist) axiomatic approach, allowing the maneavering of attack of what once belonged to the 'why' (rather than the 'how') question(s), a new form of deep holism is emerging. In this paper, our prime concern is to give a comprehensive survey of this new holism in dealing with mathematics and physics. The exhilarating work of Andréka and Németi in axiomatizing new physics, is a new venture into the realm of the axiomatic method applied to relativity theory and quantum mechanics. It raises deep questions of the form, do we need new axioms? Is such an axiom better than another? is this or that axiom a redundency? The central problem in the philosophy of natural science is when and why the sorts of facts scientists cite as evidence really are evidence, *a fortoiri* the same is true in the case of mathematics and physics. Historically, philosophers have given considerable attention to the question of when and why various forms of logical inference are truth-preserving and what is the real guide of our search of new axioms is. The word 'intution ' is vague enough as it stands, and does nothing but renaming the real problem. For some reason, there has been little attention to the understanding and classification of the sorts of facts mathematical scientists choose, let alone to the philosophical question of when and why those facts constitute evidence. The question of how the unproven can be justified is especially pressing in current set theory, where the search goes on for new axioms, that certainly goes far beyond self-evident facts, to determine the size of the continuum one way or another. (This will be further elaborated upon in a while) This is a pressing problem till this moment of time; it is is also perhaps one of the deepest, if not the deepest that contemporary mathematics presents to the contemporary philosopher of mathematics. Progress towards understanding the process of mathematical hypothesis formation and confirmation contribute to our philosophical understanding of the nature of mathematics. It might even be of solace to those mathematicians actively engaged in the axiom search. According to the Andréka and Németi unfolding project for the last decade, the same can be said out the mind boggling new advances in contemporary physics. We give a few substantial examples of how fragmentation can indeed be devastatingly counterproductve.

1.2 Fragmentation and subjectivism; Gödel and the Bourbaki group

The famous french Nicholas Bourbaki admit they they did not fully absorb the limitative results of Gödel, until in the late eighties of the last century [43]. There are numerous first class french mathematicians, to name a few, Legendre, Laplace, Lagrange, Fourier, Cauchy, Galois, Dirichlet, Poincare. This consistutes a most impressive list of scholars of the highest caliber. According to the pioneering impact of France on the development of mathematics starting in the renaissanse, in 1935 a group of young French mathematicians decided to put their efforts together and to write a number of textbooks under the joint psuedonym Nicholas Bourbaki. These textbooks aimed to cover with (French) rigour and precision what they considered to be the most important areas of mathematics. Five years earlier, in September 1930 there was a meeting at Konigsberg in during which honorary citizenship was conferred on Hilbert who had retired from his Chair at Gottingen on January 23rd of that year. On this memorable occasion, Hilbert gave his famous and profound speech *Naturerkennen und Logik* claiming that there are no such thing as 'insoluble problems.' With the delicate irony of history, Gödel had the very day before, with von Neumann but not Hilbert in the audience, announced his first incompleteness proof. A few days later, he declared his second incompleteness theorem, but after a discussion with an enthuthiastic Von Neumann who had actually arrived at the same reslt independently during the few days before the two lectures. Gödel's second incompleteness theorem says in a nutshell that we cannot prove that mathematics is consistent by methods formalizable within a substantially small fragment of set theory, unless it is inconsistent in which case we can prove anything. The proof consists of formalizing in this small fragment the proof of his first incompleteness Theorem. To that Von Neumann (who on his own reached the same conclusion) reacted: ' God exists because mathematics is consistent, but the devil also exists because we cannot prove it.'

But why did Gödel prove his incompleteess theorems? To know the reason we need to go back further in history. In August 1900 the world's best mathematicians gathered in Paris for the Second International Congress of mathematicians. Hilbert chose to use his lecture not to look back over some recent work, but rather to point the way towards the future He presented to the meeting not one but a list of twenty three major unsolved problems, problems whose solutions, if found, would each mark a significant advance in mathematics. Though the choice of the problem was obviously subjective, Hilbert was so knowledgeable and prolific, that his selection of problems practically ran the gamut of the fields of mathematics of his day, from the pure to the applied, and from the most general to the more specific. But the work on some of them led to an extraodinary amount of important mathematics in the 20 century and beyond. In fact, certain mathematicians would become famous for solving one or another of Hilbert's problems. Most of these problems were (or become) known by specific names as the continuum problem, or the Reimann problem (Reimann's hypothesis). Fermat's last Theorem also appeared on this famous list only to be solved almost a century after by the British mathematician Andrew Wiles. The problems 1,2, and 10 in his list adressed mathematical logic, better known back then as meta-mathematics, and Problem 6, on the list was simply *'Axiomatize physics.* Hilbert was initially a deductivist, but he considered certain metamathematical methods to yield intrinsically meaningful results and was a realist with respect to finitary arithmetic. Hilbert's first was concerned with the continuum hypothesis CH and the well ordering of $\mathbb{R}$. Hilbert's second problem, called for a proof of consisteny of arithmetical axioms, to secure the consistency of the whole of mathematics. In 1900 Hilbert was a bit vague in stating just which axioms he had in mind. But when he turned to logic in full scale in 1920s he made quite specific what axioms were to be considered. Not to beg the question he placed a strong restriction on the proof methods 'finitary' being one of them. In response to these questions: Gödel came up with Gödel's incompleteness in 1931 (proved in ZF without the axiom of infinity) so his proof conformed to Hilbert's restrictions, Zermelo explicity introduced the Axiom of Choice (AC) in 1918 giving the first axiomatization of set theory to show that any set *a fortiori* $\mathbb{R}$ can be well ordered, answering part of the first question. Cóhen proved the independence of the CH and the AC in 1963 answering the rest of the first question. Cóhen's proofs were the inaugural examples of a new technique, forcing, which was to become a remarkably general and flexible method for extending models of set theory. Hilbert's tenth was solved in the 1970s by Yuri Matiyasevich preceded by substantial contributions from Martin Davis, Putnam, Julia Robinson. This was done by showing that Diophantine sets and recursively enumerable sets coincide on $\mathbb{Z}$ hence Hilbert's tenth problem is algorithmically undecidable. The analagous problem for Diophantine equations on $\mathbb{Q}$ is open. Hilbert believed, that in mathematics there is not any 'ignorabimus' (statement whose truth can never be known). It seems unclear whether he would have regarded the solution of the tenth problem as an instance of ignorabimus: what is proved not to exist is not the integer solution, but (in a certain sense) the ability to discern in a specific way whether a solution exists. On the other hand, the status of the first and second problems is even more promlematic: there is not any clear mathematical consensus as to whether the results of Gödel (in the case of the second problem), or Gödel and Cohen (in the case of the first problem) give conclusive negative solutions or not, since these solutions apply to a a particular formalization of the problems, which is not necessarily the only possible one. Nevertheless, at the time, one would expect Godel's results, constituting an impossibility resullt to Hilbert's second problem to cause an earthquake. Indeed, Hilbert had presented a very positive response to the 'solvability of any problem' and oustanding disciples such as Alfred Tarski and Herbrand had in the Hilbertian spirit established cases where Hilbert was right in his highly optimistic anticipation. But then, against widespread expectations at the time, particularly by Hilbert and Bertrand Russell, Gödel shocked the mathematical community by the mind boggling fact that there were serious limitations to Hilbert's programme. Given the importance of this result for foundational studies, and for the philosophy of mathematical logic, and given the eager response of von Neumann and others to Gödel's ideas, it is natural to ask what effect Gödel had on the Bourbachistes. One searches their publications in vain for mention of Gödel's name, far less his astounding results. One then comes to the conclusion that the Bourbachistes almost completely neglected Gödel together with his results with the sole exception that the tone of a small, if not a tiny, part of their works suggests a conflict between an uneasy awareness that something drastic has happened and a semi-concious desire to pretend that it has not. Quoting A.R.D. Mathias in [43] : 'It is as though they had discovered that they were on an island with a dragon and in response chose to believe that if the dragon were given no name it would not exist.' This attitude unfortunately, due to the 'moral authority' of the Bourbaki group passed on to younger generations of mathematicians. This dissmissive and indeed harmful approach excludes awareness of perceptions of the nature of mathematics that are stupendously significant, inspiring, healthy and invigorating. It is sometimes said, now more often than before that Bourbaki has died. But has he? The answer to this question is unfolding and controversial; for exampe Matiaus claims that this is the case indeed [43]. But in case he did, Bourbaki can blame no one but himself. Matiaus goes on to declare that the Borbaki group was killed by his own limited perceptions and the sterility of his own dissmissive attitudes. In other words Bourbaki committed suicide.

Another instance of a limited view to set theory, why did the Bourbaki group not notice the inadequacy of their chosen set theory as a foundation for mathematics? The answer is simply - as brilliantly declared in [43]- because they were solely interested in areas of mathematics for which Zermelo is adequate, and that this area may broadly be described as geometry which is based on $\mathbb{R}$ whose theory is decidable, a result of Tarski's as opposed to arithmetic whose theory is undecidable as proved by Gödel. This is the result of fragmentation and sujectiveness in mathematics. Bourbaki presents a pre-Gödelian view of mathematics, and of a portion of mathematics biassed towards geometry. The ignorance of Bourbaki concerning Gödel's shattering results cannot be disputed. Furthermore, the negative repercussions of this ignorance cannot be just simply brushed aside.

Gödel's incompleteness theorems was not only a blow Hilbert's programme, but also to the logistic view envisaged by Gottlob Frege. Gottlob Frege is regarded as the greatest philosopher of logic since Aristotle for developping his 1879 *Begriffsschrift* establishing a logical foundation for arithmetic, and generally stimulating the analytic tradition in philosophy. The artichect of that tradition is Berrand Russell who in his early years influenced by Frege and Peano, wanted to found all of mathematics, on the certainty of logic. The vaulting expression of that ambition was the 1910 three voumes *Principia Mathematica* by Whitehead and Russell. But Russell was exercused by his well known paradox, one which led to the tottering of Frege's mature formal system. It remained for Hilbert to shift the ground and establish mathematical logic as a field of mathematics. Russell's philosophical disposition preluded his axiomatizing logic.

1.3 Fragmentation in physics

A typical example in Physics, that took place a long time ago, is the neglection by the scientific community of Newton's assumption about the nature of light: Newton conjectured that light is composed of discrete particles, that is, light is nothing but a stream of particles moving in a straight line. The experimental evidence culminating in the eighteenth century, however, showed that light unquestionably posessed wave properties (for example, it has been shown that light undergoes both diffraction and interference phenomena). Finally, Maxwell's work on electromagnetic phenomena was not only a confirmation of the continuous nature of light, but also proved, on solid mathematical foundations, that light is nothing but (a special type of) an electromagnetic wave moving in space (with a certain frequency). It was only at the turn of the 20-th century that Newton's ideas were again revived in Einstein's work on the photo-electric effect. To explain the emission of electrons from the surface of (some kinds of) metals when being bombarded by light, Einstein postulated that light has a grainy nature and is composed of discrete particles which he called quanta, In doing so, Einstein was actually using Planck's hypothesis proposed five years earlier on the nature of (electromagnetic) radiation. Contrary to the classical Maxwell's theory, Planck had to assume that electromagnetic radiation is composed of discrete units to explain the phenomenon of the black body radiation in an attempt to resolve (what was known as) the ultra-violet catastrophy. Had Newton's ideas been taken seriously, the dual nature of light could have been discovered much earlier giving birth to the quantum theory a long time before the turn of the 20-th century. Through the creative language of metaphores, a dynamic process of interaction and communication would be the guiding rule for the emergence of novel ideas.

These new ideas come into being, not through the breakdown or downfall of older concepts, but rather by selecting, classifying, relating, and creating different and wider contexts in which older theories may reveal original and unpredictable features. Concepts that might appear as incomensurable or mutually irrelevent and mutually exclusive in a narrow(er) context, may be combined, related, or even identified; thereby proving to be merely different manifestations of one and the same phenomenon - when viewed from a broader perspective. Such reconcilation of seemingly unrelated or even apparently contradictory phenomenon has been the mechanism by which the boundaries of scientific knowledge expand. To be sure it is the essence of scientific development: if particle and wave, (properties) appear to be mutually exclusive concepts in the classical realm (Newtonian's and Maxwell's theories), they are, however, according to the quantum theory, different aspects of the same pattern reflecting a kind of intrinsic duality in the behaviour of atomic particles. In the world of the small, they actually represent two sides of the same coin; so that together they constitute the essence of a micro-particle.It was an intuitive belief in the symmetry of nature that made the French physicist de-Broglie propose the following daring and deep idea: Not only light, under certain circumstances, may reveal corposcular nature (as Einstein has rightly discovered), *but all objects (matter) are associated by (matter) waves.* It was a few years later, through the work of Max Born, Schrodinger, Heisenberg, Dirac and others that such a revolutionary conception was further elaborated in the probabilistic language of the quantum theory resulting in a subtle form of duality between matter and radiation. In the language of metaphores, such an identification may be expressed succintly in the form a wave is a particle. Another illuminating example is revealed in the development of our understanding of electromagnetism. Throughout the eighteenth century, physicsts dealt with electrical and magnetic effects in nature as two completely seperate phenomena, described by mutually irrelevent physical theories. A partial unification of these two (mutually irrelevent) theories was first accomplished by the James Maxwell. It was not until 1905, however, when Einstein came up with his special theory of relativity, that a complete identification and unification of electromagnetic phenomena was established. If Maxwell succeeded to show that a varrying magnetic and electric fields co-exist together to form a single entity (an electromagnetic wave), Einstein went a step further, demonstrating that one and the same field may be either magnetic, electrical or a combination of both when viewed from different angles; the nature of the field revealed being dependent on the state of motion of the (inertial) observer studying it. Thus Einstein's new theory not only showed a perfect symmetry between electric and magnetic fields (a fact already partially recognized within the context of Maxwell's theory), but also established a unification of two fragments into one whole, through the metaphore: Electricity is magnetism and magnetism is electricity.

1.4 Axiomatizing set theory

The axioms of set theory should be axioms in the Greek sense, self evident statements, which form a partial basis to deductively arrive at other truths. Current reserach in set theory, however, shows that this need not be the case. The proposed additional axioms are far from being intuitive, they are judged rather externally by their pluasible consequences. In fact such expansions of ZFC can be contradictory, but each is hopefully self consistent; though it is proved that this cannot be proved. In spite of this, they may even have same plausible consequence like settling the continuum hypothesis. There is no obvious and compelling unique path of axioms that supplemented ZFC and settle important independent problems. But grossemode in current set theoretic landscape two opposing theoretic approaches dominate to solve the continuum problem and its likes. The most complete and concise of these two stems from Gódel's constructible universe L. Gódel showed that CH and AC hold in L, and in fact $\mathsf{V} = \mathsf{L}$ answers most (if not all) outstanding questions concerning sets. The other approach begins from a style of axioms for which Gódel had high hopes, namely, *large cardinal* axioms, or introducing new axioms of *determinacy*. The second theory begins from a style of axioms for which Gódel had high hopes namely large cardinal axioms. The first of those large cardinals are the inaccessible cardinals. The idea is that the two main operations for generating new sets from old postulated by ZFC replacement and power set are not enough to exhaust all ordinals. This is ripe for generalization. The heuristic of generalization from ω_0 also came to be used to motivate various large cardinals. Recalling Cantor's unitary view of the finite and the transfinite large cardinal properties satisfied by ω would be too accidental if they were not also ascribable to higher cardinals in an eternal recurrence. A cardinal κ is inaccessible if it is uncountable, it is not a sum of fewer than κ cardinals that are less than κ, and $\alpha < \kappa \implies 2^\alpha < \kappa$. A large cardinal is necessarily inaccesible. Large cardinal axioms cannot be proved from ZFC, because if they can, and κ is such, then the universe of sets V_κ will be a model of ZFC contradicting Gódel's incompleteness Theorems. in fact the following cannot be proved: If κ is a large cardinal, then "consistency of ZFC implies consistency of ZFC + the existence of this large cardinal" is consistent. While Cohen's creation of forcing transformed set theory, large cardinal hypothesis played an increasingly prominent role as a consequence. Solovay's inspiring result that if there is an inacesible cardinal, then in an inner model of a forcing extension every set of reals is measurable. Shelah proved the converse, that is to say, if all sets of reals are measurable, then thee exists an inaccessible cardinal. There are other even larger types of cardinals: Mahlo, weakly compact, ineffable, measurable, Ramsey supercompact, huge, n huge and Woodin cardinals.This is enough to make Kronecker turn in his grave infinitely many times! Kronecker believed that only the natural numbers exist, and any other numbers even the rationals is only a byproduct of our minds; they do not have an independent existence. The elaboration of the subject of large cardinals has almost outrun the names that have been introduced for various large cardinal notions, witness (in roughly increasing order of strength): 'inaccessible', 'Mahlo', 'weakly compact', 'indescribable', 'subtle', ' by set theorists involved in this development. A complicated web of relationships has been established, as witnessed by charts to be found in the recent book by Aki Kanamori, 'The Higher Innite' p. 471. A rough distinction is made between small large cardinals, and large large cardinals, according to whether they are weaker or stronger, in some logical measure or other, than measurable cardinals. The beginning threads of the subject are picked when weak inaccessibility and Mahloness arose in the study of cardinal limit processes and their strong versions. This, in turn, led to early speculations about completeness and consistency. Ulam formulated *measurability* the most prominant of all large cardinal hypothesis, out of a measure problem for sets of reals. Gódel's work on the constructible universe, is the the true beginining of axiomatic set theory as a distinctive field of mathematics , since in both reaction and generalization it shaped much of the sork on large cardinals. Weak and strong compactness, are concepts that emerged for Tarski's study of infinitary languages, and Hanf's result that are many inaccessibles below a measurable cardinal. Scott used ultrapower constructions, in his pivotal result that if there is a measutrable cardinal then not every set is constructible. The existence of a measurable cardinal (which is a large large cardinal) contradicting $\mathsf{V} = \mathsf{L}$, is the pivotal result that caused the aforementioned dichotomy of investigations in set theory. The histrocial development of descriptive set theory, that sets the stage for further results about large cardinals and projective sets, gave us a major direction of set theoretic research from the mid 1960's onwards. Solovay germinal work on Σ^1_2 sets that grew out of his measure result, and his results and conjectures on definability of sharps, was further pursued and developed by Jack Silver from Berkley. Martin used sharps to extend the methods of classical descriptive set theory to the analysis of Σ^1_3 sets.

The philosopher, Penelope Maddy, in two interesting highly informative articles called 'Believing the axioms' [36], analyzed the various kinds of arguments to accept large cardinals. The assumption of large cardinals has been fruitful through the dazzling work of Solovay, Martin, Foreman, Magidor, Shelah, Steel, Woodin and others in extending standard properties of Borel and projective subsets of the continuum, such as Lebesgue measurability, the Baire property, the perfect subset property, determinateness of associated in nite games, etc. to substantially larger classes. Indeed, the assumption of the existence of certain large large cardinals, imply certain *determinacy axioms*. The so-called full Axiom of determinacy AD refers to certain two-person topological games of length ω. AD states that every game of a certain type is determined. The axiom of determinacy is inconsistent with AC, and it implies implies that all subsets of the real numbers are Lebesgue measurable, have the property of Baire, and the perfect set property. The last implies a weak form of the CH, namely, that every uncountable set of real numbers has the same cardinality as $\mathbb{R}$. The axiom of projective determinacy (PD) states that for any two-player infinite game of perfect information of length in which the players play natural numbers, if the victory set (for either player, since the projective sets are closed under complementation) is projective, then one player or the other has a winning strategy, so that such game are not *fully deterministic*. The axiom is not a theorem of ZFC (assuming ZFC is consistent), but unlike the so-called full axiom of determinacy which is fully deterministic as the name might suggest and contradicts the axiom of choice, it is not known to be inconsistent with ZFC. PD follows from certain large cardinal axioms, such as the existence of infinitely many Woodin cardinals or the existence of a super compact cardinal. Logicians study the relative consistency of different axiomatic theories in terms of *consistency strength*. One theory has greater consistency strength than another if its consistency implies the consistency of the other. A central problem in mathematical logic is to determine the consistency strength in this ordering of any given piece of mathematics. One of the weakest systems is Peono arithmetic but even weaker systems have been found. Analysis finds its place in a a stronger theory called second order arithmetic. Still stronger theories arise when we axiomatize set theory itself. The standard version ZFC is still quite weak, although the gap between it and analysis is large, in the sense that many mathematical results require more than ordinary analysis, but less than the whole of ZFC. Beyond ZFC, large cardinals hold sway. The proofs of determinacy from large cardinal assumptions are not related by implication but rather by *relative consistency strengths. There appears to be a central axis of axioms to which all independent propositions are comparable in consistency strength. This axis is delineated by large cardinal axioms. There are no known counterexamples to this behaviour. Such results seem to support the formalist view.*

On the other hand, quoting Fefermann: 'But the striking thing, despite all this progress, is that contrary to Gödel's hopes, the Continuum Hypothesis (recall that it was abbreviated by CH) is still undecided by these further axioms, since it has been shown to be independent of all remotely plausible axioms of infinity, including that have been considered so far (assuming their consistency) That may lead one to raise doubts not only about Gödel's program but also about its very presumptions. Is CH a definite problem as Gödel and many current set-theorists believe? Is the continuum itself a definite mathematical entity? If it has only Platonic existence, how can we access its properties? Alternatively, one might argue that the continuum has physical existence in space and/or time. But then one must ask whether the mathematical structure of the real number system can be identified with the physical structure, or whether it is instead simply an idealized mathematical model of the latter, much as the laws of physics formulated in mathematical terms are highly idealized models of aspects of physical reality. (Hermann Weyl raised just such questions in his 1918 monograph Das Kontinuum) But even if we grant some kind of independent existence, abstract or physical, to the continuum, in order to formulate CH we need to refer to arbitrary subsets of the continuum and possible mappings between them, and then we are dealing with objects of a higher level of abstraction, the nature of whose existence is even more problematic than that of the continuum. Here we are skirting deep philosophical waters; let us retreat from them for the moment.'

1.5 Algebraic logic and new physics, a holistic point of view

Sometimes a holistic view becomes necessary to make progress. The unification of many scattered theories that seem incompatible is certainly warranted in any scientific field. In this case a holistic view attempting a unification of the apparently contradictory coexisting theories at hand, becomes not a luxury but a must to unlock a deadlock. This for example what happened in new physics in the 1980s in *string theory*, when Edward Witten (though a physisict is a Field Meddlar's holder) unified five seemingly incompatible string theories using duality to how that they are in essence 'dual phenomena' giving birth to Supestring theory. Given Witten's kiss of Life, Superstring theory is now mainstream new physics. There are many famous occasions in the history of both mathematics and phyics were fragmentation, and sometimes intentional overlooking or ignoring of apparently remote but relevant research areas, has led to confusion and ultimately had undesirable consequences if not calamaties. Fragmentation tends to coincide with limited progress, while a necessary condition that real progress be achieved is the existence of some holistic viewpoint, rather than drowning in the highly intricate sophisticated parts. This does not mean then one ignores the details; on the contrary, at some point to complete the picture the most intricate details must be meticulously worked out. This only entails that one should not, or rather cannot ignore the holistic overall viewpoint. The feedback between the details and the overall picture is a two way one; it is an unended quest. The holism view is primary motivated by the fact that the whole is more (revealing) than its parts.

The topic of this paper is algebraic logic in a broad sense and its novel applications to the realm of new physics. Ideas are approached as coherent aggregates whose component parts are best understood in context and in relation to one another and to the whole. Having said that, we adopt a holitic approach to a wide spectrum of topics in both modern physics and mathematics. Several branchces ranging from mathematical logic to general relativity will be dealt with in a consice but comprehensive, and hopefully comprehensible, way. Our approach though is closer to that of the pure mathematician, as our underlying philosophy is hugey influenced by fomalism with a sprinkle of realism. During the course of our investigations subtle similarities between seemingly distinct situations or/ and concepts will be highlighted. For example various representation thoreoms in several branches in mathematics is represented as adjoint situatins (using category theory jargon), and duality in a looser and broader sense will be applied to topics such as the theory of sheaves and topoi.

Initiated at the beginning of the 20th century, formal logic and its study using mathematical machinery known as metamathematical investigations, or simply metamathematics, is a crucial addition to the collection of mathematical catalysts. Traced back to the works of Frege, Hilbert, Russell, Tarski, Gödel and others, one of the branches of pure mathematics that metamathematics has precipitated is algebraic logic. Algebraic logic approaches certain special logical situations, places them in a general algebraic context, via a process called *algebraization* and via this generalization makes contact with other branches of mathematics (like set theory and topology). In algebraic logic a deeper understanding of both (universal) algebra and logic is not hidden by the largely irrelevant details of a particular logical system; it is instead illuminated and guided structurally by clear–cut causality. Hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down. One aim of the subject termed *abstract algebraic logic* is to determine the domain of validity of such and such metatheorem (e.g. the completeness theorem, the Craig interpolation theorem, or the Orey-Henkin omitting types theorem of first order logic) and to give general formulations of metatheorems in broader, or even entirely other contexts. This kind of investigation is extremely potent for applications and helps to make the distinction between what is really essential to a particular logic and what is not. Algebraic logic unifies properties of a multiplicity of logics, the trees, by developing general means and concepts which allows a uniform treatment of their meta theories; the forest with the metaphor being : 'Do not miss the forest for the trees' This is reminescent of category theory whose main concern is to highlight adjoint situations in various branches of mathematics.

During the 20th century, numerous logics have been created, to mention only a few: intuitionistic logic, modal logic, topological logic, topological dynamic logic, spatial logic, dynamic logic, tense logic, temporal logic, many-valued logic, fuzzy logic, relevant logic, para-consistent logic, non monotonic logic, etc. The rapid development of computer science, since the fifties of the 20th century initiated by work of giants like Gödel, Church and Turing, ultimately brought to the front scene other logics as well, like logics of programs and lambda calculus (the last can be traced back to the work of Church). After a while it became noticeable that certain patterns of concepts kept being repeated, albeit in different logics. But then due to the emergence of so many logical systems, the time was ripe to make an abstraction to a 'higher level' leading to *univeral logic, (abstract) algebraic logic, and abstract model theory* (Lindstrom's theorem is an example for the last abstraction.) The term "universal logic' was introduced in the 1990s by Swiss Logician Jean Yves Beziau but the field has arguably existed for many decades. Some of the works of Alfred Tarski in the early twentieth century, on metamathematics and in algebraic logic, for example, can be regarded undoubtedly, in retrospect, as fundamental contributions to both *abstract* algebraic logic and universal logic. Universal logic together with abstract algebraic logic have travelled well in both model theory and logic. The way we think of logic and model theory will never be the same as before. The development of model and proof theory in the setting of abstract algebraic logic is free of commitment to a particular logical system. When we live without concrete models, we reap the harvest of another level of abstraction and generality. Abstract algebraic logic provides a unifying algebraic framework viz the process of algebraisation. Other approaches to universal aspects of logical systems include the theory of *institutions*.

The continuous interplay between the specific and the general in algebraic logic brings a large array of new results for particular non–conventional approaches, unifies several known results, produces new results in well-studied conventional areas, and finally reveals previously unknown causality relations. Algebraic logic can be viewed as a vehicle where several ingredients interact, most notably model theory, set theory, finite combinatorics, game theory, and graph theory. *Algebraizing predicate logic*, a task primarily initiated by Tarski and further pursued and substantially enriched by the pioneering work of Andréka and Németi proved an extremely rewarding task. We can find that the theory of cylindric–like algebras is explicated primarily in three substantial monographs: Henkin, Monk and Tarski [22], and Henkin, Monk, Tarski, Andréka and Németi [23], the latter containing a sample of Andréka and Németi's substantial contribution to the subject up to the mid–eighties of the last century. The recent [5], referred to as 'Cylindric Algebras, Part 3' edited by Andréka et al, the notation of which we follow here, gives a representative picture of the research in the area over the last thirty years, emphasizing the bridges that cylindric algebra theory–in the wide sense–has built with other areas, like combinatorics, graph theory, data base theory, stochastics and other fields. Not confined to the walls of pure mathematics, algebraic logic has also found bridges to such apparently remote areas as general relativity, and hyper-computation; a lot will be said about such intriguing interactions with mind boggling and mind blowing consequences. Since the introduction of sheaves into mathematics in the 1940s, a major theme has been to study a space by studying sheaves on a space. This idea was expounded by Alexander Grothendieck by introducing the notion of a 'topos'. The main utility of this notion is in the abundance of situations in mathematics where topological heuristics are very effective, but an honest topological space is lacking; it is sometimes possible to find a topos formalizing the heuristic. This will be done below in algebraic logic. Topoi behave much like the category of sets and possess a notion of localization; they are a direct generalization of point-set topology. A topos is a category that has the following two properties: All limits taken over finite index categories exist. Every object has a power object. This plays the role of the powerset in set theory. The Grothendieck topoi find applications in algebraic geometry; the more general elementary topoi are used in logic. Below we shall extend forcing constructions from set theory to certain topoi. Topoi theory expressed in categoy theory, universal and algebraic logic and the theory of institutions are all holistic approaches to certain mathematical situations, the main aim being not to miss the forest for the trees. For example adjoint situations in algebraic logic were studied in [66]. Also several patterns are recurrent in such formalisms; but drawing analogies between these holistic approaches to different branches in mathematics, and in particular, in algebra, category theory and algebraic and universal logic, needs another paper, most probably even longer than the present paper. The temporal (historical) development of a certain topic (particularly in mathematics) does not neccesarily coincide with the most logical one, for newly discovered results often shed light on older ones, resulting in a deeper understanding of both. In (algebraic) logic new paradigms usually present themselves in the form of new systems competing with old ones. This suggests a fresh look at existing logical systems, rather than their speedy overthrow. Both developments do not even necessarily coincide with the simplest, which we aspire to achieve in this paper. The topics we focus are those related *mainstream Tarskian algebraic logic, hypercomputation, new physics as developed by*

Andréka and Németi and their students (including myself). Many references are included for those who wish to dig deeper. Our main aim is fostering and encouraging interdisciplinarity participation in in mathematics and physics. Notation used is common or /and self–explonatory; it is consistent with the notation in [5, 22].

2 Cylindric algebras

For a set V, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : \exists t \in X(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

$\langle \mathcal{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if M is a first order structure in a first order signature L with α many variables, then one manufactures a cylindric set algebra based on M as follows. Let

$$\phi^{\mathsf{M}} = \{s \in {}^{\alpha}\mathsf{M} : \mathsf{M} \models \phi[s]\},$$

(here $\mathsf{M} \models \phi[s]$ means that s satisfies ϕ in M), then the set $\{\phi^{\mathsf{M}} : \phi \in Fm^L\}$ is a cylindric set algebra of dimension α, where Fm^L denotes the set of first order formulas taken in the signature L. To see why, we have:

$$\begin{aligned} \phi^{\mathsf{M}} \cap \psi^{\mathsf{M}} &= (\phi \wedge \psi)^{\mathsf{M}}, \\ {}^{\alpha}\mathsf{M} \sim \phi^{\mathsf{M}} &= (\neg\phi)^{\mathsf{M}}, \\ \mathsf{C}_i(\phi^{\mathsf{M}}) &= \exists v_i \phi^{\mathsf{M}}, \\ \mathsf{D}_{ij} &=: (x_i = x_j)^{\mathsf{M}}. \end{aligned}$$

Following [22], Cs_α denotes the class of all subalgebras of full set algebras of dimension α. The (equationally defined) CA_α class is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations given in [22, Definition 1.1.1] that holds of course in the more concrete set algebras. (This is soundness condition).

Definition 2.1. Let α be an ordinal. By *a cylindric algebra of dimension* α, briefly a CA_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{\kappa,\lambda<\alpha}$$

where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra such that $0, 1$, and d_{ij} are distinguished elements of A (for all $j, i < \alpha$), $-$ and c_i are unary operations on A (for all $i < \alpha$), $+$ and $.$ are binary operations on A, and such that the following equations are satisfied for any $x, y \in A$ and any $i, j, \mu < \alpha$:

(C_1) $\mathsf{c}_i 0 = 0$,

(C_2) $x \leq \mathsf{c}_i x$ $(i.e., x + \mathsf{c}_i x = \mathsf{c}_i x)$,

(C_3) $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$,

(C_4) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$,

(C_5) $\mathsf{d}_{ii} = 1$,

(C_6) if $i \neq j, \mu$, then $\mathsf{d}_{j\mu} = \mathsf{c}_i(\mathsf{d}_{ji} \cdot \mathsf{d}_{i\mu})$,

(C_7) if $i \neq j$, then $\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$.

For operators on classes of algebras, **S** stands for the operation of forming subalgebras, **P** stands for that of forming products, and **H** stands for the operation of forming homomorphic images. The varieties of representable algebras of dimension α, α an ordinal, for all such classes are defined as follows: $\mathsf{RDf}_\alpha = \mathbf{SP}\mathsf{DfCs}_\alpha$ and $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$,

An algebra $\mathfrak{A} \in \mathsf{CA}_\omega$ is *locally finite*, if the dimension set of every element $x \in A$ is finite. The dimension set of x, or Δx for short, is the set $\{i \in \omega : \mathsf{c}_i x \neq x\}$. Locally finite algebras correspond to Tarski–Lindenbaum algebras of (first order) formulas; in such algebras the dimension set of (an equivalence class of) a formula reflects the number of (finite) set of free variables in this formula. Tarski proved that every locally finite ω-dimensional cylindric algebra is representable, i.e. isomorphic to a subdirect product of set algebra each of dimension ω. Let Lf_ω denote the class of locally finite cylindric algebras. Let RCA_ω stand for the class of isomorphic copies of subdirect products of set algebras each of dimension ω, or briefly, the class of ω dimensional representable cylindric algebras. Then Tarski's theorem reads $\mathsf{Lf}_\omega \subseteq \mathsf{RCA}_\omega$. This representation theorem is non-trivial; in fact it is equivalent to Gödel's celebrated Completeness Theorem [22, §4.3]. Completeness in the general case is a huge subject that has provoked extensive research.

The restrictive character of the dimension ω and local finiteness were removed in the early course of the development of the subject, and the class CA_α, of cylindric algebras of dimension α, where α is any ordinal, finite or transfinite, was introduced. Three pillars in the development of the subject, and even one can say *the* three pillars in the development of the subject are Tarski's representability result of loclly finite algebras, Henkin's characterization of the variety of representable algebras of any dimension via neat embeddings, in his celebrated Neat Embedding Theorem [22, Theorem 3.2.10], and Monk's proof that the variety of representable algebras of dimension > 2 cannot be axiomatized by a finite schema [48]. Monk's result had a shattering effect on the development of the subject. The last two results involve the central notion of neat reducts:

Definition 2.2. Let $\alpha < \beta$ be ordinals and $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$–neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \sim \alpha$, and restricting the universe to the set $Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}$.

Let α be any ordinal. If $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathsf{CA}_\beta$ $(\beta > \alpha)$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a *β–dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathsf{K} \subseteq \mathsf{CA}_\beta$, and $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{K} = \{\mathfrak{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in \mathsf{K}\} \subseteq \mathsf{CA}_\alpha$. One can show that for any ordinal α, $\mathfrak{A} \in \mathsf{RCA}_\alpha \iff \mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega}$, cf. [22, Theorem 2.6.35]. The last equivalence is Henkin's celebrated neat embedding theorem. For $2 < n < \omega$, what Monk proved is that for any $k \in \omega$, there is an algebra $\mathfrak{A}_k \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k} \sim \mathsf{RCA}_n$, such that the ultraproduct $\Pi_{k/U}\mathfrak{A}_k/U \in \mathsf{RCA}_n$ for any non–principal ultrafilter on U. This implies that RCA_n is not finitely axiomatizable. If the variety of representable cylindric algebras of dimension at least three had turned out to be axiomatized by a finite schema, algebraic logic would have evolved along a significantly different path than it did in the past fifty years, or so. This would have undoubtfully marked the end of the abstract class CA_α (α an ordinal) as a separate subject of research; after all why bother about abstract algebras, if a few nice extra axioms can lead us from those to concrete algebras consisting of genuine relations, with set theoretic operations uniformy defined over these relations. However, due to Monk's non–finitizability result, together with its improvements by various algebraic logicians (from Andréka to Venema) CA_α was here to stay and its "infinite distance' from RCA_α, when $\alpha > 2$, became an important central research topic. *Monk's non-finite axiomatizability result marked the end of an era and the beginning of a new one.*

3 Non-finite axiomatizability of RCA_n and its repercussions

3.1 First path: Andréka's splitting

Here we review Andréka's methods of splitting to obtain non finite aximatizability results for varities of representable algebras. The idea of splitting one or more atoms in an algebra to get a (bigger) superalgebra tailored to a certain purpose seems to originate with Henkin [22, p.378, footnote 1]. Let

$2 < n < \omega$. In the cylindric paradigm, Andŕeka modified such splitting methods *re-inventing* (Andréka's) splitting. In this new setting, Andréka proved a plethora of *relative non–finite axiomatizability results*. For example Andréka proved that the class of representable polyadic algebras with equality of dimension n is not finitely axiomatizable over those without equality nor over RCA_n when $2 < n < \omega$. In the former case Andréka went further excluding universal axiomatizations containing only finitely many variables, is lifted to the transfinite in [62]. The theme in splitting methods is typically of the form: *Split some (possibly all) atoms in an algebra (that need not be finite nor even atomic) each into one or more subatoms forming a bigger superalgebra that constitutes the starting point for a construction serving the purpose at hand. For cylindric–like algebras of dimension α, α an ordinal, if the atom $x \in \mathfrak{A}$ is split to the subatoms $X = (x_i : i \in I)$ in $\mathfrak{B} \supseteq \mathfrak{A}$, then X is a partition of x in the sense that $x_l \cdot x_m = 0$ for $l \neq m \in I$, and $\sum_{i\in I}^{\mathfrak{B}} x_i = x$. Furthermore, for each $i \in I$, x_i is cylindrically equivalent to x, in the sense that for all $j < \alpha$, $\mathsf{c}_j^{\mathfrak{B}} x_i = \mathsf{c}_j^{\mathfrak{B}} x$.* This roughly means that cylindrifiers, the most prominent citizens in such algebras, cannot distinguish between an atom and its splitted subatoms. So splitting, leading from $\mathfrak{A}$ to $\mathfrak{B}$ does not ruin the algebraic structure dramatically, at least as far as cylindrifiers are concerned. But at the same time, we show that the subtle technique of splitting, undetected by cylindrifiers, can lead from a representable $\mathfrak{A}$ to a non–representable $\mathfrak{B}$ (Andréka's splitting) and conversely from a non–representable $\mathfrak{A}$ to a representable $\mathfrak{B}$ (blow up and blur constructions). Such construction will be given below. We get more technical as far as Andréka's spliting is concerned proving:

Theorem 3.1. *(Andréka [1]) For any $2 < n \leq \omega$, the variety RCA_n cannot be axiomatized by a set of universal formulas containing only finitely many variables*

Proof. We prove the Theorem for $n = \omega$.

(1) Forming a non-representable algebra by splitting an atom in a set algebra: Let $2 < n < \omega$. For each k we construct a non-representable algebra, denoted by $\mathsf{split}(\mathfrak{A}, R, m)$ short for splitting an atom R in a set algebra $\mathfrak{A}$ into m abstract subatoms where m is a postive number depending on k. This algebra has the same signature as CA_n and we will show that all of its k generated subalgebras (that is to say, subalgebras generated by at most k elements) are in RCA_n. So fix such a k and let $m \geq 2^k + 1$. Let $\langle U_i : i \leq \omega\rangle$ be a system of mutually disjoint sets such that $|U_i| = m - 1$. Let $U = \bigcup U_i$, $q \in \Pi_{i\in\omega} U_i$ be arbitary and $R = \{z \in \Pi_{i\in\omega} U_i : |\{i \in I : z_i \neq q_i\}| < \omega\}$. Let $\mathfrak{A} = \mathfrak{Sg}^{\wp({}^{\alpha}U)}\{R\}$. Then R is an atom in $\mathfrak{A}$. One forms the required algebra $\mathsf{split}(\mathfrak{A}, R, m)$ by splitting the atom R in the algebra $\mathfrak{A}$ into m abstract copies. R is partitioned into a family $(R_i : i < m)$ of atoms in the bigger algebra $\mathsf{split}(\mathfrak{A}, R, m)(\supseteq \mathfrak{A})$, so that $R = \sum_{i<m} R_i$, where (recalll that) $m = |U_0| + 1$. We require that $\mathfrak{A}$ is a subalgebra of $\mathsf{split}(\mathfrak{A}, R, m)$, that the R_is $(i < m)$ are cylindrically equivalent to R in $\mathsf{split}(\mathfrak{A}, R, m)$, each element of $\mathsf{split}(\mathfrak{A}, R, m)$ is a join of element of $\mathfrak{A}$ and some of the the R_is, and that the cylindrifcations in $\mathsf{split}(\mathfrak{A}, R, m)$ distribute over the Boolean join. An algebra $\mathsf{split}(\mathfrak{A}, R, m)$ satisfying such properties exists and is unique up to isomorphism that fixes $\mathfrak{A}$ poinwise [1, Theiorem 1]. But $\mathsf{split}(\mathfrak{A}, R, m)$ will not be in RCA_n for the following reasoning: Define the term $\tau(x) = (\bigwedge_{i<m} \mathsf{s}_i^0\mathsf{c}_1 \ldots \mathsf{c}_m x \cdot \bigwedge_{i<j<m} -\mathsf{d}_{ij})$ as in [1, Top of p.157]. Then $\mathfrak{A} \models \tau(R) = 0$ hence $\mathsf{split}(\mathfrak{A}_n, R, m) \models \tau(R) = 0$ because $\mathfrak{A} \subseteq \mathsf{split}(\mathfrak{A}, R, m)$. Identifying set algebras with their domain, for an algebra $\mathfrak{A}$ and a non–zero $a \in \mathfrak{A}$, we say that a representation $h : \mathfrak{A} \to \wp({}^\omega U)$ *respects the non–zero element* a if $h(a) \neq \emptyset$. If $\mathsf{split}(\mathfrak{A}, R, m)$ were representable, then it will have a representation that respects R. But any such representation h will satisfy [1, Theorem 1] that $\tau(h(R)) \neq 0$ which is impossible.

(2) Representability of k–generated subalgebras: Now we show that the k–generated subalgebras are representable. Let $G \subseteq \mathsf{split}(\mathfrak{A}, R, m)$, $|G| \leq k$. Let $\mathfrak{P} = (R_l : l < m)$ be the abstract partition of R in the bigger algebra $\mathsf{split}(\mathfrak{A}, R, m)$ obtained by splitting R in $\mathfrak{A}$ into m (abstract) subatoms $(R_l : l < m)$. One defines the following relation on $\mathfrak{P}$: For $l, t < m$, $R_l \sim R_t \iff (\forall g \in G))(R_l \leq g \iff R_t \leq g)$. Then it is straightforward to check that $\sim$ is an equivalence relation on $\mathfrak{P}$ having $p < m$ many equivalence classes, because $|G| \leq k$, $m \geq 2^k + 1$. One next takes $B = \{a \in A_k : (\forall l, t < m)(R_l \sim R_t, R_l \leq a \implies R_t \leq a)\}$, then $G \subseteq B$, $R \in B$, and B is closed under the operations, so that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathsf{split}(\mathfrak{A}, R, m)$, where $\mathfrak{B}$ is the algebra with universe B. Furthermore, $\mathfrak{B}$ is the smallest such subalgebra of $\mathsf{split}(\mathfrak{A}, R, m)$, where for each $i, j < n$, R is partitioned into $p < m$ many parts cylindrically equivalent to R. The non-representability of the algebra $\mathsf{split}(\mathfrak{A}, R, m)$ can be pinned down to the existence of 'one more extra atom' leading to the incomptability condition $|U_0| < m$ (= number of subatoms) witnessed by the term τ using diagonal elements. Using that $|\mathfrak{P}| = m$, we showed that a representation h of $\mathsf{split}(\mathfrak{A}, R, m)$ that respects R, has to respect the atoms below it, and this forces that $|U_0| \geq m$, which contradicts the construction of $\mathfrak{A}$. But this cannot happen with $\mathfrak{B}$, because $p < m$ (by the condition $|G| \leq k$)), so that this 'one more extra atom and possibly more' vanish in $\mathfrak{B}$. Representing $\mathfrak{B}$ is done by embedding it into a representable algebra $\mathfrak{C}$ having the same top element as $\mathfrak{A}$, namely, ${}^\omega U^{(q)}$, where $R \in \mathfrak{C}$ is partitioned *concretely into* $m-1$ *real atoms*, that is, there exists $R_l \subseteq {}^\omega U^{(q)}$, $l < m-1$ *real atoms* in $\mathfrak{C}$ such that $R = \bigcup_{l<m-1} R_l = \bigcup_{l<m-1} R_l$ and $\mathsf{C}_i R_l = \mathsf{C}_i R$ for all $l < m-1$ and $i < \omega$. *This concrete partition exists because* $|U_0| = m-1$ and by the condition $|G| \leq k$, the value of p, which is the new number of subatoms of R in $\mathfrak{B}$ (depending on G) cannot exceed $m-1$.

(3) Finishing the proof: We have constructed for each positive k, a non-representable algebra having the same signture as CA_n all of whose subalgebras generated by at most k elements is representable. Using fairly basic model theoretic arguments [1] the required follows. □

Andréka's splitting argument in constructing $\mathsf{split}(\mathfrak{A}, R, m)$ $(2 < n < \omega)$ from $\mathfrak{A}$ by splitting one atom into m–many subatoms, where m is finite $> 2^k+1$, is in fact an ingenious combination of Monk–like constructions with the concept of *dilations* in the sense of [22, Construction 3.6.69]. Andréka's construction avoids 'colouring' (expressed in Monk's construction by an application of Ramsey's theorem.) Roughly, in dilations one adds atoms to an atomic algebra, if it is not down right 'impossible' to do so, witness [22, Last paragraph p. 88]. The finite number $|U_0|$ plays the role of the number of colours used in Monk's original construction of 'bad (non-representable) Monk algebras. Here too, in 'Andréka's splitting argument' the incompatibility condition between the number of atoms m and the number of colours $|U_0|$ $(m > |U_0|)$, leads to an impossibility in case there is a representation of $\mathsf{split}(A, R, m)$; for the existence of a representation *concretely represents the m atoms below R* forcing $|U_0| \geq m$. In [3] it was asked whether RCA_n is closed under Dedekind-MacNeille completions. Contrary to canonical extensions (and possibly expectations), this question was finally settled negatively. Hodkinson [33] showed that for $2 < n < \omega$, RCA_n is not *atom–canonical* answering a question posd in [3]. This result is reproved in [57], where it is shown using relation algebras having n-dimensional cylindric basis in the sense of [37] that both RRA and RCA_n are not atom-canonical. By [71], it follows that L_n-as a multimodal logic-is not Sahlqvist axiomatizable. Atom-canonicity in completely additive varieties correspond in modal logic to the notion of a formula being *dipersistent* [15, §5.6]. To strengthen Hodkinson's result, we use in what follows instances of the so–called blow up and blur construction an indicative term invented by Andréka and Németi [6].

Definition 3.2. The *atomic game $G^m_k(\mathsf{At}\mathfrak{A})$, or simply G^m_k*, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [30, Definition 3.3.2] We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$. The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the m nodes in play.

For a class $\mathbf{K}$ of BAOs, we denote by $\mathbf{K}^{\mathsf{ad}}$ the class of completely additive algebras in $\mathbf{K}$. We write $\mathfrak{A} \subseteq_d \mathfrak{B}$ for $\mathfrak{A}$ is a *dense subalgebra* of $\mathfrak{B}$ and $\mathfrak{A} \subseteq_c \mathfrak{B}$ for $\mathfrak{A}$ a *complete subalgebra* of $\mathfrak{B}$.The following Lemma can be easily proved by using ideas in [70, Lemmata 29, 26, 27].

Lemma 3.3. *Let $2 < n < m$. If $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$, then $\exists$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.*

For rainbow constructions for CAs we follow [26, 30].The complex CA_n over this atom structure will be denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension of $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$, always finite and > 2, will be clear from context. The game $\mathbf{G}^m$ lifts to a game on coloured graphs, that is like the graph games G^m_ω [26], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [26, 4.3].

Using essentially the argument in [22, Lemma 5.1.50, Theorem 5.1.51] by considering closure under infinite intersections instead of intersections, we get:

Lemma 3.4. *Let $2 < n < \omega$. If $\mathfrak{A} \in \mathsf{CA}_n$ is such that its diagonal free reduct is in RDf_n, and $\mathfrak{A}$ is generated by $\{x \in \mathfrak{A} : \Delta x \neq n\}$ (with other* CA *operations) using infinite intersections, then $\mathfrak{A} \in \mathsf{RCA}_n$.*

Theorem 3.5. *Let n be a finite ordinal > 2. Then the varieties $\mathsf{SNr}_n\mathsf{CA}_{2n}$ and RDf_n are not atom-canonical.*

Proof. **Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure** At: Take the finite cylindric algebra rainbow algebra $\mathfrak{A}_{n+1,n}$ where the reds R is the complete irreflexive graph n, and the greens are $\mathsf{G} = \{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n+1\}$, endowed with the polyadic operations. Denote its finite atom structure by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{A}_{n+1,n})$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure $\mathbf{At}$. The resulting atom structure after 'splitting the reds', namely, $\mathbf{At}$, is like the weakly (but not strongly) representable atom structure of the atomic, countable and simple algebra $\mathfrak{A}$ as defined in [33, Definition 4.1]; the sole difference is that we have $n+1$ greens and not ω–many as is the case in [33]. We denote the algebra $\mathfrak{Tm}\mathbf{At}$ by $\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ short hand for blowing up $\mathfrak{A}_{n+1,n}$ by splitting each *red graphs (atoms)* into ω many. By a red graph is meant (an equivalence class of) a surjection $a : n \to \Delta$, where Δ is a coloured graph in the rainbow signature of $\mathfrak{A}_{n+1,n}$ with at least one edge labelled by a red label (some r_{ij}, $i < j < n$). It can be shown exactly like in [33] that $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using a shade of red ρ *outside* the rainbow signature, when she is forced a red; [33, Proposition 2.6, Lemma 2.7]. Using this, one proves like in *op.cit* that $\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ is representable as a set algebra having top element ${}^n\mathsf{M}$. In the present context, after the splitting 'the finitely many red colours' replacing each such red colour r_{kl}, $k < l < n$ by ω many r^i_{kl}, $i \in \omega$, the rainbow signature for the resulting rainbow theory as defined in [27, Definition 3.6.9] consists of $\mathsf{g}_i : 1 \leq i < n-1$, $\mathsf{g}^i_0 : 1 \leq i \leq n+1$, $\mathsf{w}_i : i < n-1$, $\mathsf{r}^t_{kl} : k < l < n$, $t \in \omega$, binary relations, and $n-1$ ary relations y_S, $S \subseteq_\omega n+k-2$ or $S = n+1$.

Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathsf{At}(\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)))$: Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We

can assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_a \sim N_b$, $(M, N \in \mathsf{CGR}) \iff$ they are identical everywhere except at possibly at red edges:
$M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$.
We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M has at least one red edge. Any red atom has ω many copies that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n}(= \mathfrak{Cm}\mathbf{At}_f)$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. We check preservation of all the CA_n operations. e dkip the Boolean operations.

- Diagonal elements. Let $l < k < n$. Then:

$$\begin{aligned} M_x \leq \Theta(\mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}_f}) &\iff M_x \leq \sum_j \bigcup_{a_l = a_k} M_a^{(j)} \\ &\iff M_x \leq \bigcup_{a_l = a_k} \sum_j M_a^{(j)} \\ &\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\ &\iff M_x \in \mathsf{d}_{lk}^{\mathfrak{Cm}\mathbf{At}}. \end{aligned}$$

- Cylindrifiers. Let $i < n$. By additivity of cylindrifiers, we restrict our attention to atoms $M_a \in \mathbf{At}_f$ with $a : n \to M$, and $M \in \mathsf{CRG}_f \subseteq \mathsf{CRG}$. Then:

$$\Theta(\mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}_f} M_a) = f(\bigcup_{[c] \equiv_i [a]} M_c) = \bigcup_{[c] \equiv_i [a]} \Theta(M_c)$$
$$= \bigcup_{[c] \equiv_i [a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] \equiv_i [a]} M_c^{(j)} = \sum_j \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} M_a^{(j)}$$
$$= \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}}(\sum_j M_a^{(j)}) = \mathsf{c}_i^{\mathfrak{Cm}\mathbf{At}} \Theta(M_a).$$

We leave the proof of the preservation of the substitution operations (corresponding to tranpositions) to the reader.

$\forall$ has a winning strategy in $G^{n+3}\mathsf{At}(\mathfrak{A}_{n+1,n})$: It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private

$\mathsf{EF}_{n+1}^{n+1}(n+1,n)$ [30, Definition 16.2] since $n+1$ is 'longer' than n. Using (any) $p>n$ many pairs of pebbles avalable on the board $\forall$ can win this game in $n+1$ many rounds. In each round $0,1\ldots n$, $\exists$ places a new pebble on a new element of $n+1$. The edge relation in n is irreflexive so to avoid losing $\exists$ must respond by placing the other pebble of the pair on an unused element of n. After n rounds there will be no such element, so she loses in the next round. It is not hard to show that the winning strategy of $\forall$ in the private Ehrenfeucht–Fraïssé game lifts to a winning strategy in the graph game $G_k^{n+3}(\mathsf{At}(\mathsf{CA}_{n+1,n}))$ [26, pp.841] for some finite k. $\forall$ lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game, to the graph game on $\mathbf{At}_f=\mathsf{At}(\mathfrak{A}_{n+1,n})$ [26, p 841] forcing a win using $2n$ nodes. He needs $n-1$ points for the bases of a cone and $n+1$ appexes of cones to reveal that the reds coming from the complete irreflexive graph n are not enough. He bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). Then by Lemma 3.3, $\mathfrak{A}_{n,+1,n}\notin\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{2n}$, but $\mathfrak{A}_{n+1,n}$ is finite, so $\mathfrak{A}_{n+1,n}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{2n}$. Since $\mathfrak{A}_{n+1,n}$ embeds into $\mathfrak{CmAt}\mathfrak{A}$, then $\mathfrak{CmAt}\mathfrak{A}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{2n}$ and we are done.

RDf_n **is not atom-canonical:** It is enough to show that $\mathfrak{CmAt}\mathfrak{A}$ is generated by elements whose dimension sets have cardinality $<n$ using infinite unions. We show that for any rainbow atom $[a]$, $a:n\to\Gamma$, Γ a coloured graph, that $[a]=\prod_{i<n}\mathsf{c}_i[a]$. Clearly $\leq$ holds. Assume that $b:n\to\Delta$, Δ a coloured graph, and $[a]\neq[b]$. We show that $[b]\notin\prod_{i<n}\mathsf{c}_i[a]$ by which we will be done. Because a is not equivalent to b, we have one of two possibilities; either $(\exists i,j<n)(\Delta(b(i),b(j)\neq\Gamma(a(i),a(j))$ or $(\exists i_1,\ldots,i_{n-1}<n)(\Delta(b_{i_1},\ldots,b_{i_{n-1}})\neq\Gamma(a_{i_1},\ldots,a_{i_{n-1}}))$. Assume the first possibility (the second is similar): Choose $k\notin\{i,j\}$. This is possible because $n>2$. Assume for contradiction that $[b]\in\mathsf{c}_k[a]$. Then $(\forall i,j\in n\setminus\{k\})(\Delta(b(i),b(j))=\Gamma(a(i)a(j)))$. By assumption and the choice of k, $(\exists i,j\in n\setminus k)(\Delta(b(i),b(j))\neq\Gamma(a(i),a(j)))$, contradiction. $\square$

Metalogical repercussions: The reader is referred to [27, Definitions 13.4, 13.6] for the notions of m–flat and m–square representations for relation algebras ($m>2$) that can routinely generalized for CA_ns. [60, §5, p.14]. The main ideas used in the next Theorem can be found in [27, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27] adapted to the CA case. In all cases, the m–dimensional dilation stipulated in the statement of the Theorem, will have top element $\mathsf{C}^m(\mathsf{M})$ as defined in *op.cit*, where M is the m–relativized representation of the given algebra, and the operations of the dilation are induced by the n-clique–guarded semantics. A complete proof is in [60]. For a class $\mathbf{K}$ of BAOs, $\mathbf{K}\cap\mathsf{At}$ denotes the class odf atomi algbrs in $\mathbf{K}$.

Theorem 3.6. *[27, Theorems 13.45, 13.36]. Assume that* $2<n<m<\omega$ *and let* $\mathfrak{A}\in\mathsf{CA}_n$. *Then* $\mathfrak{A}\in\mathbf{S}\mathsf{Nr}_n\mathfrak{D}_m\iff\mathfrak{A}$ *has an* m*–square representation*

$\iff$ *$\mathfrak{A}$ has an m–flat representation. Furthermore, if $\mathfrak{A}$ is atomic, then $\mathfrak{A}$ has a complete m–square representation $\iff$ $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n(\mathfrak{D}_m \cap \mathbf{At})$.*

Fix $2 < n \leq l < m \leq \omega$. Consier the statement $\Psi(l,m)$: *There is an atomic, countable and complete L_n theory T, such that the type Γ consisting of co–atoms is realizable in every m– square model, but any formula isolating this type has to contain more than l variables.* By an m–square model of T we understand an m–square representation of the algebra $\mathfrak{Fm}_T$. Let $\mathsf{VT}(l,m)) = \neg\Psi(l,m)$, short for *Vaught's theorem holds 'at the parameters l and m'* where by definition, we stipulate that $\mathsf{VT}(\omega,\omega)$ is just Vaught's theorem for $L_{\omega,\omega}$: Countable first order atomic theories have countable atomic models. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, it is likely and plausible that *(**):* $\mathsf{VT}(l,m) \iff l = m = \omega$. In other words: *Vaught's theorem holds only in the limiting case when $l \to \infty$ and $m = \omega$ and not 'before'.* The following definition to be used next is taken from [6]:

Definition 3.7. [6, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non–identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. We say that (J,E) is a *strong n–blur*, if it (J,E) is an n–blur, such that the complex n–blur (with notation as in *op.cit*) satisfies: $(\forall V_1,\ldots V_n, W_2,\ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$.

From Theorems 3.5 and using the main construction in [6]) we get:

Corollary 3.8. *For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n, 2n)$ and $\Psi(l,\omega)$ hold. Furthermore, if for each $n < m < \omega$, there exists a finite relation algebra $\mathfrak{R}_m$ having $m-1$ strong blur and no m-dimensional relational basis, then (**) above for VT holds.*

Proof. Only the last part desreves proof. Let $\mathfrak{R}_m$ be as in the hypothesis with strong $m-1$–blur (J,E) and m-dimensional relational basis. We 'blow up and blur' $\mathfrak{R}_m$ in place of the Maddux algebra $\mathfrak{E}_k(2,3)$ blown up and blurred in [6, Lemma 5.1], where $k < \omega$ is the number of non–identity atoms and k depends recursively on l, giving the desired strong l–blurness, cf. [6, Lemmata 4.2, 4.3]. The relation algebra $\mathfrak{R}_m$ is blown up by splitting all of the atoms each to infinitely many giving a new infinite atom structure $\mathbf{At}$ denoted in [6, p.73] by At. One proves that the blown up and blurred atomic relation algebra $\mathsf{split}(\mathfrak{R}_m, J, E)$ (as denoted by $\mathfrak{Bb}(\mathfrak{R}_m, J, E)$ in [6]) with atom structure $\mathbf{At}$ is representable; in fact this representation is induced by a complete representation of its canonical extension, cf. [6, Item (1) of Theorem 3.2]. The change in notation here is to stress the fact that $\mathfrak{Bb}(\mathfrak{R}_m, J, E)$ is obtained from $\mathfrak{R}_m$ by splitting atoms. Throughout the proof the same notation is used, we always replace $\mathfrak{Bb}$ by split. Because (J,E) is a strong l–blur, then, by its definition, it is a strong j–blur for all $n \leq j \leq l$, so the atom structure $\mathbf{At}$ has a j–dimensional cylindric basis for all $n \leq j \leq l$, namely, $\mathsf{Mat}_j(\mathbf{At})$. For all such j, there is an RCA_j which we denote by $\mathsf{split}_j(\mathfrak{R}_m, J, E)$ (denoted on [6, Top of p. 9] by $\mathfrak{Bb}_j(\mathfrak{R}_m, J, E)$) such that $\mathfrak{TmMat}_j(\mathbf{At}) \subseteq \mathsf{split}_j(\mathfrak{R}_m, J, E) \subseteq \mathfrak{CmMat}_j(\mathbf{At})$ and $\mathsf{Atsplit}_j(\mathfrak{R}_m, J, E)$ is a weakly representable atom structure of dimension j, cf. [6, Lemma 4.3]. Now take $\mathfrak{A} = \mathsf{split}_n(\mathfrak{R}_m, J, E)$. We claim that $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ and thta $\mathfrak{A}$ has no m-square represenation. Since $\mathfrak{R}_m$ has a strong j–blur (J, E) for all $n \leq j \leq m-1$, then $\mathfrak{A} \cong \mathfrak{Nr}_n\mathsf{split}_j(\mathfrak{R}_m, J, E)$ for all $n \leq j \leq m-1$ as proved in [6, item (3) p.80]. In particular, taking $j = m-1$, $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_{m-1}$. Assume for contradicton that $\mathfrak{CmAtA}$ does have an m–representation representation M. Then M is infinite of course. Since $\mathfrak{R}_m$ embeds into $\mathsf{split}(\mathfrak{R}_m, J, E)$ which in turn embeds into $\mathfrak{RaCmAtA}$, then $\mathfrak{R}_m$ has an m–square representation with base M. But since $\mathfrak{R}$ is finite, $\mathfrak{R} = \mathfrak{R}^+$, so $\mathfrak{R}_m$ has an m–dimensional relational basis, contradiction.

A complete m–square representation of an atomic $\mathfrak{B} \in \mathsf{CA}_n$ induces an m–square representation of $\mathfrak{CmAtB}$ which implies by Theorem 3.6 that $\mathfrak{CmAtB} \in \mathsf{SNr}_n\mathsf{D}_m$. To see why, assume that $\mathfrak{B}$ has an m–square complete representation via $f : \mathfrak{B} \to \mathfrak{D}$, where $\mathfrak{D} = \wp(V)$ and the base of the representation $\mathsf{M} = \bigcup_{s \in V} \mathsf{rng}(s)$ is m–square. Let $\mathfrak{C} = \mathfrak{CmAtB}$. For $c \in C$, let $c\downarrow = \{a \in \mathsf{At}\mathfrak{C} : a \leq c\} = \{a \in \mathsf{At}\mathfrak{B} : a \leq c\}$. Define, representing $\mathfrak{C}$, $g : \mathfrak{C} \to \mathfrak{D}$ by $g(c) = \sum_{x \in c\downarrow} f(x)$, then g is the required homomorphism into $\wp(V)$ having base M. We prove $\Psi(m-1, m)$, hence the required, namely (**). Now by [22, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–square model, for if M is an m–square model omitting Γ, then M would be the base of a complete m–square representation of $\mathfrak{A}$, and so by Theorem 3.6 $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathfrak{D}_m$ which is impossible. Suppose for contradiction that ϕ is an $m-1$ witness, so that $T \models \phi \to \alpha$, for all $\alpha \in \Gamma$, where recall that Γ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss that $\mathfrak{A}$ is a set algebra with base M say. Let $\mathsf{M} = (M, R_i)_{i \in \omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [22, §4.3]. Let ϕ^{M} denote the set of all assignments satisfying ϕ in M. We have $\mathsf{M} \models T$ and $\phi^{\mathsf{M}} \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_{m-1}$. But $T \models \exists x\phi$, hence $\phi^{\mathsf{M}} \neq 0$, from which it follows that ϕ^{M} must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let ψ be the formula, such that $\psi^{\mathsf{M}} = \alpha$. Then it cannot be the case that $T \models \phi \to \neg\psi$, hence ϕ is not a witness, contradiction and we are done. □

Coming back full circle we reprove strong non-finite axiomatizibility results refining Monk's obtained by Maddux and Biro. Let $2 < n \leq l < m \leq \omega$. In $\mathsf{VT}(l, m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. Using elementary calculas terminology one can view $\mathsf{lim}_{l \to \infty}\mathsf{VT}(l, \omega) = \mathsf{VT}(\omega, \omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$ with strong l–blur (J_l, E_l) and $f(l) \geq l$ as specified in [6, Lemma 5.1] (denoted by k therein). Let $\mathcal{R}_l = \mathsf{split}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathsf{split}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$. Then $(\mathsf{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. Let LCA_n denote the class of CA_ns satisfying the Lyndon conditions.

Corollary 3.9. *(Monk, Maddux, Biro, Hirsch and Hodkinson) Let $2 < n < \omega$. Then the set of equations using only one variable that holds in each of the varieties RCA_n and RRA, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety [14, 37]. Furthermore, LCA_n is not finitely axiomatizable.*

Remark 3.10. Due to its importance we give an entirely different proof to Theorem 3.9. Let $2 < n < \omega$. We start off by proving a statement that could be of interest intself giving atom structures of relation and cylindric algebras of dimension n that are weakly but not strongly representable in one go. We claim that here exists an atomic countable relation algebra $\mathfrak{R}$, such that $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ forms an n–dimensional cylindric basis, $\mathfrak{A} = \mathfrak{TmMat}_n(\mathsf{At}\mathfrak{R}) \in \mathsf{RQEA}_n$, while even the diagonal free reduct of the Dedekind-MacNeille completion of $\mathfrak{A}$, namely, $\mathfrak{CmMat}_n(\mathsf{At}\mathfrak{R})$ is not representable. To prove the alleged statement, let $\mathcal{G}$ be a graph. Let ρ be a 'shade of red'; we assume that $\rho \notin \mathcal{G}$. Let L^+ be the signature consisting of the binary relation symbols (a, i), for each $a \in \mathcal{G} \cup \{\rho\}$ and $i < n$. Let T denote the following (Monk) theory in this signature: $\mathsf{M} \models T \iff$ for all $a, b \in \mathsf{M}$, there is a unique $p \in (\mathcal{G} \cup \{\rho\}) \times n$, such that $(a, b) \in p$ and if $\mathsf{M} \models (a, i)(x, y) \wedge (b, j)(y, z) \wedge (c, k)(x, z)$, $x, y, z \in \mathsf{M}$, then $|\{i, j, k\} > 1$, or $a, b, c \in \mathcal{G}$ and $\{a, b, c\}$ has at least one edge of $\mathcal{G}$, or exactly one of a, b, c – say, a – is ρ, and bc is an edge of $\mathcal{G}$, or two or more of a, b, c are ρ.
We denote the class of models of T which can be seen as coloured undirected graphs (not necessarily complete) with labels coming from $(\mathcal{G} \cup \{\rho\}) \times n$ by $\mathbf{G}$. Now specify $\mathcal{G}$ to be either:
(i) the graph with nodes $\mathbb{N}$ and edge relation E defined by $(i, j) \in E$ if $0 < |i - j| < N$, where $N \geq n(n-1)/2$ is a postive number,
(ii) the ω disjoint union of N cliques, same N.
In both cases the countably infinite graphs contain infinitey many N cliques. In the first they overlap, in the second they do not. One shows that there is a countable (n–homogeneous) coloured graph (model) $\mathsf{M} \in \mathbf{G}$ with the following property [33, Proposition 2.6]:
If $\triangle \subseteq \triangle' \in \mathbf{G}$, $|\triangle'| \leq n$, and $\theta : \triangle \to \mathsf{M}$ is an embedding, then θ extends to an embedding $\theta' : \triangle' \to \mathsf{M}$. Here the choice of $N \geq n(n-1)/n$ is not haphazard of course; it bounds the number of edges of any graph Δ of size $\leq n$. This is crucial to show that for any permutation χ of $\omega \cup \{\rho\}$, Θ^χ is an n-back-and-forth system on M [57], so that the countable atomic set algebra $\mathfrak{A}$ based on M whose top element is obtained from ${}^n\mathsf{M}$ by discarding

assignments whose edges are labelled by one of n–shaded of reds $((\rho, i) : i < n))$ forming $W \subsetneq {}^n\mathsf{M}$, is classically representable. The classical semantics of $L_{\omega,\omega}$ formulas and relativized semantics (restricting assignmjents to W), coincide, so that $\mathfrak{A}$ is isomorphic to a set algebra with top element ${}^n\mathsf{M}$. Consider the following relation algebra atom structure $\alpha(\mathcal{G}) = (\{\mathsf{Id}\} \cup (\mathcal{G} \times n), R_{\mathsf{Id}}, \breve{R}, R_;)$, where: The only identity atom is Id. All atoms are self converse, so $\breve{R} = \{(a, a) : a \text{ an atom }\}$. The colour of an atom $(a, i) \in \mathcal{G} \times n$ is i. The identity Id has no colour. A triple (a, b, c) of atoms in $\alpha(\mathcal{G})$ is consistent if $R;(a, b, c)$ holds ($R;$ is the accessibility relation corresponding to composition). Then the consistent triples are (a, b, c) where: One of a, b, c is Id and the other two are equal, or none of a, b, c is Id and they do not all have the same colour, or $a = (a', i), b = (b', i)$ and $c = (c', i)$ for some $i < n$ and $a', b', c' \in \mathcal{G}$, and there exists at least one graph edge of G in $\{a', b', c'\}$. The algebra $\mathfrak{C}$ is not representable because $\mathfrak{Cm}(\alpha(\mathcal{G}))$ is not representable and $\mathsf{Mat}_n(\alpha(\mathcal{G})) \cong \mathsf{At}\mathfrak{A}$. To see why, for each $m \in \mathsf{Mat}_n(\alpha(\mathcal{G}))$, let $\alpha_m = \bigwedge_{i,j<n} \alpha_{ij}$. Here α_{ij} is $x_i = x_j$ if $m_{ij} = \mathsf{Id}$ and $R(x_i, x_j)$ otherwise, where $R = m_{ij} \in L$. Then the map $(m \mapsto \alpha_m^W)_{m \in \mathsf{Mat}_n(\alpha(\mathcal{G}))}$ is a well - defined isomorphism of n-dimensional cylindric algebra atom structures. Non-representability follows from the fact that $\mathcal{G}$ is a 'bad' graph, that is, in the sense that its chromatic number, namely, $\chi(\mathcal{G})$ is finite; in fact $\chi(G) = N(< \infty)$ [27, Definition 14.10, Theorem 14.11]. Non-representability of the diagonal free reduct of $\mathfrak{C}$ follows from Lemma 3.4. The relation algebra atom structure specified above is exactly like the one in Definition 14.10 in *op.cit*, except that we have n colours rather than just three. Having the above contruction at hand, let $\mathfrak{A}_l$ be the atomic $\mathsf{RCA_n}$ constructed from $\mathcal{G}_l$, $l \in \omega$ where $\mathcal{G}_l$ is a disjoint countable union of N_l cliques, such that for $i < j \in \omega$, $n(n-1)/n \leq N_i < N_j$. Then $\mathfrak{Cm}\mathfrak{A}_l$ with $\mathfrak{A}_l$ based on $\mathcal{G}_l$, as constructed above is not representable. So $(\mathfrak{Cm}(\mathfrak{A}_l) : l \in \omega)$ is a sequence of non–representable algebras, whose ultraproduct $\mathfrak{B}$, being based on the ultraproduct of graphs having arbitrarily large chromatic number, will have an infinite clique, and so $\mathfrak{B}$ will be completely representable [30, Theorem 3.6.11]. Applying Lós Theorem we get the required.

Theorem 4.2 implies that any complete axiomatization of any algebraizable (in the standard Blok-Pigozzi sense) formalism of $L_{\omega,\omega}$ in a signature containing infinitely many relation symbols must contain infinitely many relational atomic formulas. We mention another incompleteness result. We denote by L_ω the 'classical' more basic algebraizable typeless extension of $L_{\omega,\omega}$ with usual Tarskian square semantics dealt with in [22, §4.3]. For provability we use the basic proof system in [22, p. 157, §4.3] which is a natural extension of a complete calculas for $L_{\omega,\omega}$ expressed in terms of so-called restricted formula.

Theorem 3.11. *For any $k \geq 1$, there is no finite schemata of L_ω whose set Σ of instances satisfies $\Sigma \vdash_{\omega+k} \phi \Longleftrightarrow \vdash_{\omega+k+1} \phi$.*

Proof. It suffices to show that for any positive $k \geq 1$, the variety $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+k+1}$ is not axiomatizable by a finite schema over $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+k}$. We start by the finite dimensional case, then we lift the construction to the transfinite. Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [27, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [27, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$. Then for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_n$, $\mathfrak{C}(m,n,r) \notin \mathsf{SNr}_m\mathsf{CA}_{n+1}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$, cf. [27, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13]. Take

$$x_n = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega); m \leq j < n \to \exists i < m, f(i,j) = \mathsf{Id}\}.$$

Then $x_n \in \mathfrak{C}(n,n+k,r)$ and $\mathsf{c}_i x_n \cdot \mathsf{c}_j x_n = x_n$ for distinct $i,j < m$. Furthermore (*), $I_n : \mathfrak{C}(m,m+k,r) \cong \mathfrak{Rl}_{x_n}\mathfrak{Rd}_m\mathfrak{C}(n,n+k,r)$ via the map, defined for $S \subseteq H_m^{m+k+1}(\mathfrak{A}(m+k,r),\omega))$, by

$$I_n(S) = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega) : f \upharpoonright {}^{\leq m+k+1}m \in S,$$
$$\forall j(m \leq j < n \to \exists i < m, f(i,j) = \mathsf{Id})\}.$$

We have proved the (known) result for finite ordinals > 2. To lift the result to the transfinite, we proceed like in [32]. Let α be an infinite ordinal. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let F be an ultrafilter on I such that $\forall \Gamma \in I$, $M_\Gamma \in F$. For each $\Gamma \in I$, let ρ_Γ be an injective function from $|\Gamma|$ onto Γ. Let $\mathfrak{C}_\Gamma^r$ be an algebra similar to CA_α such that $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}_\Gamma^r = \mathfrak{C}(|\Gamma|, |\Gamma|+k, r)$ and let $\mathfrak{B}^r = \Pi_{\Gamma/F \in I}\mathfrak{C}_\Gamma^r$. Then we have $\mathfrak{B}^r \in \mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k}$ and $\mathfrak{B}^r \notin \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+k+1}$. These can be proved exactly like the proof of the first two items in [32, Theorem 3.1]. The second part uses that the element x_n satisfies for all $i \neq j \in m$, $\mathsf{c}_i x_n \cdot \mathsf{c}_j x_n = x_n$ and $\mathsf{s}_i^j x_n x \cdot \mathsf{s}_j^i x_n = x_n$. This is crucial to guarantee that in the algebra obtained after relativizing to x_n, we do not lose commutativity of cylindrifiers; the relativized algebra stays inside CA_n. We know from the finite dimensional case that $\Pi_{r/U}\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}_\Gamma^r = \Pi_{r/U}\mathfrak{C}(|\Gamma|, |\Gamma|+k, r) \subseteq \mathfrak{Nr}_{|\Gamma|}\mathfrak{A}_\Gamma$, for some $\mathfrak{A}_\Gamma \in \mathsf{CA}_{|\Gamma|+\omega} = \mathsf{CA}_\omega$. Let $\lambda_\Gamma : \omega \to \alpha+\omega$ extend $\rho_\Gamma : |\Gamma| \to \Gamma$ ($\subseteq \alpha$) and satisfy $\lambda_\Gamma(|\Gamma|+i) = \alpha+i$ for $i < \omega$. Let $\mathfrak{F}_\Gamma$ be a $\mathsf{CA}_{\alpha+\omega}$ type algebra such that $\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma = \mathfrak{A}_\Gamma$. Then $\Pi_{\Gamma/F}\mathfrak{F}_\Gamma \in \mathsf{CA}_{\alpha+\omega}$, and we have proceeding like in the proof of item 3 in [32, Theorem 3.1]: $\Pi_{r/U}\mathfrak{B}^r = \Pi_{r/U}\Pi_{\Gamma/F}\mathfrak{C}_\Gamma^r \cong \Pi_{\Gamma/F}\Pi_{r/U}\mathfrak{C}_\Gamma^r \subseteq \Pi_{\Gamma/F}\mathfrak{Nr}_{|\Gamma|}\mathfrak{A}_\Gamma = \Pi_{\Gamma/F}\mathfrak{Nr}_{|\Gamma|}\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma = \mathfrak{Nr}_\alpha\Pi_{\Gamma/F}\mathfrak{F}_\Gamma$. But $\mathfrak{B} = \Pi_{r/U}\mathfrak{B}^r \in \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega}$ because $\mathfrak{F} = \Pi_{\Gamma/F}\mathfrak{F}_\Gamma \in \mathsf{CA}_{\alpha+\omega}$ and $\mathfrak{B} \subseteq \mathfrak{Nr}_\alpha\mathfrak{F}$, hence it is representable (here we use the neat embedding theorem). The rest follows using a standard Lós argument. □

3.2 Second path; positive results:

Finitizability attempts and guarding: The second path is to try and sidestep complex axiomatizations of RCA_α for $\alpha > 2$, often referred to as *taming methods.* One seeks to find a variety of 'representable concrete algebras' that is *finitely axiomatizable by a recursive set of equations*, is known in the literature as the *finitizability problem* (FP) [49, 55, 50, 60]. Henkin, Monk and Tarski formulated the FP this way: *Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based variety* [1, 8, 50, 55, 13, 70, 27, 60, 22, 62]. The solution for this problem for first order logic without equality is due to Sain [50] using the so-called semigroup approach. A solution for first order logic with equality is in [60].

Another highly fruitful and effective way of getting rid of such 'severe incompleteness', resulting from non-finite axiomatizability results is to *relativize* semantics and *guard* syntax. The following theorem relates the semantics of a (possibly infinitary) formula ϕ in a *generalized model* to the semantics of its *guarded version*, denoted by $\mathsf{guard}(\phi)$, in the standard part of the model expanded with the guard. Let $\alpha \leq \omega$. Let L_α denote the algebriazable formalism corresponding to CA_α as defined in [22].

Theorem 3.12. *Let L be a signature taken in L_α. Let (M, V) be a generalized model in L, that is, M is an L–structure and $V \subseteq {}^\alpha\mathsf{M}$ is the set of admissible assignments. Assume that R is an α–ary relation symbol outside L. For ϕ in L, let $\mathsf{guard}(\phi)$ be the formula obtained from ϕ by relativizing all quantifiers to one and the same atomic formula $R(\bar{x})$ and let $\mathsf{Guard}(\mathsf{M}, V)$ be the model expanding M to $L \cup \{R\}$ by interpreting R via $R(s) \iff s \in V$. Then the following holds:*

$$\mathsf{M}, V, s \models \phi \Longleftrightarrow \mathsf{Guard}(\mathsf{M}, V), s \models \mathsf{guard}(\phi),$$

where $s \in V$ and ϕ is a formula.

Proof. By induction on complexity of formulas. □

The philosophy of guarding is to try and tame the wild and often unruly behavior of L_n with standard semantics which manifests itself in a long list of negative results for RCA_n to name a few: Infinite axiomatizability, any equational axiomatization must contain infinitely many variables and infinitely many non-canonical equations, *a fortiori* non-Sahlqvist ones [16], and Theorem 3.5, undecidability of telling whether a finite CA_n has a representation ..., and the list goes on [16, 37, 1, 30, 60]. The core of the idea of guarding is to find a semantics that give just the right action while additional effects of square set–theoretic representations are separated out as negotiable decisions of formulation that can threaten completeness and decidability. Using square semantics is a voluntary commitment to one particular mathematical implication whose complexity seems to be an overkill. An insidious term that often confuses this issue is the 'concreteness of set theoretic models' and the pre–assumption of the canonicity of 'simple' square ones; guarding intriguingly reveals the exact opposite. The square ones are the most complicated 'concealing' many far better well behaved multi-modal logics. In the guarded fragment, put forward by Andréka, Van Benthem and Németi, one looks at quantification patterns. Only relativized quantification (along the accessibility relation of the Kripke frame) is allowed. In modal formula of the guarded fragment complexity results deciding validity of sentences is complete for double exponential time, but the n-variable fragments of the guarded fragments are $\mathsf{EXPTIME}$ complete, and some 2-variable fragments are even in PSPACE, cf. [13] for a thorough overview.

Fix $n < \omega$. The notion of guarding syntax accompanied by relativizing semantics can be traced back to the classical Andréka–Resek–Thompson result [8] reproved in [60, 46] which says that every n–dimensional algebra that has the same signature as CA_n, satisfying a certain finite set of equations together with the the so–called *merry go round identies*, is representable by set algebras whose top elements are *diagonizable* in the following sense: If V $(\subseteq {}^nU)$ is the top element of a given set algebra of dimension n, then $s \in V \implies s\circ[i|j] \in V$. We say that $V \subseteq {}^nU$ is *locally square* if whenever $s \in V$ and $\tau : n \to n$, then $s \circ \tau \in V$. Let D_n (G_n) be the class of set algebras whose top elements are diagonizable (locally square) and operations are defined like cylindric set algebra of dimension n relativized to the top element V.

Theorem 3.13. *[8, 46, 4]. Fix $2 < n < \omega$. Then D_n and G_n are finitely axiomatizable and have a decidable universal (hence equational) theory.*

Proving decidability for guarded fragments of first order logic went historically via the *mosaic method* of Németi's, later developed for guarded fragments to so–called quasi–models, which is a mixture of *filtration*, a well known technique for proving decidability results in modal logic, mosaics and semantic tableaus for first order logic. A more recent proof of decidability depends on the decidability of the loosely guarded fragment of first order logic [60, 62]. For $\mathfrak{A} \in \mathsf{G}_n$, let $L(\mathfrak{A})$ be the first order signature consisting of an n–ary relation symbol for each element of $\mathfrak{A}$. Then it can be shown that for $\mathfrak{A} \in \mathsf{G}_n$, for any $\psi(x)$ a quantifier free formula of the signature of G_n and $\bar{a} \in \mathfrak{A}$ with $|\bar{a}| = |\bar{x}|$, there is a loosely guarded $L(\mathfrak{A})$ sentence $\tau_{\mathfrak{A}}(\psi(\bar{a}))$ whose relation symbols are among $\bar{a}$ such that for any relativized representation M of $\mathfrak{A}$, $\mathfrak{A} \models \psi(\bar{a}) \iff M \models \tau_{\mathfrak{A}}(\psi(\bar{a}))$. Yet another ingenious proof of decidability given in [4] uses a subtle model theoretic combinatorial result of Herwig's building on work of Huroshovski. Roughly Herwig's result says that any finite structure has an extension that is still finite such that any partial isomorphism in the smaller structure extends to an automorphism of the big structure. The proof of finite base property is really a mixture of filtration and Herwig's result. By filtration one gets an approximate finite structure then applies Herwig's theorem to fine tune the constructed structure to get the desired finite base on which the representation is based. The thrust of this line of research suggests that the *genuine logical core* of first order logic may well be decidable and that undecidability resulted from using 'more than needed'. One could trace such positive decidability results to the seminal result of Németi's that the universal theory of Crs_n for any finite n is decidable.

3.3 Godél's incompleteness property in guarded decidable fragments

The two justly celebrated results of Godel are the completeness and incompleteness theorems, proved by the young Austro-Hungain logician in 1930, the incompleteness theorem, though, was published in 1931. Both theorems, deal with the infinite one way or another. The first shows that the essentially infinitely many valid formulas can be captured by a finite Hilbert style axiomatization, while the second theorem implies that infinitary methos are unavoidable in proving consistency of strong enough theories (like set theory). It is known that Gödel's second incompleteness is obtained when the theory in question is strong enough to encode the proof of its first incompleteness theorem. Such a theory cannot prove its consistency by methods formalizable in theory. Gödel's second incompleteness theorem says, officially, that given a set of axiom A and rules by which you can deduce (prove) theorems from the axioms, if you can deduce all the laws of good old elementary arithmetic from the axioms A, then you can't prove that the axioms are consistent (i.e. that they aren't self-contradictory).

Guarding semantics has led to the discovery of a whole landscape of multimodal logics having nice modal behaviour (like decidability) with the multimodal logic whose modal algebras are the class of relativized set algebras 'at the bottom' and Tarskian semantics whose modal algebras is the variety RCA_n with its undesirable properties (like undecidability) is only the top of an iceberg. Below the surface a treasure of nice multimodal logics was discovered. This dynamic viewpoint enriches and unifies our view of a mutiplicity of disciplines sharing a cognitive slant. However, recent research in algebraic logichas revealed that some guarded fragments of first order logic, surviving undecidability and other undesirable properties, made contact with a deep and subtle reformulation of Gódel's incompleteness theorem. For a class K of algebras, and a cardinal $\beta > 0$, $\mathfrak{Fr}_\beta K$ stands for the β-generated free K algebra. In particular, $\mathfrak{Fr}_\beta \mathsf{CA}_n$ denotes the β-generated free cylindric algebra of dimension n. The following is known: If $\beta \geq \omega$, Pigozzi proved that $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is atomless (has no atoms) [22, Theorem 2.5.13]. Assume that $0 < \beta < \omega$. If $n < 2$ then $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is finite, hence atomic, [22, Theorem 2.5.3 (i)]. $\mathfrak{Fr}_\beta \mathsf{CA}_2$ is infinite but still atomic, a result of Henkin's [22, Theorems 2.5.3 (ii), .4.7 (ii)]. If $3 \leq n < \omega$, Tarski proved that $\mathfrak{Fr}_\beta \mathsf{CA}_n$ has infinitely many atoms [22, Theorem 2.5.9] and it was posed as an open question, cf [22, Problem 2.14] whether it is atomic or not. Recent research in algebraic logic has revealed however that some guarded fragments of first order logic, surviving undecidability and other undesirable properties, made contact with a deep and subtle *algebraic reformulation* of Gódel's incompleteness theorem. This was proved first by Németi for the 'unguarded' L_n $(n \geq 3)$ with square Tarskian semantics by translating a form of Gódel's incompleteness property to non-atomicity of the free algebras and then proving for any finite m, $\mathfrak{Fr}_m\mathsf{CA}_n$ is not atomic. The key idea is that if T is a finite consistent complete L_n theory, then the equivalence class of $\bigwedge T$ will be an atom in the formula algebra of pure logic, built up of the symbols occurring in formulas in T. Here completeness and consistency are defined with respect to provability using only n variables. Now if one finds a formula that *cannot* be extended to such a theory then there will be no atoms below the equivalence class of this formula, and here is where Gódel's Theorem intervenes. The idea in Német's proof uses a translation function of $L_{\omega,\omega}$ into L_3 together with the pairing technique of Tarski's suitably re-defined to adapt the case of three variables rather than four; the latter being the natural habitat of relation algebras. Using similar but more sophisticated methods, Andréka and Németi later proved the analogous result for the $\mathfrak{Fr}_m\mathsf{Df_n}$ free algebras, solving a long standing open problem in algebraic logic, posed by Tarski, Maddux, Németi and others. This was deduced from the fact that the whole of ZFC can be coded in $\mathsf{Df_3}$; in other words, Df_3, which is substantially weaker than CA_3 *a fiortiori* strictly weaker than the calculas of relations is an adequate vehicle for the whole of mathematics. Tarski and Givant had formalized set theory in the calculas of relations establishing an intriguing 'variable free' approach to meta-mathematics. Their joint work in this fascinating topic is published in the monograph *A formalization of Set Theory without varibales.* In *op.cit* it is shown that in principle mathematics can be developed in the very simple framework of equations and substitution of equals for equals rather than the customary basis using set theory formalized in first order logic, which is, to say the least, an impressive tour de force with profound metamathematical and philosophical repercussions. The first chapter of Andréka et al [5] gives an excellent account of these results. Mohamed Khaled [43], another student of Andréka and Németi, proved that the free Crs_ns with finitey many generators are not atomic and developed his proof to show that many guarded logics has a form of Gödel's incompleteness property: There is a formula ϕ (in the signature of the guarded logic under investigation) that cannot be extended to a finite complete recursive theory. Such 'incompletable formulas are called *inseparable formulas* by Németi. This solves another long standing open problem in algebraic logic posed by Németi in the early eighties of the last century. This, in turn, shows that after all 'the taken for granted' implication 'Godel's incompleteness $\implies$ undecidability' does not always hold. Results of this kind are open to huge philosophical considerations, revisions and repercusions, and are far from being fully understood. Indeed such results tend to raise more questions than answers. The question that bears a lot of discussion and reflection in this connection is how faithfull the algebraic translation to non-atomocity of the free algebras, vis a vis the in built dependence of the proof of Gódel's incompleteness Theorem in Peano arithmetic (without the axiom of infinity) using Gódel numbering. Is Gódel's celebrated incompleteness theorem intrinsic to arithmetic (and richer formal systems like set theory), or can it lend itself to different, possibly more general frameworks? Is the idea of Gódel numbering–mirroring statements abouts numbers to statements about other statements of numbers, possibly themselves- applicaple only to the entities known as natural numbers? The two ingenious components in Gódel's proof are *diagonalization* and *self-reference*. Such methods and antimonies were known before Gódel. Diagonalization is implemented in Cantor's proof of the uncountability of $\mathbb{R}$ and self reference appeared (philosophically) with the liar paradox, later getting a more mathematical manifestation in the famous hugely influential Russell's parardox with several scattered re-incarnations in interdiscipilinary literature between mathematics, logic and philosophy [68]. But combining the two is certainly a master stroke proving one of the most important Theorems in mathematics in the 20th century and beyond. These two ingredients of Gódel's proof simply vanish in the algebraic proofs for Crs_n proving the non-atomicity of their finitely generated free algebras. The new proofs use an ingenious purely algebraic method [44]. Non-atomicity of finitely generated free algebras are also proved for D_n and G_n [45]. The consistency of a form of Gödel's incompleteness theorem and decidability (for guarded fragments of L_n) is certainly an exciting and a telling co-existence.

3.4 The interaction with modal logic to obtain non-orthodox completeness theorems for unguarded logics

Standard Hilbert style axiomatizations, being excluded by Monk's resullt, Venema succeeded to obtain a sound and complete proof system for finite variable fragments of first order logic (disguised in a modal formalism) with at least three variables; using instead non–orthodox derivation rules, cf. Venema' chapter entitled *Cylndric Modal Logic* in [5]. There is another algebraic expression of such non–orthdox derivation rules by so–called *density conditions*. This approach also has its roots in the prophetic monograph [22], cf. [22, Theorem 3.2.14]. Removing the condition of atomicity, Andréka et al. [3] show that every *rectangulary dense* algebra is representable, where by *rectangularly density* is meant that below every element there is a rectangle, that is not necessarily an atom. The rectangles in a rectangulary dense algebra can be associated with so–called 0–thin elements [22, Definition 3.2.1]. Such elements have a double facet, a geometric one and a metalogical one. The metalogical interpretation is that these elements *abstract the notion of individual constants*

[22, Remark 3.2.2]. Geometrically, in a set algebra 0–thin elements are obtained by fixing the first component of assignments by a constant. That is, if $\mathfrak{A}$ is a set algebra with top element ${}^nU$ say, and $X \in \mathfrak{A}$ is 0 thin, then $X = \{s \in {}^\alpha U : s_0 = u\}$ for some $u \in U$. *Thinnes here means that literally there is a thin line between the dimension of* X *and the dimension of* $\mathfrak{A}$. There is 'enough supply' of such elements in a rectangularly dense algebra. This algebraic notion of *richness* actually reflects the notion of *so-called rich theories in Henkin constructions.* Rich theories occuring in Henkin's completeness proof eliminate existential quantifiers in existential formulas via individual constants more commonly referred to as *witnesses*. Algebraically, every cylindrifier is elimintated, or *witnessed* by a 0–thin element [22, Definition 3.2.1]. But then by algebraising the rest of Henkin's proof, we get that rich algebras are representable. This connection manifests itself blatantly in the proof of [22, Theorem 3.2.5] where the base of the representation actually consists only of 0-thin elements, whereas the generic canonical models in Henkin constructions consist of individual constants. More succintly, richness and rectangular density are *saturation conditions*. Geometrically: rectangular density means that below every non–zero element there is a rectangle, while richness means that below every element there is *a square*, a special kind of rectangle as the name suggests. It ws proved in [3] that both notions are essentially equivalent and both notions *can be transferred to non–orthodox derivation rules using the difference operator* achieving completeness. In *op.cit* it is proved that $\mathfrak{A}$ is rectangularly dense (and quasi–atomic) $\iff$ $\mathfrak{A}$ is rich. For Df_n one uses rectangular density to prove representability results [70] since equality is not expressible.

4 Methods of constructing counterexamples in modal and algebraic logic

$\mathbf{K}^n$ is the logic of n-ary product frames, of the form $(W_i, R_i)_{i<n}$ where for each $i < n$, R_i is any any relation on W_i. On the other hand, $\mathbf{S5}^n$ can be regarded as the logic of n–ary product frames of the form $(W_i, R_i)_{i<n}$ such that for each $i < n$, R_i is an equivalence relation. It is known that logics between $\mathbf{K}^n$ and $\mathbf{S5}^n$ are quite complicated, cf. [35] for a detailed overview. Theorem 4.1 to be proved in a moment adds to their complexity.

It is known that modal languages can come to grips with a strong fragment of second order logic. Modal formulas translate to second order formulas, *their correspondants* on frames. Some of these formulas can be *genuinely second order*; they are not equivalent to first order formulas. An example is the *McKinsey formula*: $\Box\Diamond p \to \Diamond\Box p$. This can be proved by showing that its correspondant violates the downward Löwenheim- Skolem Theorem. The next proposition bears on the last two issues. But first a lemma. Using essentiually the argument in [22, Lemma 5.1.50, Theorem 5.1.51] by considering closure under infinite intersections instead of intersections, we get:

Proposition 4.1. *Let $2 < n < \omega$. There is no axiomatization of $\mathbf{S5}^n$ with formulas having first order correspondence. For any canonical logic $\mathfrak{L}$ between $\mathbf{K}^n$ and $\mathbf{S5}^n$, it is undecidable to tell whether a finite frame is a frame for $\mathfrak{L}$, $\mathfrak{L}$ cannot be finitely axiomatized in kth order logic (for any finite k), and $\mathfrak{L}$ cannot be axiomatized by canonical formulas, a fortiori Sahlqvist formulas. In fact any axiomatization of $\mathfrak{L}$ must contain infinitely many non-canonical formulas.*

Proof. The first part follows from that the class of strongly representable Df_n atom structures is not elementary [16]. With lemma 3.4 at our disposal, a slightly different proof can be easily distilled from the construction adressing CAs in [30] or [29]. We adopt the construction in the former reference, using the Monk–like CA_ns $\mathfrak{M}(\Gamma)$, Γ a graph, as defined in [30, Top of p.78]. For a graph $\mathcal{G}$, let $\chi(\mathcal{G})$ denote it chromatic number. Then it is proved in *op.cit* that for any graph Γ, $\mathfrak{M}(\Gamma) \in \mathsf{RCA}_n \iff \chi(\Gamma) = \infty$. By lemma 3.4, $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma) \in \mathsf{RDf}_n \iff \chi(\Gamma) = \infty$, because $\mathfrak{M}(\Gamma)$ is generated by the set $\{x \in \mathfrak{M}(\Gamma) : \Delta x \neq n\}$ using infinite unions. Now we adopt the argument in [30]. Using Erdos' probabalistic graphs [19], for each finite κ, there is a finite graph G_κ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let Γ_κ be the disjoint union of the G_l for $l > \kappa$. Then $\chi(\Gamma_\kappa) = \infty$, and so $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma_\kappa)$ is representable. Now let Γ be a non-principal ultraproduct $\Pi_D\Gamma_\kappa$ for the Γ_κs. For $\kappa < \omega$, let σ_κ be a first-order sentence of the signature of the graphs stating that there are no cycles of length less than κ. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Loś's Theorem, $\Gamma \models \sigma_\kappa$ for all κ. So Γ has no cycles, and hence by $\chi(\Gamma) \leq 2$. Thus $\mathfrak{Rd}_{df}\mathfrak{M}(\Gamma)$ is not representable. (Observe that the the term algebra $\mathfrak{Tm}\mathsf{At}(\mathfrak{M}(\Gamma))$ is representable (as a CA_n), because the class of weakly representable atom structures is elementary [27, Theorem 2.84].) Since Sahlqvist formulas have first order correspondants, then $\mathbf{S5}^n$ is not Sahlqvist. In [29], it is proved that it is undecidable to tell whether a finite frame is a frame for $\mathfrak{L}$, and this gives the non–finite axiomatizability result required as indicated in *op.cit*, and obviously implies undecidability. The rest follows by transferring the required results holding for $\mathbf{S5}^n$ [16, 29] to $\mathfrak{L}$ since $\mathbf{S5}^n$ is finitely axiomatizable over $\mathfrak{L}$, and any axiomatization of RDf_n must contain infinitely many non-canonical equations. □

4.1 Monk algebras, the good and the bad

From good to bad: Fix $2 < n < \omega$. Because RCA_n is a variety, an atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n \iff$ all equations axiomatizing RCA_n holds in $\mathfrak{A}$. From the point of view of $\mathsf{At}\mathfrak{A}$, the atom structure of $\mathfrak{A}$, each equation in the signature of RCA_n corresponds to a certain universal monadic second–order statement, in the signature of $\mathsf{At}\mathfrak{A}$ where the universal quantifiers are restricted to ranging over the set of atoms that lie below elements of $\mathfrak{A}$. Such a statement fails $\iff$ $\mathsf{At}\mathfrak{A}$ is partitioned into finitely many $\mathfrak{A}$–definable sets (sets definable using atoms of $\mathfrak{A}$ as parameters) with certain 'bad' properties. Call this a *bad partition*. A bad partition of a graph is a *finite colouring*, namely, a partition of its sets of nodes into finitely many independent sets. A typical Monk argument is to construct for finite dimension > 2 a sequence of non–representable algebras based on graphs with bad partitions (having finite chromatic number) converging to one that is based on a graph having *infinite chromatic number*, hence, representable. It follows immediately that the variety of representable algebras of dimension > 2 is not finitely axiomatizable. We call the non–representable algebras in the sequence *bad* Monk algebras. Based on graphs having finite chromatic number, they are not representable as subdirect products of set algebras. This is proved by an application of Ramsey's theorem. But these bad algebras converge to a *good* infinite Monk algebra that is based on a graph that has infinite chromatic number. This (limit) algebra is *good in the sense that it permits a representation*; it is a subdirect product of set algebras. We borrow the terminolgy 'bad and good' from [28] where such notions are applied to the *graph used in the construction of the Monk algebra*. A graph is *good* if it has infinite chromatic number, othewise it is, as mentioned above, bad, that is, it has a finite colouring. Constructing algebras based on graphs having arbitrarily large chromatic number converging to one that is based on a graph having infinite chromatic number seems a plausible thing to do, and indeed it can be done, witnessing non–finite axiomatizability of the class of representable algebras of finite dimension > 2. Monk's original proof [48] and Maddux's proof in [37], as well as the proofs presented in Theorem 3.9 and Remark 3.10 can be seen this way. Monk used *finite* bad Monk algebras converging to a good *infinite one*. This is easy to visualize; graphs having larger and larger chromatic number 'converging' to one with an infinite chromatic number.

From bad to good: Conversely, one can form an *anti–Monk ultraproduct*, of a sequence $(\mathfrak{A}_i : i \in \omega)$ of *good* infinite Monk algebras (based on graphs with infinite chromatic number) converging to an infinite bad atomic algebra $\mathfrak{A}$, namely, one that is based on a graph that is *only* 2*–colourable* [28]. This last algebra is representable, but only *weakly*. This means that its subalgebra generated by the atoms is representable, but its Dedekind-MacNeille minimal completion, which is the complex algebra of its atom structure, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, is not. In this sense, this limit $\mathfrak{A}$ is not *strongly* representable. But every element of the sequence $\mathfrak{A}_i$ $(i \in \omega)$ is strongly representable, in the same sense, meaning that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_i$ is representable, The aforementioned technique of proof, showing that the class of strongly representable cylindric algebras is not elementary is due to Hirsch and Hodkinson [28]. The *ultraproduct* that Monk used in his 1969 seminal result is a 'reverse' to this one, and is more intuitive, since as indicated it is plausible that a sequence of graphs having arbitrarily finite chromatic numbers getting larger and larger, converges to one that has infinite chromatic number, but the 'other way round' is hard to visualize, let alone implement. So how did Hirsch and Hodkinson prove their result? Fix $2 < n < \omega$. Recall that a CA_n atom structure $\mathbf{At}$ *strongly representable* $\iff$ the complex algebra $\mathfrak{Cm}\mathbf{At}$ is representable. So that an atomic algebra $\mathfrak{A}$ is strongly representable (in the sense that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$) $\iff$ its atom structure is strongly representable. One shows that an atom structure is strongly representable $\iff$ it has *no bad partition using any sets at all.* So, here, we want to find atom structures, with no bad partitions, with an ultraproduct that *does have a bad partition.* From a graph Γ, one can create an atom structure that is strongly representable $\iff$ the graph is good, namely, it has has no finite colouring; this atom structure is denoted by $\rho(\mathfrak{I}(\Gamma))$ in [30] and its complex algebra $\mathfrak{Cm}(\rho(\Gamma))$ is denoted by $\mathfrak{M}(\Gamma)$ in [30, Proposition 3.6.8]. These structues were used in Theorem 4.1. So the problem reduces to finding a sequence of graphs with no finite colouring, with an ultraproduct that does have a finite colouring. We want graphs of *infinite chromatic numbers*, having an ultraproduct with *finite chromatic number.* It is not obvious, *a priori*, that such graphs actually exist. And here Erdos' probabilistic methods offer solace. Graphs like this can be found using the probabilistis methods of Erdös, for those methods render finite graphs of arbitrarily large chromatic number and girth [19, 28]. By taking disjoint unions, we obtain graphs of infinite chromatic number (no bad partitions) and arbitarly large girth. A non–principal ultraproduct of these has no cycles, so has chromatic number 2 (bad partition), cf Theorem 4.1 and the proof of [30, Corollary 3.7.2]. Now let $\mathcal{G}$ be a graph. Consider the relation algebra atom structure $\alpha(\mathcal{G})$ defind exactly like in the proof of Theorem 3.10. Then the relation complex algebra based on this atom structure will have an n–dimensional cylindric basis and, in fact, the cylindric atom structure of $\mathfrak{M}(\mathcal{G})$ is isomorphic (as a cylindric algebra atom structure) to the atom structure $\mathsf{Mat}_n(\alpha(\mathcal{G}))$ of all n–dimensional basic matrices over the relation algebra atom structure $\alpha(\mathcal{G})$. It is plausible that one can prove that $\alpha(\mathcal{G})$ is strongly representable $\iff$ $\mathfrak{M}(\mathcal{G})$ is representable $\iff$ $\mathcal{G}$ has infinite chromatic number, so that one gets the result, that the class of strongly representable atom structure both RAs and CAs of finite dimension at least three, is not elementary in one go. The underlying idea here is that shade of red ρ will appear in the *ultrafilter extension* of $\mathcal{G}$, if it has infinite chromatic number, as a reflexive node [30, Definition 3.6.5] and its n–copies (ρ, i), $i < n$, can be used to completely represent $\mathfrak{M}(\mathcal{G})^+$.

4.2 Rainbow algebras vs Monk-like algebras; pros and cons

The model–theoretic ideas used in Theorem 3.5 and the construction in [57] recalled in Remark 3.10 are quite similar. In the overall structure; they follow closely the model–theoretic framework in [33]. In both cases, we have finitely many shades of red outside a Monk-like and rainbow signature, that were used as labels to construct an n– homogeneous model M in the expanded signature. Though the shades of reds are *outside* the signature, they were used as labels during an ω–rounded game played on labelled finite graphs–which can be seen as finite models in the extended signature having size $\leq n$– in which $\exists$ had a winning strategy, enabling her to construct the required M as a countable limit of the finite graphs played during the game. The construction, in both cases, entailed that any subgraph (substructure) of M of size $\leq n$, is independent of its location in M; it is uniquely determined by its isomorphism type. A relativized set algebra $\mathfrak{A}$ based on M was constructed by discarding all assignments whose edges are labelled by these shades of reds, getting a set of n–ary sequences $W \subseteq {}^n\mathsf{M}$. This W is definable in ${}^n\mathsf{M}$ by an $L_{\infty,\omega}$ formula and the semantics with respect to W coincides with classical Tarskian semantics (on ${}^n\mathsf{M}$) for formulas of the signature taken in L_n (but not for formulas taken in $L^n_{\infty,\omega}$). This was proved in both cases using certain n back–and–forth systems, thus $\mathfrak{A}$ is representable classically, in fact it (is isomorphic to a set algebra that) has base M. *The heart and soul of both proofs is to replace the reds label by suitable non–red binary relation symbols within an n back–and–forth system, so that one can adjust that the system maps a tuple $\bar{b} \in {}^n\mathsf{M}\backslash W$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing the non–red symbols that are 'moved' by the system. In fact, all injective maps of size $\leq n$ defined on M modulo an appropriate permutation of the reds will form an n back–and–forth system.*

This set algebra $\mathfrak{A}$ was further atomic, countable, and simple (with top element ${}^n\mathsf{M}$). The subgraphs of size $\leq n$ of M whose edges are not labelled by any shade of red are the atoms of $\mathfrak{A}$, expressed syntactically by MCA formulas. The Dedekind-MacNeille of $\mathfrak{A}$, in symbols $\mathfrak{CmAt}\mathfrak{A}$, has top element W, but it is not in $\mathsf{SNr}_n\mathsf{CA}_{n+3}$ in case of the rainbow construction, least representable, and it is (only) not representable in the case of the Monk–like algebra. In case of both constructions 'the shades of red' – which can be intrinsically identified with non–principal ultrafilters in $\mathfrak{A}$, were used as colours, together with the principal ultrafilters to represent completely $\mathfrak{A}^+$, inducing a representation of $\mathfrak{A}$. Non–representability of $\mathfrak{CmAt}\mathfrak{A}$ in the Monk case, used Ramsey's theory. The non neat–embeddability of $\mathfrak{CmAt}\mathfrak{A}$ in the rainbow case, used *the finite number of greens* that gave us information on when $\mathfrak{CmAt}\mathfrak{A}$ stops to be representable. The reds in both cases have to do with representing $\mathfrak{A}$. The model theory used for both constructions is almost identical. Nevertheless, from the algebraic point of view, there is a crucial difference. The non–representability of $\mathfrak{CmAt}\mathfrak{A}$ was tested by a game between the two players $\forall$ and $\exists$. The winning strategy's of the two players are independent, this is reflected by the fact that we have two 'independent parameters' G (the greens) and R (the reds) that are more were finite irreflexive complete graphs. In Monk–like constructions like the one used in [57] to show that RCA_n is not atom–canonical by constructing a countable atomic $\mathfrak{A}$ the non represenatbility of $\mathfrak{CmAt}\mathfrak{A}$ was also tested by a game between $\exists$ and $\forall$. But in *op.cit* winning strategy's are interlinked, one operates through the other; hence only one parameter is the source of colours. This parameter is a graph $\mathcal{G}$ (a countable disjoint unon of N cliques $n(n-1)/2 \leq N < \omega$). Representability of the complex algebra $\mathfrak{CmAt}\mathfrak{A}$ in this case depends only on the *chromatic number of* $\mathcal{G}$, via an application of Ramseys' theorem. *In both cases two players operate using 'cardinality of a labelled graph'.* $\forall$ *trying to make this graph too large for* $\exists$ *to cope, colouring some of its edges suitably.* For the rainbow case, it is a red clique formed during the play as we have seen in Theorem 3.5. It might be clear in both cases (rainbow and Monk–like algebras), to see that $\exists$ cannot win the infinite game, but what might not be clear is *when does this happens; we know it eventually does happen in finitely many round, but till which round* $\exists$ *has not lost yet.* The non–representability of $\mathfrak{CmAt}\mathfrak{A}$ amounts to that $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{SNr}_n\mathsf{CA}_{n+k}$ for some finite k because $\mathsf{RCA}_n = \bigcap_{k<\omega} \mathsf{SNr}_n\mathsf{CA}_{n+k}$. Can we pin down the value of k? getting an estimate that is not 'infinitely' loose which is the case with the contructions in [57, 33]. We say that an algebra $\mathfrak{A} \in \mathsf{CA}_n$ *does not stop to be representable* $\iff$ $\mathfrak{A}$ can be be neatly embedded into an cylindric algebra having arbitrary finite extra dimensions $\iff$ $\mathfrak{A} \in \mathsf{SNr}_n\mathsf{CA}_{n+k}$ for all $k \geq 0 \iff \mathfrak{A} \in \mathsf{SNr}_n\mathsf{CA}_{n+\omega} \iff \mathfrak{A} \in \mathsf{RCA}_n \iff \mathfrak{A}$ is simply representable. By adjusting the number of greens in the proof of Theorem 3.5 to be $n+1$ one gets a finer result than Hodkinson's [33] where there were infinitely many greens. By truncating the greens to be $n+1$, we could tell when $\mathsf{SNr}_n\mathsf{CA}_{n+k}$, $3 \leq k \leq \omega$ is not atom–canonical; this occured when $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{SNr}_n\mathsf{CA}_{n+3}$, that is to say, when $\mathfrak{CmAt}\mathfrak{A}$ *stopped to be representable.*

4.3 More non-finite axiomatizability results for other cylindric-like algebras

Sometimes Monk–like algebras, without a rainbow intervention using the independent parameter (of greens) G are efficient in controlling ' excuding a pre-assignded number of spare dimensions ' in a certain construction. Let us elaborate some more: Hirsch and Hodkinson [27, Theorem 15.1] solve [22, Problem 2.12] which was referred to in the literature (originally by Monk) : *The neat embedding problem.* In [32], the analogue of the neat embedding problem is approached for diagonal free cylindric algebras Dfs, Pinter's substitution algebras Scs, polyadic algebras PAs, quasi–polyadic algebras QA, PAs with equality PEAs, and QAs with equality QEAs. For K any of these classes and α any ordinal, we write K_α for the variety of α–dimensional K algebras, and RK_α for the class of representable K_αs, which happens to be a variety too (that is not finitely axiomatizable for $\alpha > 2$ except for $\mathsf{K} = \mathsf{RQA}$ and $\alpha \geq \omega$). The class of completely representable (in the sense of [26]) K_αs is denoted by CRK_α. The standard reference for all the classes of algebras mentioned previously is [22], cf. the appendix of [32]. We recall the concrete verions of such algebras. Given a ordinal α and $i, j \in \alpha$, we let $[i|j]$ denote the replacement that sends i to j and is the identity on $\alpha \sim \{i\}$, while $[i, j]$ will denote the transposition that swap i and j. For a set X, $\mathfrak{B}(X)$ denotes the Boolean field of sets $\langle \wp(X), \cap, \sim\rangle$. Let $\tau : \alpha \to \alpha$ and $X \subseteq {}^\alpha U$, then

$$\mathsf{S}_\tau X = \{s \in {}^\alpha U : s \circ \tau \in X\},$$

- A *diagonal free cylindric set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i\rangle_{i,j<\alpha}$. The class of all diagonal free cylindric set algebras of dimension α is denoted by DfCs_α.

- A *Pinter's substitution set algebra of dimension* α. or simply a substitution algebra of dimension α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}\rangle_{i,j<\alpha}$. The class of all cylindric set algebras of dimension α is denoted by Ss_α.

- A *quasi-polyadic set algebra of dimension* α is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}\rangle_{i,j<\alpha}$. The class of all quasi-polyadic set algebras of dimension α is denoted by Ps_α.

- A *quasi-polyadic equality set algebra* is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]}, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$. The class of all quasi-polyadic equality set algebras of dimension α is denoted by Pes_α.

According to a widespread custom, we denote $\mathsf{S}_{[i|j]}$ for $i \neq j$, by S^i_j. For operators on classes of algebras, $\mathbf{S}$ stands for the operation of forming subalgebras, $\mathbf{P}$ stands for that of forming products, and $\mathbf{H}$ stands for the operation of forming homomorphic images. The varieties of representable algebras of dimension α, α an ordinal, for all such classes are defined as follows: $\mathsf{RDf}_\alpha = \mathbf{SP}\mathsf{DfCs}_\alpha$, $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$, $\mathsf{RSc}_\alpha = \mathbf{SP}\mathsf{Sc}_\alpha$, $\mathsf{RQA}_\alpha = \mathbf{SP}\mathsf{Ps}_\alpha$, and $\mathsf{RQEA}_\alpha = \mathbf{SP}\mathsf{Pes}_\alpha$. It is known that RDf_α and RQA_α, for $\alpha > 2$, cannot be axiomatized by a *finite schema* of equations.

Andréka proved a plethora of *relative non–finite axiomatizability results* in the following sense. Let K be a variety having signature t, and let V be a variety having signature $t' \subseteq t$, such that if $\mathfrak{A} \in \mathsf{K}$ then the reduct of $\mathfrak{A}$ obtained by discarding the operations in $t \sim t'$, $\mathfrak{Rd}_{t'}\mathfrak{A}$ for short, is in V. We say that a set of first order formulas Σ in the signature t axiomatizes K over V, if for any algebra $\mathfrak{A}$ in the signature t whenever $\mathfrak{A} \models \Sigma$ and $\mathfrak{Rd}_{t'}\mathfrak{A} \in \mathsf{V}$, then $\mathfrak{A} \in \mathsf{K}$. This means that Σ 'captures' the properties of the operations in $t \sim t'$. A relative non–finite axiomatizability result is of the form: There is no set 'of a special form' of first order formulas satisfying a 'finitary condition' that axiomatizes K over V. Such special forms may be equations, or universal formulas. By finitary, we exclusively mean that Σ is finite (this makes no sense if the signature at hand is infinite), or Σ is a finite schema in the sense of Monk's schema [22, Definition 5.6.11-5.6.12], or Σ contains only finitely many variables. The last two cases apply equally well to varieties having infinite signature like RCA_ω. In the last case a finite schema is understood in the sense of [22, Definition 4.1.4] namely, in a two-sorted sense, one sort for ordinals $< \omega$, the other sort for the usual first order situation. finitely many variables. For example Andreka proved that RQEA_n is not finitely axiomatizable over RQA_n nor RCA_n. In the former case Andréka went further excluding universal axiomatizations containing only finitely many variables, a result that we lift to the transfinite in moment. We let $\mathfrak{Rd}_{ca}$ denote 'cylindric reduct' and $\mathfrak{Rd}_{qa}$ denote 'quasi—polyadic reduct.'

Theorem 4.2. *The variety* RQEA_ω *cannot be axiomatized by a set of universal formulas containing finitely many variables over* RQA_ω.

Proof. For each positive k we construct $\mathfrak{A} \notin \mathsf{RQEA}_\omega$ such that its k- subalgebras (subalgebras generated by at most k–elements) are in RQEA_ω, $\mathfrak{Rd}_{ca}\mathfrak{A} \notin \mathsf{RCA}_\omega$ and $\mathfrak{Rd}_{qa}\mathfrak{A} \in \mathsf{RQA}_\omega$. But for fixed k not only one splitting is done, but infinitely many each (to an atom) in a different set algebra; the resulting algebras (obtained after splitting) form a chain; their directed union will be the $\mathfrak{A}$ we want. This can (and will) be done for each positive k. Accordingly, throughout the proof fix a positive k.

(1) Splitting a single atom getting non-representable algebras: For fixed $2 < n < \omega$, take a finite $m \geq 2^{k\times n!+1}$. Suppose that the signature consists of ω–many cylindrifier, $\mathsf{c}_i : i < \omega$, diagonal constants $\mathsf{d}_{ij}, i < j < \omega$, and $n^2 - n$ substitutions $\mathsf{s}_{[i,j]} : i < j < n$. One forms, for each such n, an algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ in this specified signature, by splitting the ω–ary relation $R = \prod_{i\in\omega} U_i$ with $U_0 = m-1$ and $|U_i| = m$ for $0 < i < \omega$ in the algebra $\mathfrak{A}_n = \mathfrak{Sg}^{\wp({}^\omega U)}\{R\}$, where $U = \bigcup_{i\in I} U_i$ into m abstract copies. Observe that here R depends on n, because m depends on n and R depends on $U_0 = m - 1$. The resulting algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ therefore has the signature expanding CA_ω by the finitely many substitution operators $\mathsf{s}_{[i,j]}$, $i < j < n$. Here the set algebra $\wp({}^\omega U)$ is taken in the specified signature with operations interpreted the usual way as in set algebras, e.g. $\mathsf{S}_{[0,1]}\{R\} = \{s \in {}^\omega U : (s_1, s_0) \in R\}$. It can be easily checked that for all $i < j < n$, $\mathsf{S}_{[i,j]}R$ is an atom in $\mathfrak{A}_n$. In particular, R is partitioned into a family $(R_i : i < m)$ of atoms in the bigger algebra $\mathsf{split}(\mathfrak{A}_n R, m)(\supseteq \mathfrak{A}_n)$, so that $R = \sum_{i<m} R_i$, where $m = |U_0| + 1$. Furthermore in $\mathsf{split}(\mathfrak{A}_n, R, m)$, we have $\mathsf{s}_{[l,j]}R = \sum_{i<m} \mathsf{s}_{[l,j]}R_i$ and $\mathsf{c}_t\mathsf{s}_{[l,j]}R_i = \mathsf{c}_t\mathsf{s}_{[l,j]}R$ for all $l, j < n$, $i < m$ and $t < \omega$; so that, in particular, R is *cylindrically* *equivalent* to its abstract copies. The algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ is determined uniquely (up to isomorphism) by $\mathfrak{A}_n, R$ and m, hence the notation, and it will not be representable. Even more, the algebra $\mathfrak{Rd}_{ca}\mathsf{split}(\mathfrak{A}_n, R, m)$ having CA_ω signature will not be representable for the same reason used in the proof of Theorem 4.2: One defines the term $\tau(x) = (\bigwedge_{i<m} \mathsf{s}_i^0\mathsf{c}_1 \ldots \mathsf{c}_m x \cdot \bigwedge_{i<j<n} -\mathsf{d}_{ij})$ as in [1, Top of p.157]. Then $\mathfrak{A}_n \models \tau(R) = 0$ hence $\mathsf{split}(\mathfrak{A}_n, R, m) \models \tau(R) = 0$ because $\mathfrak{A}_n \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$. Identifying set algebras with their domain, for an algebra $\mathfrak{A}$ and a non–zero $a \in \mathfrak{A}$, we say that a representation $h : \mathfrak{A} \to \wp({}^\omega U)$ *respects the non–zero element* a if $h(a) \neq \emptyset$. If $\mathsf{split}(\mathfrak{A}_n, R, m)$ were representable, then it will have a representation that respects R. But any such representation h will satisfy that $\tau(h(R)) \neq 0$ which is impossible.

(2) Representability of k–generated subalgebras: Now we show that the k–generated subalgebras are representable. Let $G \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$, $|G| \leq k$. Let $\mathfrak{P} = (R_l : l < m)$ be the abstract partition of R in the bigger algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ obtained by splitting R in $\mathfrak{A}_n$ into m (abstract) subatoms $(R_l : l < m)$. The reasoning is essentially the same as that used in the previous proof, with a minor technical deviation due to the presence of the substitution operators. We have to take them into account when defining the following relation on $\mathfrak{P}$: For $l, t < m$, $R_l \sim R_t \iff (\forall g \in G)(\forall i, j < n)(\mathsf{s}_{[i,j]}R_l \leq g \iff \mathsf{s}_{[i,j]}R_t \leq g)$. iThen, like before, it is straightforward to check that $\sim$ is an equivalence relation on $\mathfrak{P}$ having $p < m$ many equivalence classes, because $|G| \leq k$, $n^2 - n < n!$ and (recall that) $m \geq 2^{k\times n!+1}$. One next takes $B = \{a \in B_{k,n} : (\forall l, t < m)(\forall i, j \in n)(R_l \sim R_t, \mathsf{s}_{[i,j]}R_l \leq a \implies \mathsf{s}_{[i,j]}R_t \leq a)\}$, then $G \subseteq B$, $R \in B$, and B is closed under the operations, so that $\mathfrak{A}_n \subseteq \mathfrak{B} \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$, where $\mathfrak{B}$ is the algebra with universe B. Furthermore, $\mathfrak{B}$ is the smallest such subalgebra of $\mathsf{split}(\mathfrak{A}_n, R, m)$, where for each $i, j < n$, $\mathsf{s}_{[i,j]}R$ is partitioned into $p < m$ many parts cylindrically equivalent to $\mathsf{s}_{[i,j]}R$. The non–representability of the algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ can be pinned down to the existence of 'one more extra atom' leading to the incomptability condition $|U_0| < m$ (= number of subatoms) witnessed by the term τ using diagonal elements. Using that $|\mathfrak{P}| = m$, we showed that a representation h of $\mathsf{split}(\mathfrak{A}_n, R, m)$ that respects R, has to respect the atoms below it, and this forces that $|U_0| \geq m$, which contradicts the construction of $\mathfrak{A}_n$. But this cannot happen with $\mathfrak{B}$, because $p < m$ (by the condition $|G| \leq k$)), so that this 'one more extra atom and possibly more' vanish in $\mathfrak{B}$. In representing $\mathfrak{B}$, we use the following *optimal compatibility condition* between the cardinality of $|U_0|$ and the number of *concrete* copies of R represented genuinely in $\mathfrak{B}$: *(*) If* $\mathfrak{m}$ *is any cardinal,* α *is an ordinal* $\geq \omega$*,* $(U_i : i \in \alpha)$ *is a system of sets each having cardinality* $\geq \mathfrak{m}$*, and* $U \supseteq \bigcup\{U_i : i \in \alpha)$*, then there is a partition* $(R_j : j < \mathfrak{m})$ *of* $R = \prod_{i\in I} U_i$ *such that* $\mathsf{c}_i^U R_j = \mathsf{c}_i^U R$ *for all* $i < \alpha$ *and* $j < \mathfrak{m}$ *[1, Lemma 3].* Representing $\mathfrak{B}$ is done by embedding it into a representable algebra $\mathfrak{C}$ having the same top element as $\mathfrak{A}_n$, namely, ${}^\omega U$, where $R \in \mathfrak{C}$ is partitioned *concretely into* $m - 1$ *real atoms*, that is, there exists $R_l \subseteq {}^{\omega}U$, $l < m-1$ *real atoms* in $\mathfrak{C}$ such that for all $i < j < n$, $\mathsf{S}_{[i,j]}R = \mathsf{S}_{[i,j]}\bigcup_{l<m-1} R_l = \bigcup_{l<m-1} \mathsf{S}_{[i,j]}R_l$ and $\mathsf{C}_i R_l = \mathsf{C}_i R$ for all $l < m-1$ and $i < \omega$. *This concrete partition exists by (*) because* $|U_0| = m-1$ and by the condition $|G| \leq k$, the value of p, which is the new number of subatoms of R in $\mathfrak{B}$ (depending on G) cannot exceed $m-1$.

(3) Forming the directed union getting the required algebra: For fixed k, obtaining the algebras $\mathfrak{B}_{k,n} = \mathsf{split}(\mathfrak{A}_n, R, m)$ for each each $2 < n < \omega$ we proceed as follows. The constructed non–representable algebras form a chain in the following sense: For $2 < n_1 < n_2$, $\mathfrak{B}_{k,n_1}$ embeds into $\mathfrak{B}_{k,n_2}$, where the last algebra is the reduct obtained by discarding substitution operations not in the signature of the former, that is the substitution operations $\mathsf{s}_{[i,j]}$: $i, j \geq n_1, i \neq j$. Take the directed union $\mathfrak{B}_k = \bigcup_{n\in\omega\sim 3} \mathfrak{B}_{k,n}$ having the signature of QEA_ω. The cylindric reduct of $\mathfrak{B}_k$ is not representable because the cylindric reduct of every $\mathfrak{B}_{k,n}$ is not representable. Using that the k generated subalgebras of $\mathfrak{B}_{k,n}$ for each $2 < n < \omega$ are repesentable, it follows without difficulty that the k–generated subalgebras of $\mathfrak{B}_k$ remain representable. One constructs such algebra $\mathfrak{B}_k$ having the signature of QEA_ω for each positive k. But one can even go further, by showing that the diagonal free reduct of $\mathfrak{B}_k$ so constructed is in RQA_ω for each k, by showing that this is the case for every $\mathfrak{B}_{k,n}(n > 2)$. Recall that for fixed positive k and $2 < n < \omega$, the algebra $\mathfrak{B}_{k,n}$ is not representable because of the incompatibity of $|U_0| <$ number of subatoms. One now adds 'one extra element or more' to $|U_0|$ forming W_0 to compensate for such an incompatibility. The diagonal free reduct of $\mathfrak{B}_{k,n}$ can now be represented by a set algebra $\mathfrak{C}$ obtained by splitting an ω–ary relation $R = W_0 \times \prod_{i\in\omega} U_i$ where $|W_0| \geq m+1$ and (as before) $|U_i| = m$, $i \in \omega$, in a set algebra generated by R, into m *real atoms*, as described in (*). Here, in the *absence of diagonal elements*, we cannot count the elements in $|W_0|$ (like we did with $|U_0|$ using the term τ defined above), so adding this element to U_0 does not clash with the concrete interpretation of the other operations. In short, $\mathfrak{Rd}_{qa}\mathfrak{B}_{k,n}$ can be represented via $\mathfrak{C}$. Using basic model theory available in [1] one obtains the relative non–finite axiomatizability required result.

(4) Relative non–finite axiomatizability; the required result: For a set X we denote by $\mathfrak{B}(X)$ the Boolean algebra $\langle \wp(X), \cup, \cap, \sim \rangle$. Now we show that any universal axiomatization of RQEA_ω, must contain a formula with more than k variables and containing at least one diagonal constant getting the required relative non–finite axiomatizability result. Fix $n \in \omega \sim 3$. Let Σ_n^v be the set of universal formulas using only n substitutions and k variables valid in RQEA_ω, and let Σ_n^d be the set of universal formulas using only n substitutions and no diagonal elements valid in RQEA_ω. By n substitutions we understand the set $\{\mathsf{s}_{[i,j]} : i, j \in n\}$. Then $\mathfrak{B}_{k,n} = \mathsf{split}(\mathfrak{A}_n, R, m) \models \Sigma_n^v \cup \Sigma_n^d$. $\mathfrak{B}_{k,n} \models \Sigma_n^v$ because the k generated subalgebras of $\mathfrak{B}_{k,n}$ are representable, while $\mathfrak{B}_{k,n} \models \Sigma_n^d$ because $\mathfrak{B}_{k,n}$ has a representation that preserves all operations except for diagonal elements. Indeed, let $\phi \in \Sigma_n^d$, then there is a representation of $\mathfrak{B}_{k,n}$ in which all operations are the natural ones except possibly for the diagonal elements. This means that (after discarding the diagonal elements) there is a injective homomorphism $h : \mathfrak{A}^d \to \mathfrak{P}^d$ where $\mathfrak{A}^d = \langle B_{k,n}, +, \cdot, \mathsf{c}_k, \mathsf{s}_i^j, \mathsf{s}_{[i,j]}\rangle_{k\in\omega,i,j\in n}$ and $\mathfrak{P}^d = \langle \mathfrak{B}({}^\omega W), \mathsf{C}_k, \mathsf{S}_i^j, \mathsf{S}_{[i,j]}\rangle_{k\in\omega,i,j\in n}$, for some infinite set W.

Now let $\mathfrak{P} = \langle \mathfrak{B}({}^\omega W), \mathsf{C}_k, \mathsf{S}_i^j, \mathsf{S}^W_{[i,j]}, \mathsf{D}_{kl}\rangle_{k,l\in\omega,i,j\in n}$. Then we have that $\mathfrak{P} \models \phi$ because ϕ is valid and so $\mathfrak{P}^d \models \phi$ due to the fact that no diagonal elements occur in ϕ. It thus follows that $\mathfrak{A}^d \models \phi$, because $\mathfrak{A}^d$ is isomorphic to a subalgebra of $\mathfrak{P}^d$ and ϕ is quantifier free. Therefore $\mathfrak{B}_{k,n} \models \phi$.

Let $\Sigma^v = \bigcup_{n\in\omega\sim 3}\Sigma_n^v$ and $\Sigma^d = \bigcup_{n\in\omega\sim 3}\Sigma_n^d$. It follows that $\mathfrak{B}_k = \bigcup_{n\in\omega\sim 3}\mathfrak{B}_{k,n} \models \Sigma^v \cup \Sigma^d$. For if not, then there exists a quantifier free formula $\phi(x_1, \ldots, x_m) \in \Sigma^v \cup \Sigma^d$, and $b_1, \ldots, b_m$ such that $\phi[b_1, \ldots, b_n]$ does not hold in $\mathfrak{A}_k$. We have $b_1, \ldots, b_m \in \mathfrak{B}_{k,i}$ for some $2 < i < \omega$. Take n large enough $\geq i$ so that $\phi \in \Sigma_n^v \cup \Sigma_n^d$. Then ϕ does not hold in $\mathfrak{B}_{k,n}$, which is a contradiction. Let Σ be a set of quantifier free formulas axiomatizing RQEA_ω, then $\mathfrak{B}_k$ does not model Σ since $\mathfrak{B}_k$ is not representable, so there exists a formula $\phi \in \Sigma$ such that $\phi \notin \Sigma^v \cup \Sigma^d$. Then ϕ contains more than k variables and a diagonal constant occurs in ϕ.

□

Using the above methods all complexity results in [1] generalize to RQEA_ω and using transfinite induction such results lift to an arbitrary infinite ordinal α, cf. [62].

Corollary 4.3. *1. The variety RQEA_ω cannot be axiomatized by a set of universal formulas containing finitely many variables over RSc_ω nor over RDf_ω.*

2. [1] The variety RCA_ω cannot be axiomatized by a set of universal formulas containing finitely many variables over RSc_ω nor over RDf_ω.

Proof. For each positive k, using the notation in the previous proof, $\mathfrak{Rd}_{sc}\mathfrak{B}_k \in \mathsf{RSc}_\omega$ and $\mathfrak{Rd}_{df}\mathfrak{B}_k \in \mathsf{RDf}_\omega$, while $\mathfrak{B}_k \notin \mathsf{RQEA}_\omega$. This gives the required in the first item By observing that in fact, for each positive k, $\mathfrak{Rd}_{ca}\mathfrak{B}_k \notin \mathsf{RCA}_\omega$, we get the required in the second item. □

In [62] it is proved that for any infinite ordinal α, RQEA_α cannot be axiomatized by a finite schm over RCA_α using the construction in [7]. In *op.cit* a non–representable QEA_ω whose CA reduct is an ω–dimensional weak set algebra, briefly, a Ws_ω, is constructed. The unit of the weak set algebra are sequences agreeing cofinitely with $\prod_{i\in\omega} Z_i$ where $Z_0 = Z_1 = 3 = \{0, 1, 2\}$ and $Z_i = \{2i-1, 2i\}$ for $i > 1$; denote this QEA_ω with representable CA reduct by $\mathfrak{A}_3$. For $4 \leq n < \omega$, take $Z_0^n = Z_1^n = n = \{0, 1, 2, \ldots n-1\}$ and $Z_i^n = \{(n-1)i-1, (n-1)i\}$ for $i > 1$. In exactly the same way, one can construct an algebra $\mathfrak{A}_n \in \mathsf{QEA}_\omega \sim \mathsf{RQEA}_\omega$ and $\mathfrak{Rd}_{ca}\mathfrak{A}_n \in \mathsf{Ws}_\omega$ with top element agreeing cofinitely with sequences in $\prod_{i\in\omega} Z_i^n$. The proof for $\mathfrak{A}_n$ is identical to the proof for $\mathfrak{A}_3$ replacing 3 by n. It can also be shown that $\Pi_{i\in\omega\sim 2}\mathfrak{A}_i/F \in \mathsf{RQEA}_\omega$ for any non–principal ultrafilter on ω, proving the required.

4.4 The neat embedding problem

Sometimes Monk–like algebras, without a rainbow intervention using the independent parameter (of greens) G are efficient in controlling ' excuding a pre-assignded number of spare dimensions ' in a certain construction. Let us elaborate some more: Hirsch and Hodkinson [27, Theorem 15.1] solve [22, Problem 2.12] which was referred to in the literature (originally by Monk) : *The neat embedding problem.* In [32], the analogue of the neat embedding problem is approached for diagonal free cylindric algebras Dfs, Pinter's substitution algebras Scs, quasi–polyadic algebras QA, and QAs with equality QEAs. For K any of these classes and α any ordinal, we write K_α for the variety of α–dimensional K algebras, and RK_α for the class of representable K_αs, which happens to be a variety too (that is not finitely axiomatizable for $\alpha > 2$) The class of completely representable (in the sense of [26]) K_αs is denoted by CRK_α. The standard reference for all the classes of algebras mentioned previously is [22], cf. the appendix of [32]. For operators on classes of algebras, $\mathbf{S}$ stands for the operation of forming subalgebras, $\mathbf{P}$ stands for that of forming products, and $\mathbf{H}$ stands for the operation of forming homomorphic images. The varieties of representable algebras of dimension α, α an ordinal, for all such classes are defined as follows: Recall that $\mathsf{RDf}_\alpha = \mathbf{SP}\mathsf{DfCs}_\alpha$, $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$. Analogously, we set: $\mathsf{RSc}_\alpha = \mathbf{SP}\mathsf{Ssc}_\alpha$, $\mathsf{RQA}_\alpha = \mathbf{SP}\mathsf{Ps}_\alpha$, and $\mathsf{RQEA}_\alpha = \mathbf{SP}\mathsf{Pes}_\alpha$. It is known that RDf_α, RSc_α, RQA_α and RQA_α, for $\alpha > 2$, cannot be axiomatized by a *finite schema* of equations. It is proved in *op. cit* that (like the CA case recalled in Theorem 3.11) and settled by Hirsch and Hodkinson) for any class K between Sc and QEA, for any positive k, and for any ordinal $\alpha > 2$, the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{K}_{\alpha+k+1}$ is not axiomatizable by a finite schema over $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{K}_{\alpha+k}$. The case CA_α for infinite α was not tackled Hirsch and Hodkinson.

Recall that for an atomic relation algebra $\mathfrak{R}$ and $l > 3$, recall that we denote by $\mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ the set of all l–dimensional basic matrices on $\mathfrak{R}$. For $\tau : l \to l$ we write $(f\tau)$ for the function defined by $(f\tau)(x, y) = f(\tau(x), \tau(y))$. It is always the case that $f\tau \in \mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ for any $f \in \mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ and any $\tau : l \to l$, so if $\mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ is an l–dimensional cylindric basis, then $\mathfrak{Cm}\mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ can be expanded to a QEA_l, by defining for $X \subseteq \mathsf{Mat}_l(\mathsf{At}\mathfrak{R})$ and transposition $\tau : l \to l$: $\mathsf{s}_\tau(X) = \{f \in \mathsf{Mat}_l(\mathsf{At}\mathfrak{R}) : f\tau \in X\}$. It is not hard to adapt the proof of Theorem 3.11 to QEAs as follows: Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m, n, r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n, r), \omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [27, Definition

12.21] on $\mathfrak{A}(n,r)$ as defined in [27, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$, and it can be easily expanded to a QEA_m, since $\mathfrak{C}(m,n,r)$ is 'symmetric', in the sense that it allows a polyadic equality expansion by defining substitution operations corresponding to transpositions. This follows by observing that $\mathbf{H}$ is obviously symmetric in the following exact sense: For $\theta : m \to m$ and $N \in \mathbf{H}$, $N\theta \in \mathbf{H}$, where $N\theta$ is defined by $(N\theta)(x,y) = N(\theta(x), \theta(y))$. Hence, the binary polyadic operations defined on the atom structure $\mathbf{H}$ the obvious way (by swapping co–ordinates) lifts to polyadic operations of its complex algebra $\mathfrak{C}(m,n,r)$. In more detail, for a transposition $\tau : m \to m$, and $X \subseteq \mathbf{H}$, define $\mathsf{s}_\tau(X) = \{N \in \mathbf{H} : N\tau \in X\}$. Furthermore, for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{QEA}_n$, $\mathfrak{Rd}_{ca}\mathfrak{C}(m,n,r) \notin \mathsf{SNr}_m\mathsf{CA}_{\mathsf{n+1}}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RQEA}_m$ by easily adapting [27, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] to the QEA context. Let $2 < n < m <\leq \omega$. The notion of m-flat representation of a CA_n is given in [60]

Theorem 4.4. *Let $2 < m < n < \omega$.*

1. *For any K any variety beween CA and QEA, any positive $k \geq 1$, and any finite $l \geq 0$, the variety $\mathsf{SNr}_m\mathsf{K}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathsf{SNr}_m\mathsf{K}_{m+k}$ and RK_m is not finitely axiomatizable over $\mathsf{SNr}_m\mathsf{K}_{m+l}$ for any $0 < l < \omega$.*

2. *The variety of CA_ms having n–flat representations is not finitely axiomatizable over the variety of CA_ms having n–square representations.*

Proof. Item (1) follows from the discussion preceding the theorem. We prove item (2). Write $\mathfrak{C}_r$ for $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$ (used in the proof of theorem 4.4) not to clutter notation. The parameters m and n will be clear from context. Given positive k, then for any $r \geq k^2$, $\exists$ has a winning strategy in $G^k_r(\mathsf{At}(\mathfrak{A}(n,r))$ [27, Remark 15.13]. This implies using ultraproducts and an elementary chain argument that $\exists$ has a winning strategy in the ω–rounded game, in an elementary substructure of $\Pi_{r/U}\mathfrak{A}(n,r)/F$, hence the former is representable, and then so is the latter because RRA is a variety. Now $\exists$ has a winning strategy in $G^k_\omega(\mathfrak{A}(n,r))$ when $r \geq k^2$, hence, $\mathfrak{A}(n,r)$ embeds into a complete atomic relation algebra having a k–dimensional relational basis by [27, Theorem 12.25]. But this induces a winning strategy for $\exists$ in the game $G^{k'}_\omega(\mathsf{At}(\mathfrak{C}_r))$ with k' nodes and ω rounds, for $k' \geq k$, $k' \in \omega$ so that $\mathfrak{C}_r$ has a k' –square represenation, when $r \geq k'^2$. So if $n \geq m+2$, $k \geq 3$, and $r \geq k'^2$, then $\mathfrak{C}_r$ has an $n+1$–square representation, an n–flat one but does not have an $n+1$–flat one. But $\Pi_{r/U}\mathfrak{C}_r/F \in \mathsf{RCA}_m(\supseteq \mathsf{SNr}_m\mathsf{CA}_{n+1})$ by [27, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] and we are done. □

We now outline a similar construction; but due to the absence of diagonal elements here (the analogoue of) r appearing in $\mathfrak{A}(n,r)$ above is not merely a number, but it carries a linear order. Things here are a little bit more complicated technically but the idea in essence is very similar to that used for CAs. But the above result for CAs and QEAs is not re–obtained in its full stength for diagonal free algebras. In this respect, the third item in our coming theorem 4.5, which is (the main theorem) [32, Theorem 1.1] is *strictly weaker* than the result obtained in theorem 4.4, namely (using the notation precedng *op.cit*), that $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RQEA}_m$ (upon replacing $\mathfrak{C}(m,n,r)$ by $\mathfrak{D}(m,n,r)$.)

Theorem 4.5. *For $3 \leq m \leq n$ and $r < \omega$ there exists finite algebras $\mathfrak{D}(m,n,r) \in \mathsf{QEA}_m$.*

I. $\mathfrak{D}(m,n,r) \in \mathsf{Nr}_m\mathsf{QEA}_\mathsf{n}$,

II. $\mathfrak{Rd}_{\mathsf{Sc}}\mathfrak{D}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{Sc}_{n+1}$,

III. $\Pi_{r/U}\mathfrak{D}(m,n,r)$ *is elementarily equivalent to a polyadic equality algebra* $\mathfrak{C} \in \mathsf{Nr}_m\mathsf{QEA}_{\mathsf{n+1}}$.

We define the algebras $\mathfrak{D}(m,n,r)$ for $3 \leq m \leq n < \omega$ and r and then give a sketch of (II) given in detail in [32, p. 211–215]. We start with.

Definition 4.6. Define a function $\kappa : \omega \times \omega \to \omega$ by $\kappa(x,0) = 0$ (all $x < \omega$) and $\kappa(x,y+1) = 1 + x \times \kappa(x,y))$ (all $x,y < \omega$). For $n,r < \omega$ let

$$\psi(n,r) = \kappa((n-1)r,(n-1)r) + 1.$$

This is to ensure that $\psi(n,r)$ is sufficiently big compared to n,r for the proof of non-embeddability to work. The second parameter $r < \omega$ may be considered as a finite linear order of length r. For any $n < \omega$ and any linear order r, let

$$\mathfrak{B}(n,r) = \{\mathsf{Id}\} \cup \{a^k(i,j) : i < n-1; j \in r,\ k < \psi(n,r)\}$$

where $\mathsf{Id}, a^k(i,j)$ are distinct objects indexed by k,i,j. (So here every atom $a(i,j)$ is split into $\psi(n,r)$ subatoms). The *forbidden* triples) are:

$$\{(\mathsf{Id},b,c) : b \neq c \in \mathfrak{B}(n,r)\}$$
$$\cup$$
$$\{(a^k(i,j),a^{k'}(i,j),a^{k^*}(i,j')) : k,k',k^* < \psi(n,r),\ i < n-1,\ j' \leq j \in r\}.$$

Let $3 \leq m \leq n < \omega$. The set of m–basic matrices on $\mathfrak{R}$ is is a QEA_m atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. $\mathfrak{D}(m,n,r)$ is defined to be the complex algebra of the m–dimensional atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$, that is, $\mathfrak{D}(m,n,r) = \mathfrak{Cm}\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. Unlike the algebras $\mathfrak{C}(m,n,r)$ used to prove theorem 4.4, the algebras $\mathfrak{D}(m,n,r)$ are now finite. It is not hard to see that $3 \leq m,\ 2 \leq n$ and $r < \omega$ the algebra $\mathfrak{D}(m,n,r)$ satisfies all of the axioms defining QEA_m except, perhaps, the commutativity of cylindrifiers which it satisfies because

$\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$ is a (symmetric) cylindric basis, so that overlapping matrices amalgamate. Furthermore, if $3 \leq m \leq m'$, then $\mathfrak{D}(m,n,r) \cong \mathsf{Nr}_m\mathfrak{D}(m',n,r)$ via $X \mapsto \{f \in \mathsf{Mat}_{m'}(\mathsf{At}\mathfrak{R}) : f\restriction_{m\times m} \in X\}$.

We give a sketch of proof of 4.5(II), which is the heart and soul of the proof. Assume hoping for a contradiction that $\mathfrak{Rd}_{\mathsf{Sc}}\mathfrak{D}(m,n,r) \subseteq \mathsf{Nr}_m\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{Sc}_{n+1}$, some finite m,n,r. Then for $1 \leq t \leq n+1$, it can be shown inductively that there must be a 'large set' S_t of distinct elements of $\mathfrak{C}$, satisfying certain inductive assumptions, which we outline next. Here largness depends on t and weakens as t increases; for example S_n has only two elements. For each $s \in S_t$ and $i,j < n+2$ there is an element $\alpha(s,i,j) \in \mathfrak{B}(n,r)$ obtained from s by cylindrifying all dimensions in $(n+1)\setminus\{i,j\}$, then using substitutions to replace i,j by $0,1$. It can be shown that the triple $(\alpha(s,i,j), \alpha(s,j,k), \alpha(s,i,k))$ is consistent (not forbidden). The induction hypothesis says chiefly that $\mathsf{c}_n s$ is constant, for $s \in S_t$, and for $l < n$ there are fixed $i < n-1,\ j < r$ such that for all $s \in S_t,\ \alpha(s,l,n) \leq a(i,j)$. This defines, like in the proof of theorem 15.8 in [30] p.471, two functions $I : n \to (n-1),\ J : n \to r$ such that $\alpha(s,l,n) \leq a(I(l),J(l))$ for all $s \in S_t$. The *rank* $\mathsf{rk}(I,J)$ of (I,J) (as defined in definition 15.9 in [30]) is the sum (over $i < n-1$) of the maximum j with $I(l) = i,\ J(l) = j$ (some $l < n$) or -1 if there is no such j. From S_t one constructs a set S_{t+1} with index functions (I', J'), still relatively large (large in terms of the number of times we need to repeat the induction step) where the same induction hypotheses hold but where $\mathsf{rk}(I',J') > \mathsf{rk}(I,J)$. By repeating this enough times (more than nr times) we obtain a non-empty set T with index functions of rank strictly greater than $(n-1)\times(r-1)$, an impossibility. We sketch the induction step. Since I cannot be injective there must be distinct $l_1, l_2 < n$ such that $I(l_1) = I(l_2)$ and $J(l_1) \leq J(l_2)$. We may use l_1 as a "spare dimension" (changing the index functions on l will not reduce the rank). Since $\mathsf{c}_n s$ is constant, we may fix $s_0 \in S_{t-1}$ and choose a new element s' below $\mathsf{c}_l s_0 \cdot \mathsf{s}^n_l \mathsf{c}_l s$, with certain properties. Let $S_{t+1} = \{s' : s \in S_t \setminus \{s_0\}\}$. Re-establishing many of the induction hypotheses for S_{t+1} is not too hard. Also, it can be shown that $J'(l) \geq J(l)$ for all $l < n$. Since $(\alpha(s,i,j), \alpha(s,j,k), \alpha(s,i,k))$ is consistent and by the definition of the forbiden triples either $\mathsf{rng}(I')$ properly extends $\mathsf{rng}(I)$ or there is $l < n$ such that $J'(l) > J(l)$, hence $\mathsf{rk}(I',J') > \mathsf{rk}(I,J)$. The idea of constructing S_{t+1} from S_t is given pictorially on [31, Figure 2, p. 8] in the context of CAs. The essence of the ideas used in [31, 32] is the same. Suppose we are at stage t. Then every $x \in S_t$ gives a set of colours (atoms) denoted in [31] by $x(i,t)($ $i < t)$. One gets S_{t+1} from S_t by first 'glueing together' any two elements x, z of S_t, using $t+1$ as a spare dimension, first moving the tth co–ordinate of x to $t+1$ forming $\mathsf{s}^t_{t+1}x$. By fixing z and varying x one gets a huge number of different elements. Their $(t,t+1)$th colours cannot be controlled yet; they may not be the same. To get over this hurdle, one uses the pigeon–hole principal to pick the *still large set* S_{t+1} in which the $(t,t+1)$th colour is fixed to be the same. 'Largness' enables one to do so.

We summarize next the essence of the idea used in the solution of [22, Problem 2.12]:
In figure 2 in [31] there is a top element that is connected by coloured edges to the intermediate elements that are all connected to a bottom element. The number of elements (in this figure) is the number of colours plus one. So one gets the same control as rainbow algebras provided by (the second independent parameter) G. *The key idea here is that the proof of Ramsey in this context does not require an uncontrollable Ramsey number of 'spare dimensions', which were the versions used by Monk and Maddux before proving non finite axiomatizability [48, 37], but only one more than the number of colours used.*

For the above non–representable Monk–style algebras denoted by $\mathfrak{A}(n,r)$, $3 \leq m < n < \omega$ and $r \in \omega$, it is easy to see that $\exists$ cannot win the usual infinite atomic game. But this time one can use 'a hyperbasis game' denoted by $G_r^{m,n+1}$ in [27] with r denoting the number of rounds, to pin point the leask $k > n$ for which $\mathfrak{A}(n,r)$ 'stops to be representable' getting the sharper result we want. The game $G_r^{m,n+1}$ is stronger than G_ω, involving additional amalgamation moves played on $n+1$–dimensional m–wide hypernetworks. One can show that $\forall$ has a winning strategy in $G_r^{m,n+1}(\mathsf{At}\mathfrak{A}(n,r))$, using exactly $n+1$ nodes (for any $r < \omega$), getting the same control we get from rainbows using the parameter G, and in fact the best possible. This is the approach adopted in [30]. Here $\mathfrak{A}(n,r)$ has an n–dimensional cylindric basis, but no $n+1$*–dimensional hypebasis*. Worthy of note, is that the last condition is strictly stronger than 'not having an $n+1$–dimensional cylindric basis'. Relation algebras having n–dimensional cylindric basis but no $n+1$–dimensional cylindric basis were constructed by Maddux. We refer to [31] for more. In the proof of theorem 4.4, one uses that $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RQEA}_m$. As stated in the last item of theorem 4.5, we do not guarantee that the ultraproduct on r of the $\mathfrak{D}(m,n,r)$s $(2 < m < n < \omega)$ is representable. A standard Lös argument shows that $\Pi_{r/U}\mathfrak{C}(m,n,r) \cong \mathfrak{C}(m,n,\Pi_{r/U}r)$ and $\Pi_{r/U}r$ contains an infinite ascending sequence. Here one extends the definition of ψ by letting $\psi(n,r) = \omega$, for any infinite linear order r. The infinite algebra $\mathfrak{D}(m,n,J) \in \mathbf{El}\mathsf{Nr}_n\mathsf{QEA}_{n+1}$ when J is the infinite linear order as above. Since $\Pi_{r/U}r$ is such, then we get $\Pi_{r/U}\mathfrak{D}(m,n,r) \in \mathbf{El}\mathsf{Nr}_m\mathsf{QEA}_{n+1}(\subseteq \mathbf{S}\mathsf{Nr}_m\mathsf{QEA}_{n+1})$, cf. [32, pp.216-217]. This suffices to show that for any K having signature between Sc and QEA, for any $2 < m < \omega$, and for any positive k, the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{K}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{K}_{m+k}$.

The lifting technique via ultraproducts used in Theorem 3.11 can be used to show the following by lifting the result in theorem 4.4 to the transfinite.

Theorem 4.7. *Let α be any ordinal > 2 possibly infinite. Then for any $r \in \omega$, and $k \geq 1$, there exists $\mathfrak{A}_r \in \mathbf{S}\mathsf{Nr}_\alpha\mathsf{QEA}_{\alpha+k}$ such that $\mathfrak{Rd}_{ca}\mathfrak{A}_r \notin \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k+1}$ and $\Pi_{r/U}\mathfrak{A}_r \in \mathsf{RQEA}_\alpha$ for any non–principal ultrafilter U on ω.*

On the other hand, including diagonal free algebras like Scs, in [32, Theorem 3.1] only the following is proved lifting the (weaker) result in theorem 4.5 to the transfinite:

Theorem 4.8. *Let $\alpha > 2$. Then for any $r \in \omega$, for any finite $k \geq 1$, there exist $\mathfrak{B}^r \in \mathsf{SNr}_\alpha\mathsf{QEA}_{\alpha+k}$, and $\mathfrak{Rd}_{\mathsf{Sc}}\mathfrak{B}^r \notin \mathsf{SNr}_\alpha\mathsf{Sc}_{\alpha+k+1}$ such $\Pi_{r/U}\mathfrak{B}^r \in \mathsf{SNr}_\alpha\mathsf{QEA}_{\alpha+k+1}$.*

5 Recent results: Counting varieties and models:

Let α be an infinite ordinal. In [9] Andréka and Németi, among so many other things, count the number of varieties of RCA_α obtaining their infinitary combinatorial results in ZFC. This paper that appeared in the Transaction of the Amerian Mathematical Society in August 2017, solves a long standing open problem in algebraic logic [22, Problem 4.2]. As a byproduct it solves another longstanding open problem [22, Problem 2.13], and provides yet another solution to *the central problem in algebraic logic*, namely, [22, Problem 4.1]. The first problem asks for the number of subvarieties of RCA_α, the second asks whether the class described in item (iii) of [22, Theorem 2.6.50], exhausts the variety RCA_α. Algebras in this class are given a new name in this paper, namely, *Endo dimension-complemented* algebras, and the class consisting of all such algebras of dimension α is denoted by Edc_α. There are at least continuum many subvarieties of RCA_α and because there are $|\alpha|$ many equations in its signature, there are at most $2^{|\alpha|}$ many varieties. In [9] it is proved that this maximum is attained.

The kind of investigation in [9] is blatantly reminiscent of Shelah's investigation in model theory of counting models of a first order theory (better known as stability theory). The idea of stability theory is to find dichotomy between theories. There are two cases; in the first class we can find a classification theorem of the number of models of the theory, and in the second we convince ourselves that are the theories with no reasonable characterization. The first case includes κ-stable, stable, superstable and so-called totally trancendental theories. [1]. Peano arithmetic, Set Theory are *unstable*; they belong to the second category. A branch that is wide open, is the interaction of stability theory and algebraic logic. In what follows we present briefly recent results in this direction [61]: An instance of this kind of investigations was recently applied to Vaught' conjecture which states that the numbr of countable models of a complete countable theory is either countable or the continuum. Recent work in algebraic logic has found contact with Vaught's conjecture [51, 10]; we give

[1]Totally transcendental theories are those such that every formula has Morley rank less than infinity. This is equivalent to being ω-stable.

only a tiny sample. We assume familiarity with basic descriptive set theory, notions like Polish spaces, G_δ sets, Borel sets, analytic sets will be used without warning. Let $\mathfrak{A}$ be any Boolean algebra, we denote its Stone space by $\mathfrak{A}^*$. It is easy to see that if A is countable, then $\mathfrak{A}^*$ (its Stone space) is *Polish*, (i.e., separable and completely metrizable). Now, suppose $\mathfrak{A} \in \mathsf{Lf}_\omega$ is countable. Let $\mathcal{H}'(\mathfrak{A}) = \bigcap_{i<\omega, x\in A}(N_{-\mathsf{c}_i x} \cup \bigcup_{j<\omega} N_{\mathsf{s}^i_j x})$ and let $\mathcal{H}(\mathfrak{A}) = \mathcal{H}'(\mathfrak{A}) \cap \bigcap_{i\neq j\in\omega} N_{-\mathsf{d}_{ij}}$. Recall that N_x is the clopen set $\{F \in \mathsf{Uf}(\mathfrak{A}) : x \in F\}$ in the Stone topology. Observe that $\mathcal{H}(\mathfrak{A})$ are is a G_δ dense subset of $\mathfrak{A}^*$ and is therefore a Polish space, too. For $\mathfrak{B} \in \mathsf{Lf}_\omega$, $x \in \mathfrak{B}$ and $\tau : \omega \to \omega$, the substitution operator $\mathsf{s}^+_\tau x$ is defined as in [22, Definition 1.11.13]; so that particular s^+_τ is a Boolean endomorphism. Assume $F \in \mathfrak{A}^*$. For any $x \in A$, define the function rep_F to be $\text{rep}_F(x) = \{\tau \in {}^\omega\omega : \mathsf{s}^+_\tau x \in F\}$. For a theory T, (recall that) $\mathfrak{Fm}_T \in Lf_\omega$ denotes the Lindenbaum-Tarski quotient algebra corresponding to T. We have the following results due to G. Sági and D. Sziráki [51]:

Theorem 5.1. *1. If $F \in \mathfrak{A}^*$ then rep_F is a homomorphism from $\mathfrak{A}$ onto an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{\mathsf{reg}}_\omega$, with base ω. Conversely, if h is a homomorphism from $\mathfrak{A}$ onto an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{\mathsf{reg}}_\omega$ with base ω, then there is a unique $F \in \mathfrak{A}^*$ such that $h = \text{rep}_F$.*

2. Let T be a consistent first order theory in a countable language. Let M_0 and M_1 be two models of T whose universe is ω. Suppose $F_0, F_1 \in (\mathfrak{Fm}_T)^*$ are such that rep_{F_i} are homomorphisms from $\mathfrak{Fm}_T$ onto $\mathsf{Cs}^{\mathsf{M}_i}_\omega$, for $i = 0, 1$. If $\rho : \omega \longrightarrow \omega$ is a bijection, then the following are equivalent:

(i) $\rho : \mathsf{M}_0 \longrightarrow \mathsf{M}_1$ *is an isomorphism.*

(ii) $F_1 = \mathsf{s}^+_\rho F_0 = \{\mathsf{s}^+_\rho x : x \in F_0\}$.

These last theorem allows us to study models and count them via corresponding ultrafilters. Consider the following action of S_∞ on the space $\mathfrak{H}(\mathfrak{A})$, $(\tau, F) \longmapsto \mathsf{s}^+_\tau F$. Then Vaught's conjecture [34] states that the previous action has either countably many orbits or else continuum many. Let $\mathfrak{A} \in \mathsf{Lf}_\omega$ be countable. For an ultrafilter F of $\mathfrak{A}$ and $a \in A$, define $Sat_F(a) = \{t|_{\Delta a} : t \in {}^\omega\omega : \mathsf{s}^+_t a \in \mathcal{F}\}$. Let $\mathcal{E}$ be the following equivalence relation on $\mathcal{H}(\mathfrak{A})$:

$$\mathcal{E} = \{(F_0, F_1) : (\forall a \in A)(|Sat_{F_0}(a)| = |Sat_{F_1}(a)|)\}.$$

Then it is proved in [10] that $\mathcal{E}$ is Borel in the product space $\mathcal{H}(\mathfrak{A}) \times \mathcal{H}(\mathfrak{A})$. We say that $F_0, F_1 \in \mathfrak{H}(\mathfrak{A})$ are *distinguishable* if $(F_0, F_1) \notin \mathcal{E}$. We also say that two models of a theory T are *distinguishable* if their corresponding ultrafilters in $\mathfrak{Fm}_T$ are distinguishable. That is, two models are distinguishable if they disagree in the number of realizations they have for some formula.

Theorem 5.2 (Harrington-Kechris-Louveau [34]). *Let X be a Polish space and E a Borel equivalence relation on X. Then E has at most many countably many orbits or perfectly many.*

Corollary 5.3. *Let T be a first order complete theory in a countable language. If T has an uncountable set of countable models that are pairwise distinguishable, then actually it has such a set of size 2^ω.*

We next give a new proof of Morley's theorem.[2]

Theorem 5.4. *(Morley) Suppose T is a first order complete theory in a countable language with equality.* (Morley) *If T has more than ω_1 countable models, then it has 2^ω countable models.*

Proof. Let T be a first order theory in a countable language with equality, and let $\mathfrak{A} = \mathfrak{Fm}_T$. Then S_∞ is a Polish group with respect to composition of functions and the topology it inherits from the Baire space ${}^\omega\omega$. Consider the map $J : \mathsf{S}_\infty \times \mathfrak{H}(\mathfrak{A}) \longrightarrow \mathcal{H}(\mathfrak{A})$ defined by $J(\rho, F) = \mathsf{s}_\rho^+ F$ for all $\rho \in \mathsf{S}_\infty$, $F \in \mathcal{H}(\mathfrak{A})$. Then J is a well defined action of S_∞ on $\mathcal{H}(\mathfrak{A})$. Also J is a continuous map from $\mathsf{S}_\infty \times \mathcal{H}(\mathfrak{A})$ (with the product topology) to $\mathcal{H}(\mathfrak{A})$ because for an arbitrary $a \in A$, $J^{-1}(N_a \cap \mathcal{H}(\mathfrak{A})) = \bigcup_{\tau \in \mathsf{S}_\infty} \left(\{\mu^{-1} : \mu \in \mathsf{S}_\infty,\ \mu|_{\Delta a} = \tau|_{\Delta a}\} \times [N_{\mathsf{s}_\tau^+ a} \cap \mathcal{H}(\mathfrak{A})]\right)$. To see why, let $f : S_\infty \longrightarrow S_\infty$ be the map given by $f(\tau) = \tau^{-1}$. Observe that f is continuous and open. Hence, $\{\mu^{-1} : \mu \in S_\infty,\ \mu|_{\Delta a} = \tau|_{\Delta a}\} = f^*(\{\mu \in S_\infty : \mu|_{\Delta a} = \tau|_{\Delta a}\})$ is the image of an open set via an open map, and is therefore open. Now, let (ρ, F) be an arbitrary element in $S_\infty \times \mathcal{H}(\mathfrak{A})$. We have the following:

$$\begin{aligned}
(\rho, F) \in \bigcup_{\tau \in \mathsf{S}_\infty} & \left(\{\mu^{-1} : \mu \in S_\infty,\ \mu|_{\Delta a} = \tau|_{\Delta a}\} \times [N_{\mathsf{s}_\tau^+ a} \cap \mathcal{H}(\mathfrak{A})]\right) \\
& \Longleftrightarrow (\exists \tau \in \mathsf{S}_\infty)[(\exists \mu \in S_\infty)(\mu|_{\Delta a} = \tau|_{\Delta a} \wedge \rho = \mu^{-1}) \wedge \mathsf{s}_\tau^+ a \in F] \\
& \Longleftrightarrow (\exists \tau \in \mathsf{S}_\infty)[(\exists \mu \in S_\infty)(\mu|_{\Delta a} = \tau|_{\Delta a} \wedge \rho^{-1} = \mu) \wedge \mathsf{s}_\tau^+ a \in F] \\
& \Longleftrightarrow (\exists \tau \in \mathsf{S}_\infty)[\rho^{-1}|_{\Delta a} = \tau|_{\Delta a} \wedge \mathsf{s}_\tau^+ a \in F] \\
& \Longleftrightarrow (\mathsf{s}_\rho^+)^{-1} a = \mathsf{s}_{\rho^{-1}}^+ a \in F \\
& \Longleftrightarrow a \in \mathsf{s}_\rho^+ F \\
& \Longleftrightarrow J(\rho, F) = \mathsf{s}_\rho^+ F \in N_a \cap \mathcal{H}(\mathfrak{A}) \\
& \Longleftrightarrow (\rho, F) \in J^{-1}(N_a \cap \mathcal{H}(\mathfrak{A})).
\end{aligned}$$

It follows that the the orbit equivalence relation is analytic [12, Theorem 3.2]. By Burgess' Theorem [20] if there are more than ω_1 orbits, then there are 2^ω orbits. But the number of orbits here is exactly the number of non-isomorphic countably infinite models of T. This completes the proof. □

6 Omitting types and complete representations

The next theorem is an algebraic version of an omitting types Theorem proving [58, Theorem 3.2.9-3.2.10]. But first a definition.

[2] An analogous proof is obtained independently by Gabor Sagi.

Definition 6.1. Let λ be a cardinal. If $\mathfrak{A} \in \mathsf{RCA}_n$ and $\mathbf{X} = (X_i : i < \lambda)$ is family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ *is omitted in* $\mathfrak{C} \in \mathsf{Gs}_n$, if there exists an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. If $X \subseteq \mathfrak{A}$ and $\prod X = 0$, then we refer to X as a *non-principal type* of $\mathfrak{A}$.

Theorem 6.2. *Let $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ be countable. Let $\lambda < 2^\omega$ and let $\mathbf{X} = (X_i : i < \lambda)$ be a family of non-principal types of $\mathfrak{A}$. If $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and the X_is are non–principal ultrafilters, then $\mathbf{X}$ can be omitted in a Gs_n.*

Proof. We assume that $\mathfrak{A}$ is simple. We have $\prod^{\mathfrak{B}} X_i = 0$ for all $i < \kappa$ because, $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. Since we can assume that $\mathfrak{B}$ is a locally finite, then $\mathfrak{B} = \mathfrak{Fm}_T$ for some countable consistent theory T. For each $i < \kappa$, let $\Gamma_i = \{\phi/T : \phi \in X_i\}$. Let $\mathbf{F} = (\Gamma_j : j < \kappa)$ be the corresponding set of types in T. Then each Γ_j $(j < \kappa)$ is a non-principal and *complete n-type* in T, because each X_j is a maximal filter in $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{B}$. We use [**?**, Theorem 5.16, Chapter IV] (+): *Assume that λ is an infinite regular cardinal. Suppose that T is a first order theory, $|T| \leq \lambda$ and ϕ is a formula consistent with T, then there exist models $\mathsf{M}_i : i < {}^\lambda 2$, each of cardinality λ, such that ϕ is satisfiable in each, and if $i(1) \neq i(2)$, $\bar{a}_{i(l)} \in \mathsf{M}_{i(l)}$, $l = 1, 2,$, $\mathsf{tp}(\bar{a}_{l(1)}) = \mathsf{tp}(\bar{a}_{l(2)})$, then there are $p_i \subseteq \mathsf{tp}(\bar{a}_{l(i)})$, $|p_i| < \lambda$ and $p_i \vdash \mathsf{tp}(\bar{a}_{l(i)})$ ($\mathsf{tp}(\bar{a})$ denotes the complete type realized by the tuple $\bar{a}$).*
By (+) let $(\mathsf{M}_i : i < 2^\omega)$ be a set of countable models for T that overlap only on principal maximal types. Asssume for contradiction that for all $i < 2^\omega$, there exists $\Gamma \in \mathbf{F}$, such that Γ is realized in M_i. Let $\psi : 2^\omega \to \wp(\mathbf{F})$, be defined by $\psi(i) = \{F \in \mathbf{F} : F \text{ is realized in } \mathsf{M}_i\}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq j$, $\psi(i) \cap \psi(j) = \emptyset$, for if $F \in \psi(i) \cap \psi(j)$, then it will be realized in M_i and M_j, and so it will be principal. This implies that $|\mathbf{F}| = 2^\omega$ which is impossible. Hence we obtain a model $\mathsf{M} \models T$ omitting $\mathbf{X}$ in which ϕ is satisfiable. The map f defined from $\mathfrak{A} = \mathfrak{Fm}_T$ to $\mathsf{Cs}_n^{\mathsf{M}}$ (the set algebra based on M [22, 4.3.4]) via $\phi_T \mapsto \phi^{\mathsf{M}}$.

□

Using the full power of (+) one can replace ω in Theorem 6.2 by any regular uncountable cardinal μ.

Definition 6.3. Let $n < \omega$. Then $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable*, if there exists $\mathfrak{B} \in \mathsf{Gs}_n$ and an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such for all $X \subseteq \mathfrak{A}$, $f(\prod X) = \bigcap_{x \in X} f(x)$ whenever $\prod X$ exists.

The maximality condition cannot be removed as we show (algebraically) in the following theorem.

Theorem 6.4. *Let κ be an infinite cardinal. Then there exists a $\mathfrak{C} \in \mathsf{CA}_\omega$ such that for all $2 < n < \omega$, $|\mathfrak{Nr}_n\mathfrak{C}| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$, but $\mathfrak{Nr}_n\mathfrak{C}$ is not completely representable. Thus the non–principal type of co–atoms of $\mathfrak{Nr}_n\mathfrak{C}$ cannot be omitted.*

Proof. One uses the ideas in [56] replacing ω and ω_1 by κ and 2^κ, respectively, constructing $\mathfrak{C}$ from a relation algebra. The resulting (new) relation algebra $\mathfrak{R}$ has an ω dimensional amalgamation class S, cf. [56, Lemma 3]. Using the notation in [56, Lemma 6], let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Ca}(S)$ generated by X'; the latter is defined just before the lemma. Then $\mathfrak{R} = \mathfrak{Ra}(\mathfrak{C})$, cf. [56, Lemmata 6, 7], but $\mathfrak{R}$ has no complete representation [56, Lemma 2]. Then $\mathfrak{Nr}_n\mathfrak{C}$ $(2 < n < \omega)$ is atomic, but has no complete representation. By Lemma 3.3, $\exists$ has a winning strategy in $\mathbf{G}_\omega(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$, hence she has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$, *a fortiori* in $G_k(\mathsf{At}\mathfrak{Nr}_n\mathfrak{C})$ for all $k \in \omega$, hence by coding the winning strategy's of the G_k's in first order sentences, we get that $\mathfrak{Nr}_n\mathfrak{C}$ satisfies these first order sentences which are precisely (by definition) the Lyndon conditions. We show that the ω–dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. Then $\mathfrak{Nr}_n\mathfrak{C}$ $(2 < n < \omega)$ is atomic, but has no complete representation.

□

6.1 Complete representations

In the previous construction $\mathsf{At}\mathfrak{R}$ also satisfies the Lyndon conditions by [70, Theorem 33] but is not completely representable. Since the Lyndon conditions characterize the elementary closure of the class of completely represntable algebras for RAs and CAs, we get:

Corollary 6.5. *[26] Let $2 < n < \omega$. Then the classes* CRRA *and* CRCA_n *are not elementary.*

The next Theorem follows from the proof of Theorem 6.4 and [59, Theorem 5.3.6].

Theorem 6.6. *For* $2 < n < \omega$ $\mathsf{CRCA}_n \subseteq \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. *At least two of the previous three classes are distinct but the elementary closure of each coincides with* LCA_n. *Furthermore, all three classes coincide on the class of atomic algebras having countably many atoms.*

We first define a game $\mathbf{H}$ that involves certain *hypernetworks*. A λ–neat hypernetwork is roughly a network endowed with hyperdeges of length $\neq n$ allowed to get arbitrarily long but are of finite length, and such hyperedges get their labels from a non–empty set of labels Λ; such that all so–called *short hyperedges* are constantly labelled by $\lambda \in \Lambda$. The board of the game consists of λ-neat hypernetworks:

Definition 6.7. For an n–dimensional atomic network N on an atomic CA_n and for $x, y \in \mathsf{nodes}(N)$, set $x \sim y$ if there exists $\bar{z}$ such that $N(x, y, \bar{z}) \leq \mathsf{d}_{01}$. Define the equivalence relation $\sim$ over the set of all finite sequences over $\mathsf{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$. (It can be easily checked that this indeed an equivalence relation). A *hypernetwork* $N = (N^a, N^h)$ over an atomic CA_n consists of an n–dimensional network N^a together with a labelling function for hyperlabels $N^h : {}^{<\omega}\mathsf{nodes}(N) \to \Lambda$ (some arbitrary set of hyperlabels Λ) such that for $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(N)$ if $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$. If $|\bar{x}| = k \in \mathbb{N}$ and $N^h(\bar{x}) = \lambda$, then we say that λ is a k-ary hyperlabel. $\bar{x}$ is referred to as a k–ary hyperedge, or simply a hyperedge. A hyperedge $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ is *short*, if there are $y_0, \ldots, y_{n-1}$ that are nodes in N, such that $N(x_i, y_0, \bar{z}) \leq \mathsf{d}_{01}$ or $\ldots N(x_i, y_{n-1}, \bar{z}) \leq \mathsf{d}_{01}$ for all $i < |x|$, for some (equivalently for all) $\bar{z}$. Otherwise, it is called *long.* This game involves, besides the standard cylindrifier move, two new amalgamation moves. Concerning his moves, this game with m rounds ($m \leq \omega$), call it $\mathbf{H}_m$, $\forall$ can play a cylindrifier move, like before but now played on λ— neat hypernetworks (λ a constant label). Also $\forall$ can play a *transformation move* by picking a previously played hypernetwork N and a partial, finite surjection $\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N, θ). $\exists$'s response is mandatory. She must respond with $N\theta$. Finally, $\forall$ can play an *amalgamation move* by picking previously played hypernetworks M, N such that $M{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N{\restriction}_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$, and $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. This move is denoted (M, N). To make a legal response, $\exists$ must play a λ_0–neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.

Theorem 6.8. *Let α be a countable atom structure. If $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then there exists a complete $\mathfrak{D} \in \mathsf{RCA}_\omega$ such that $\mathfrak{Cm}\alpha \cong \mathfrak{Nr}_n\mathfrak{D}$ and $\alpha \cong \mathsf{At}\mathfrak{Nr}_n\mathfrak{D}$.*

Proof. Fix some $a \in \alpha$. The game $\mathbf{H}_\omega$ is designed so that using $\exists$ s winning strategy in the game $\mathbf{H}_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of λ–neat hypernetworks where M_0 is $\exists$'s response to the initial $\forall$-move a, such that: If M_r is in the sequence and $M_r(\bar{x}) \leq \mathsf{c}_i a$ for an atom a and some $i < n$, then there is $s \geq r$ and $d \in \mathsf{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y}_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if M_r is in the sequence and θ is any partial isomorphism of M_r, then there is $s \geq r$ and a partial isomorphism θ^+ of M_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(M_r)$ (This can be done using $\exists$'s responses to amalgamation moves). Now let $\mathfrak{M}_a$ be the limit of this sequence, that is $\mathfrak{M}_a = \bigcup M_i$, the labelling of $n-1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the M_is are nested. Let L be the signature with one n-ary relation for each $b \in \alpha$, and one k–ary predicate symbol for each k–ary hyperlabel λ. *Now we work in $L_{\infty,\omega}$.* For fixed $f_a \in {}^{\omega}\mathsf{nodes}(\mathfrak{M}_a)$, let $\mathfrak{U}_a = \{f \in {}^{\omega}\mathsf{nodes}(\mathfrak{M}_a) : \{i < \omega : g(i) \neq f_a(i)\} \text{ is finite}\}$. We make $\mathfrak{U}_a$ into the base of an L relativized structure $\mathcal{M}_a$ like in [70, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0 \ldots, i_{k-1} < \omega$, k–ary hyperlabels λ, and all L-formulas

ϕ, ϕ_i, ψ, and $f \in U_a$:

$$\begin{aligned}
\mathcal{M}_a, f \models b(x_{l_0} \ldots, x_{l_{n-1}}) &\iff \mathcal{M}_a(f(l_0), \ldots, f(l_{n-1})) = b, \\
\mathcal{M}_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff \mathcal{M}_a(f(i_0), \ldots, f(i_{k-1})) = \lambda, \\
\mathcal{M}_a, f \models \neg\phi &\iff \mathcal{M}_a, f \not\models \phi, \\
\mathcal{M}_a, f \models (\bigvee_{i \in I} \phi_i) &\iff (\exists i \in I)(\mathcal{M}_a, f \models \phi_i), \\
\mathcal{M}_a, f \models \exists x_i \phi &\iff \mathcal{M}_a, f[i/m] \models \phi, \text{ some } m \in \mathsf{nodes}(\mathcal{M}_a).
\end{aligned}$$

For any such L-formula ϕ, write $\phi^{\mathcal{M}_a}$ for $\{f \in \mathfrak{U}_a : \mathcal{M}_a, f \models \phi\}$. Let $D_a = \{\phi^{\mathcal{M}_a} : \phi \text{ is an } L\text{-formula}\}$ and $\mathfrak{D}_a$ be the weak set algebra with universe D_a. Let $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$. Then $\mathfrak{D}$ is a generalized *complete* weak set algebra [22, Definition 3.1.2 (iv)]. Now we show that $\alpha \cong \mathsf{At}\mathfrak{Nr}_n\mathfrak{D}$ and $\mathfrak{Cm}\alpha \cong \mathfrak{Nr}_n\mathfrak{D}$. By density, we get that $\mathfrak{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. □

Theorem 6.9. *For $2 < n < \omega$, any class $\mathbf{K}$ such that $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary.*

Proof. We use the construction in [60, Theorem 5.12] based on ideas in [70]. [3] The algebra $\mathfrak{C}_{\mathbb{Z},\mathbb{N}}(\in \mathsf{RCA}_n)$ based on $\mathbb{Z}$ (greens) and $\mathbb{N}$ (reds) denotes the rainbow-like algebra where the reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega(= \mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathbb{N}$, but now the triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$. It can be shown that $\forall$ has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C})$ played on coloured graphs [26]. The rough idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathbb{N}$. In ω many rounds $\forall$ forces a win, so by lemma 3.3, $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. In more detail, in the initial round $\forall$ plays a graph M with nodes $0, 1, \ldots, n-1$ such that

[3] The article [70] claims that it proves that any class K such that $\mathsf{RaCA}_\omega \cap \mathsf{CRRA} \subseteq \mathsf{K} \subseteq \mathbf{S}_c\mathsf{RaCA}_5$ is not elementary. But there a mistake in the proof which is assuming that the implication $\mathsf{At}\mathfrak{A} \in \mathsf{AtRaCA}_\omega \implies \mathfrak{A} \in \mathsf{RaCA}_\omega$ which is not true in general. This implication is also not true for RRA and RCA_n by Theorem 3.5. However, for CRRA, CRCA_n, LCA_n and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ $(n < m)$ it is true; for e.g. $\mathsf{At}\mathfrak{A} \in \mathsf{AtCRCA}_n \implies \mathfrak{A} \in \mathsf{CRCA}_n$. The mistake was corrected in [25] proving the weaker result of the non-first order definability of any class K such that $\mathbf{S}_c\mathsf{RaCA}_\omega \cap \mathsf{CRRA} \subseteq \mathsf{K} \subseteq \mathbf{S}_c\mathsf{RaCA}_5$. This was strengthened by the present author by replacing $\mathbf{S}_c\mathsf{RaCA}_\omega$ by $\mathbf{S}_d\mathsf{RaCA}_\omega$. In the cylindric (present) case adjoining the auxillary construction in [52], we could prove the CA analogue of the stronger result alleged in [70].

$M(i,j) = \mathsf{w}_0$ for $i < j < n-1$ and $M(i,n-1) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, $M(0,n-1) = \mathsf{g}_0^0$ and $M(0,1,\ldots,n-2) = \mathsf{y}_{\mathbb{Z}}$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0,\ldots,n-2)$ and demands a node n with $M_2(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, and $M_2(0,n) = \mathsf{g}_0^{-1}$. $\exists$ must choose a label for the edge $(n+1,n)$ of M_2. It must be a red atom r_{mk}, $m,k \in \mathbb{N}$. Since $-1 < 0$, then by the 'order preserving' condition we have $m < k$. In the next move $\forall$ plays the face $(0,\ldots,n-2)$ and demands a node $n+1$, with $M_3(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, such that $M_3(0,n+2) = \mathsf{g}_0^{-2}$. Then $M_3(n+1,n)$ and $M_3(n+1,n-1)$ both being red, the indices must match. $M_3(n+1,n) = r_{lk}$ and $M_3(n+1,r-1) = r_{km}$ with $l < m \in \mathbb{N}$. In the next round $\forall$ plays $(0,1,\ldots n-2)$ and re-uses the node 2 such that $M_4(0,2) = \mathsf{g}_0^{-3}$. This time we have $M_4(n,n-1) = \mathsf{r}_{jl}$ for some $j < l < m \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$. In more detail, in the initial round $\forall$ plays a graph M with nodes $0,1,\ldots,n-1$ such that $M(i,j) = \mathsf{w}_0$ for $i < j < n-1$ and $M(i,n-1) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, $M(0,n-1) = \mathsf{g}_0^0$ and $M(0,1,\ldots,n-2) = \mathsf{y}_{\mathbb{Z}}$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0,\ldots,n-2)$ and demands a node n with $M_2(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, and $M_2(0,n) = \mathsf{g}_0^{-1}$. $\exists$ must choose a label for the edge $(n+1,n)$ of M_2. It must be a red atom r_{mk}, $m,k \in \mathbb{N}$. Since $-1 < 0$, then by the 'order preserving' condition we have $m < k$. In the next move $\forall$ plays the face $(0,\ldots,n-2)$ and demands a node $n+1$, with $M_3(i,n) = \mathsf{g}_i$ $(i = 1,\ldots,n-2)$, such that $M_3(0,n+2) = \mathsf{g}_0^{-2}$. Then $M_3(n+1,n)$ and $M_3(n+1,n-1)$ both being red, the indices must match. $M_3(n+1,n) = r_{lk}$ and $M_3(n+1,r-1) = r_{km}$ with $l < m \in \mathbb{N}$. In the next round $\forall$ plays $(0,1,\ldots n-2)$ and re-uses the node 2 such that $M_4(0,2) = \mathsf{g}_0^{-3}$. This time we have $M_4(n,n-1) = \mathsf{r}_{jl}$ for some $j < l < m \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$. We have proved the required. Since $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ for all $k \in \omega$, so that $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \in \mathbf{El}\mathsf{CRCA}_n$. With some more effort it can be proved that $\exists$ has a winning strategy σ_k say in $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ for all $k \in \omega$. We can assume that σ_k is deterministic. Let $\mathfrak{D}$ be a non–principal ultrapower of $\mathfrak{C}_{\mathbb{Z},\mathbb{N}}$. Then $\exists$ has a winning strategy σ in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$ — essentially she uses σ_k in the k'th component of the ultraproduct so that at each round of $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. We can also assume that $\mathfrak{C}_{\mathbb{Z},\mathbb{N}}$ is countable by replacing it by the term algebra. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in this manner. One defines $\mathfrak{A}_{i+1}$ be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that σ selects in a play of $G(\mathsf{At}\mathfrak{D})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$. Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$ and $\mathfrak{B} \equiv \mathfrak{C}_{\mathbb{Z},\mathbb{N}}$. Thus by Lemma 6.8 $\mathsf{At}\mathfrak{B} \in \mathsf{AtNr}_n\mathsf{CA}_\omega$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. (This does not imply that $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, cf.

example 6.10). Since $\mathfrak{B} \subseteq_d \mathfrak{CmAt}\mathfrak{B}$, $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$, so $\mathfrak{B} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$. Being countable, it follows by [59, Theorem 5.3.6] that $\mathfrak{B} \in \mathsf{CRCA}_n$. But $\forall$ has a winning strategy in $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$, hence by Lemma 3.3, $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Let $\mathbf{K}$ be a class between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Then $\mathbf{K}$ is not elementary, because $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}(\supseteq \mathbf{K})$, $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n(\subseteq \mathbf{K})$, and $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \equiv \mathfrak{B}$. We now use the construction in [52], where two atomic algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_n$ are constructed such that, $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$ where $\mathbf{S}_d$ is the operation of forming dense subalgebras. Thus $\mathfrak{B} \in \mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \sim \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \nsubseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$, there can be no elementary class between $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. Having already eliminated elementary classes between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we are done. □

The next example alerts us to the fact that if $\mathfrak{E} \in \mathsf{Cs}_n$ is atomic with atom structure α satifying $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$ and $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{E}$ itself may not be in $\mathsf{Nr}_n\mathsf{CA}_\omega$.

Example 6.10. Assume that $1 < n < \omega$. Let $V = {}^n\mathbb{Q}$ and let $\mathfrak{A} \in \mathsf{Cs}_n$ have universe $\wp(V)$. Then $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Let $y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}$ and $\mathfrak{E} = \mathfrak{Sg}^{\mathfrak{A}}(\{y\} \cup X)$, where $X = \{\{s\} : s \in V\}$. Now $\mathfrak{E}$ and $\mathfrak{A}$ having same top element V, share the same atom structure, namely, the singletons, so $\mathfrak{CmAt}\mathfrak{E} = \mathfrak{A}$. Thus $\mathsf{At}\mathfrak{E} \in \mathsf{AtNr}_n\mathsf{CA}_\omega$ and $\mathfrak{A} = \mathfrak{CmAt}\mathfrak{E} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathfrak{E} \subseteq_d \mathfrak{A}$, so $\mathfrak{E} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$, but as proved in [67] $\mathfrak{E} \notin \mathbf{El}\mathsf{Nr}_m\mathsf{CA}_{n+1} \subseteq \mathsf{Nr}_m\mathsf{CA}_{n+1} \supseteq \mathsf{Nr}_n\mathsf{CA}_\omega$.

7 The long and winding road from Hilbert's problems in 1900 to axiomatizing general relativity in first order logic

Hilbert gave a second order many sorted axiomatization of Euclidean geometry. Tarski went further. In his 1926-27 lectures at the university of Warsaw, Tarski gave an axiomatic development of elementary Euclidean geometry in one sorted first order logic. He proved that his system of geometry admits elimination of quantifiers, the theory is complete and decidable. Later, he gave a first order axiomatization to different geometries whether Euclidean or not [69]. Suppes, and before him Reinenbach suggested that theories like the special theory of relativity whose mathematical model is Minkowski's geometry can be, and indeed, should be formalized in first order logic. Andréka, Madárasz, and Németi applied this logistic viewpoint to axiomatizing Einstein's *general* theory of relativity in first order logic. The investigations triggered off by Andréka and Németi to deal with Problem 6 in Hilbert's list, are flourashing till the present day. Mádarasz [38] gives an excellent lucid acount of the Andréka and Németi axiomatic approach to the special theory of relativity, dealing also with so-called accelarating observers as an intermediate phase between special and general relativity. In her strictly first order axiomatization of special relativity, Mádarasz deduces bizzare consequences demonstrated by Einstein such as masses increase, clocks run slower, measuring rods shrink when physical systems move at speeds close to that of light and simultaneity is frame dependent from simple and transparent axioms formulated in first order logic; these effects are not postulated *a priori*.

7.1 Challenging the Church-Turing Thesis using general relativity:

Godels' incompleteness Theorem implies that the concept of mathematical truth cannot be encapsulated in any formalistic scheme. Mathematical truth is something that goes beyond mere formalism. Gódel's incompleteness theorems thus seemed to be a blow to the logistic view envisaged by Gottlob Frege. The pioneer representative of logisicsm is Frege. Logicism, basically asserts that mathematics is reducible to logic and hence is nothing but a chapter of logic. Logicists hold that mathematics can be known a priori, but suggest that our knowlege of mathematics is just a part of our knowlege in logic, and thus is analytic, not requiring any special faculty of mathematical intuition. In this view logic is a proper foundation of mathematics, and all mathematical statements are necessary logical truths. Hilbert brought mathematical logic under scrutiny as he did Euclidean geometry by establishing the axiomatic context and raising the crucial questions of consistency and completeness. This largely syntactical approach was soon given a superstructure, when Hilbert developed proof theory and proposed his program establishing the consistenc of classical mathematics with his metamathematics. However, Godel's incompleteness theorems, knocked down Hilbert's Program. But it remained to be seen whether Hilbert's Consistency Program is still viable in any way, either by restricting its scope or by somehow enlarging the methods of proof admitted. Taking into account the laws of the general theory of relativity, more specifically, the (infinite) time dilation in strong graviational fields of (rotating) black holes, then Andréka and Németi show that one can imagine thought experiments in which Church thesis is no longer valid!

Theorem 7.1. *(Andréka and Németi). It is consistent with Einstein's equations that by certain kinds of relativstic experiments, future generations can find the answers to non-computable questions like the Halting problem, Hilberts tenth and the consistency of* ZFC.

Idea: A slowly *rotating* black hole has two event horizons. The gravitational pull of such black holes grows without limit as one approaches the (outside) event horizon. Assume that two observers H and L are hovering over the outside event horizon, with H being higher up. Then L's clock runs slower than H's clock since he is experiencing a stronger gravitational field (this is a prediction of general relativity). Moreover, as L moves towards the horizon, this discrepancy between the ticking of both clocks gets larger and larger. In fact, by lowering L appropriately, this time lag can be controlled. Now, if a programmer P gets very close to the outside event horizon while leaving his computer C higher up, then in a few days time relative to the programmer, the computer does a few million's year's job. Accordingly, we can reach an infinite speed up by lowering P to the right position; hence breaking the Turing barrier. The rotation of the black hole induces a repelling effect (a centrifugal force in the language of Newtonian mechanics) that counter-balances the strong gravitational pull of the black hole. In this way, L can slow down as desired without being crushed. It is possible for an observer L to stay at a fixed distance from the center of the rotating black hole. In this way, the infinite time dilation speed-up of the computer C safely outside the outer event horizon with respect to the programmer P, is accomplished. The creation of a computer that can compute tasks beyond the Turing limit can be achieved as follows. The programmer P leaves earth in a spaceship towards a huge slowly rotating black hole. As P is heading towards his target, C checks one by one the theorems of set theory. If C finds a contradiction, he sends a signal to P. Otherwise, he does nothing. From C's point of view, as the programmer P approaches the event horizon, his clock will be ticking slower and slower relative to C's clock. At the limit, that is, when P reaches the inner horizon, his clock *freezes* coming to a halt relative to C. From the point of view of P, however, the C's clock appears to be running faster and faster. Moreover, assuming that the black hole is huge, P will safely cross the inner event horizon. If P recieves a light signal from C, P will know that C has found an inconsistency in ZFC. Otherwise, P concludes that ZFC is consistent. Because the black hole is huge, the center of the black hole is relatively far from the event horizon. More importantly, The matter content, that is, the singularity is not a point (as the case of a static black hole), but is actually a ring (one fascinating property of rotating black holes). This makes P comfortably pass through the middle of the ring without being torn apart! We emphasize that the last Theorem is not Science fiction. It is a scientific thought experiment that becomes more plausible especially after the first black hole was actually seen!.

The two major paradigms of computing arising from new physics are *quantum computing* and *general relativistic computing*. Quantum computing challenges complexity bariers in computability, while general relativistic computing challenges the *Physical Church Thesis (*PhCT*)*, namely, that recursiveness is the mathematical equivalence of computability. This was formulated and accepted in the 1930's as being well supported by rigorous mathematics and common sense. But 'common sense today' means 'physics a century ago'. Due to the major paradigm shift in our physical world view this thesis can now be challenged by thought experiments nvolving huge rotating black holes. Andréka and Németi's (exotic) results, depending essentially on the non-absoluteness of time in general relativity, in the last theorem is a cry away from Newtonian philosophy and a counterblow to Gódel's theorems on incompleteness and impossibility of proving ZFC consistent!

7.2 From the Vienna circle to The Budapest Group

A great influence on the philosophy of Science in the early 20th century that ultimately led to the formalist view is the school of Logic positivism in philosophy. Logical positivism also known as logical empirisicm or logical neo-positivism was a philosophical movement risen in Austria and Germany in the 1920's, primarily concerned with the logical anlysis of scientific knowledge, which affirmed that statements about metaphysics, religion and ethics are void of cognitive reasoning; only statements in mathematics, logic and natural science have a definitive meaning. The cheif influences on the early logical positivists were Rudolf Carnap, The advocates of logicical positivism including at one piont of time Bertrand Russell and Albet Einstein became known as the Vienna circle. Ernst Mach and Ludwig Wittgenstein. Wittgenstein's *Tractus Logica Philophicus* introduced the conception of philosophy as a critique of language. Leibniz taught us through his theory of so-called *monads* to reject any reference to *a-priori* and immutable structure, such as Newton's absolute space and time. But he did not tell us what to replace them with. Mach did, for he showed us that every use of such an absolute entity hides an implicit reference to something real and tangible that has so far been left out of the picture. What we feel pushing against us when we accelerate cannot be absolute space, for there is no such thing. It must somehow be the whole of the matter of the universe. Einstein took a third step in the transformation from an absolute to a relational conception of space and time. In this step, the absolute elements, identified by Mach as the distance galaxies, are tied into an interwoven, dynamical cosmos. The final result is that the geometry of space and time - which was for Newton absolute and eternal - became dynamical, contingent and lawful. Mach's philosophy well prepared for Einstein's (non-Newtonian) relativistic philosophy. In his special theory of relativity, Einstein was saying that the *mathematical structure* of a physical law must *not change* as we go from one observer to the other. Laws of nature have the same form relative to all inertial observers. Space is different for different (inertial) observers, time is different for different (inertial) observers, but *spacetime is the same for all (inertial) observers.* In general relativity, Einstein's field equations say that *matter curves spacetime and spacetime tells matter how to move.* One must only imagine the experience of falling and recall that those who fall have no sensation of weight. In the hands of Einstein, this everyday fact became the openning to a profound shift in our way of understanding the world: while you can abolish the effects of gravity locally, by freely falling, this can never be done over a large region of spacetime. Therefore, while curved space(time) can be approximated by a patchwork of small flat regions, these regions will always have discontinuities where we try to join them at their edges. This could be taken to mean that the overall space is curved. The very fact of this failure to join smoothly *is the curvature of space*. Einstein has specified the mechanism by which gravity is transmitted: the wrapping of spacetime. Einstein tells us that the gravitational pull holding the earth in orbit is not, as Newton claimed, a mysterious instantaneous action of the sun; rather, it is the wrapping of the spatial fabric induced by the sun. This dynamic relation between matter and geometry; this feedback loop, is coded in the following very elegant (and simple!) equation $\mathsf{G}_{ab} = 8\pi\mathsf{T}_{ab}$. On the right, stands the *source* of curvature, namely, the energy-momentum tensor. On the left, stands the *receptacle* of curvature in the form of what one wants to know, the metric coefficients twice differentiated (the Einstein tensor). The matter distribution T_{ab} determines the geometry G_{ab}, and hence is a source of inertial effects. When Einstein had created his general theory of relativity, he is supposed to have said that while the left hand side had been curved in marble, the right hand side was built out of straw. The left hand side of Einstein's equations (refering to the actual geometry of spacetime) is surely one of the great insights of science. The right hand side (describing how the mass and energy produces this curvature) did not follow with such elegance as the geometric part of the field equations. The field equations show how the stress energy of matter generates an average curvature in its neighbourhood. It governs the external spacetime curvature of a static and dynamic source, the generation of gravitational waves (ripples in the curvature of spacetime) by stress energy in motion, the external spacetime geometry of a (static and rotating) black hole and, last but not least, the expansion and the contraction of the universe. it is fair to say that all modern theories in physics nowaday, namely, string theory, Penrose's twistor geometry are attempts to fully understand the right hand side of Einstein's field equations!

Einstein's theory of relativity exerted a great influence over the origin of positivism. A scientific theory, according to positivism, is an axiomatic system which acquires an empirical interpretation from suitable statements, called coordinative definitions, which establish a correlation between real objects or processes and the abstract concepts of the theory. Pragmatic aspects of scientific research was usually dismissed by positivists, who were not interested in the real process of discovering, in so much as they were concerned with rational reconstruction of scientific knowlege, that is the study of the logical relationships between statements, hypothesis and empirical evidence. Positivists were interested in clarifying the philosophical significance of the theory of relativity.

Schlick wrote in 1915 and 1917 two essays on relativity. Reinbach (who attended Einstein's lectures on theory of relativity in Berlin University in 1917), wrote several books on relativity. These writings largely influenced Suppes, Tarski and much later Andréka and Németi. in their axiomatic (formalist) approach to new physics. Axiomatizations offered by the Budapest group are based on quite novel primitive concepts such as the *world view relation* connecting different observers bodies and coordinates, which they have shown to be capable of encapsulating the entire structure and behaviour of spacetime arena, and subjecting it to intricate mathematical analysis reaching Einstein's field's equations in an absolutely breathtaking achievement. The insisting of using first order logic here is very useful and innovative, and allows the basic assumptions of relativity to be reduced of simple and transparent principles clarifying what the predictions of physical reality depend on. This is a very valuable contribution to knowledge in a subject whose concepts have been notoriously counterproductive and difficult to grasp. This programme is proving to be a very productive and unique research programme with great potential and highly realistic achievable valuable goals; the main one is to prove deep theorems of the special and the general theory of relativity from a small number of simple, easily understandable and convincing axioms. The underlying idea here is that general relativity can be defined (in the very broad sense of the word), or obtained, as patching together its local parts, which reflect the laws of special relativity, in this way obtaining ultimately a *manifold* that represents or is a mathematical model for general relativity. Category theory intervines here; one can view such a spacetime manifold as a *colimit* of its local 'special relativistic' parts. Locally spacetime is apparently flat but globally it is curved. The patching here involves subtle very deep elaborations on definability issues in logic. Indeed, in her dissertation [38] Judit Madarasz presents special relativity in light of a definability theory. Having established this methodology and demonstrated its effectiveness the research team now are extending it to new directions, such as the study of space time with rotating black holes and wormholes.

7.3 Platonism versus formalism

Andréka and Németi are closer to being formalists in the sense that they are tolerant and inviting to new approaches in logic as long as the rule of the games are sound and the games are exciting and challenging. However, their motivations are drawn from existing mathematical or philosophical concerns, so the games are not completely arbitrary manipulation games. Gódel was a realist platonist who believed in an objective mathematical reality that could be perceived in a manner analogous to sense perceptions. A platonist insists upon the absolute, immutable nature of mathematics, it still has an a *priori* aspect. Many mathematicians have been mathematical realists: discovering naturally occuring objects like the Hungarian hugely prolific mathematician Paul Erdos. Paul Erdos envisaged that there is book written by God, in which beautiful proofs in mathematics are present, and he referred to beautiful proofs as proofs from the book. For example triangles and circles from the viewpoint of a platonist are real entities, not creations of the mind. Platonism is the form of realism that suggests that mathematical entities are abstract, have no spiatotemporal or causal properties, and are eternal and unchanging. This is often claimed to be the view most people have of natural numbers. According to extreme platonism, mathematical objects are real, real as any thing in the world we live in. For example infinite sets exist, not just as a mental constuct, but in a real sense, perhaps in a hyperworld. Similarly uncountable sets exist, real numbers, choice functions, Lebesgue measure and the category of all categories. Since all of the mathematical objects are real, the job of a mathematician is empirical as that of a geologist or physicst; the mathematician looks at a special aspect of nature and tries to discover (not invent) some of the facts. For the platonist independence results are not about mathematics but rather about the formalism of mathematics. A Formalist on the other hand, does not believe that any mathematical objects have a real existence independent of himself. For him, mathematics is just a collection of axioms, theorems and formal proofs. Of course the activity of mathematics is not just randomly writing down formal proofs for random theorems. The choice of axioms, of problems, of research directions are influenced by a variety of considerations, practical, artistic, mystical but really not mathematical. A formalist is not concerned with 'what is' but rather with 'what would be if'. Acccording to this view mathematics is a purely logical discipline and, like logic, is carried on entirely within the confines of language; it has nothing whatever to do with reality, or with pure intuition; on the contrary it deals exclusively with the use of signs or symbols. These signs or symbols can be used as we like, in conformity with rules that we have set. The only restriction on our freedom is that we may under no circumstances contradict the self established rules. The final criterion of mathematical existence thus becomes freedom from contradiction; that is to say, mathematical existence can be ascribed to every concept whose use does not enmesh us in contradiction. According to Formalism, mathematical truths are not about numbers and sets and the like- in fact they aren't about anything at all. A formalist would be content with Cohen's 'independence solution' of CH and AC; all resulting set theories are on equal footing from his viewpoint.

For a formalist, there is no preference of $\mathsf{ZF}+\neg\mathsf{AC}$ to $\mathsf{ZFC}+\neg\mathsf{CH}$, say. The formalist is indifferent to such inquiries concerning which set theory is 'better' and if so on what grounds. But a platonist would stipulate that AC is 'true', but CH is 'false' *objectively*; neither can be both in two different contradictory worlds existing among many, for there is only one 'platonic immutable objective' world where choice functions exist for any relation and $\omega_1 < 2^\omega$. A platonist would view such results not about the mathematics but rather about the formalism mathematics. Some leading mathematicians-who it seems are platonists at heart- such as Woodin and Shelah argue that Cohen's solution is not final, trying to settle the CH one way or another by modifying the axioms of Set Theory introducing so-called determinacy axioms (like projective determinacy) that turn out equiconsistent with the existence of certain large large cardinals like the so-called super compact ones. There is no obvious and compelling unique path of axioms that supplemented ZFC and settle important independent problems. It frequently happens that controversy arises in a particular physical problem, due to undefined basic concepts. In Roger Penrose's *The Emperor's new mind* the central thesis is that nature produces essentially harnessable non-computable processes, but at the quantum level only; and Penrose goes on to speculate that human intelligence may be such a process. Like Roger Penrose, Andréka and Németi are also concerned with questions concerning artificial intelligence; can we devise a machine that "thinks"? Gódel's incompleteness theorem seems to tell us that Turing machines cannot. However, the recent work of Andréka and Németi in Theorem 7.1 says that (certain relativitic) machines can compute non-computable functions if we are willing to replace the underlying Newtonian physics by the general theory of relativity; surely not an unreasonable; in fact a highly plausible, thing to do. To understand the human mind, conciousness and intelligence, we need new insights in both physics and mathematics. The huge project initiated by Andréka and Németi in the last two decades provides just that!

7.4 Godel's limiting self-referential solution to Einstein's equation; a strange loop in time

Gódel's completeness theorem actually provided the reverse of a limitation result, since what he demonstrated was on the level of formalisation of first order predicate logic, first presented in Frege's *begriffsschrift.* Here then proof can indeed simulate truth, and thus one cannot impose a limit on either interpretation of the formalism. In contrast, for Hilbert's ideal, the complete axiomatisation of each branch of mathematics and the replacement of mathematical truth with that of provability in an appropriate formal system (itself to be proved consistent by finitistic methods), Gódel's result was genuinely limitative. The incompleteness theorem showed that here proof could not simulate truth. Gódel demonstrated an intrinsic limitation on the semantic or contentful interpretation that can be imposed on the syntactic predicates in Hilbert's formal systems. Gódel's results relied on the very mathematization of logic and metalogic, introduced by Frege and Hilbert. Exploiting next, Einstein's geometrization of time- itself a kind of formalisation, Gódel was able to construct a mathematical model of a possible universe in which by his lights, there are essential constraints on the possible contentful, intuitive interpretations the model can bear. Since in such a universe there are closed timelike world lines, and, in a certain sense, the possibility of time travel, Gódel concludes from this that the standard interpretation of t here as denoting time, genuine successive time, which represents the objectivity of becoming, is clearly impermissible. This reading of the model relies on the fact that Einstein has provided not just a time geometrry but rather a geometrization of the temporal. The distinction is crucial. Many domains can be decribed as occupying a certain logical space, which in turn may have a distinctive geometric signature. The geometrization of such a domain, however, consists in the elimination or reduction of all qualitative features in favor of the geometrical or structural. This is how Gódel reads the Einstein Minkowski geometrisation of time. Nevertheless, he is well aware that there are those who pesist in maintaining the compatibility of relativity theory with the interpretation of time as successive, intuitive, unfolding time. With the derivation of te world model of Gódel's universe, however a solution to the field equations of general elativity, Gódel has constructed a limit case for the realitivistic geometriization of time. That is, he has produced a formal model that essentially limits the possible intuitive, contentful interpretation it can support. Thus just as the incompleteness theorem demonstrated in regard to the Hilbert program that, in that context, intuitive mathematical truth cannot be simulated by formal proof, so in regard to Einstein, the construction of a formal mathematical model of the Gódel universe demonstrated by Godels lights, that in this contect time can be given an intuititive contentful interpretation of denoting that successive, unfolding time that ushers in objective temporal becoming. Now a natural question arises at this point as to why Gódel did not reach the same conclusion in regard to Einstein's geometrization of time as he did for Hilbert's attempted formalisations of mathematics. Namely, that his surprising discoveries demonstrated the limits of formalism, rather than the disappearing of an intuitive content not just a possible interpretation of this system, but, as it were, absolutely. What is to the point here, is that in his papers on Einstein , the technical and he philosophical- Gódel appears to have brought a classic and recalcitrant, informal question, the question of the objectivity of temporal becoming, to a sudden and dramatic formal resolution, and again by means of the construction of an ingenious limit case that essentially constrains the possible intuitive content that can be expressed by this form. Put differently , why then did he not conclude with regard to Einstein's idealstic methodology, only that the form of relativity theory as standardly interpreted was limited in its ability to represent the sought for content, i.e intuitive time, instead of viwwing his results as a kind of verification of Kant's idealism i.e of the non existence of the intuitive content. One could say that in the case of Gódels incompleteness theorem, the result is achieved in the context of full intuitive classical number theory, in which the unprovable formula is seen to be true. But in case of cosmology papers, it would seem that there is no comparable uncontroversial well developed theoretical context in which can be seen that there is, though invisible to relativity theory - an objective lapse of time. But the recent work of Andŕeka and Németi on axiomatizations of General Relativity, can once again give a more formal content to this informal query, and I believe can be ground for fruitful research in the line of Gódel making informal questions succesptible to mathematical rigorous investigations, opening new avenues of research, and providing novel realms of conception. The formalist approach to physics, one can argue, might 'freeze' physic, but in fact just the contrary is happening here, since Andréka and Németi are experimenting with new axioms, making their subtle investigations completely flexible with no prejudice to any physical constraints, gaining more insight, hence progress of the subject matter at hand. Furthermore, being extraordinary scientists, they avoid the mistake that occurs often in physics, and this is jumping from one theory to another without specifying the supposedly new vocabularly, and this can ultimately lead to mistakes and inconsistencies. It frequently happens that controversy arises in a particular physical problem, due to undefined basic concepts.

8 Algebraic geometry, Topoi, and Sheaf Theoretic Duality for Cylindric algebras

Another motivating principle behind formalism is the desire to inter-relate different parts of mathematics, one may cite the ties among sentential logic, Boolean algebra and topology, or that between first order logic and Tarski's cylindric algebras, and Sheaf theoretic duality for cylindric algebras is worded out by Comer [17, 18]. Physicists and mathematicians use the term *duality*, admittedly often in a loose sense, but to basically describe theoretical models (abstract structures) that appear - when viewed from a certain percpective - to be different or mutually irrelevent; however, upon deeper scrutiny, can be shown to describe exactly the same phenomenon (mathematics). Accordingly, the two descriptions represent one and the same pattern, viewed from different angles. In other words, they represent two sides of the same coin. Both descriptions are valid, and we are free to move from one description to the other using whatever description we find more convenient. In some cases, solving the problem under investigation in one context proves to be extremely difficult; while moving to the dual picture facilitates the solution considerably. This is one of the aspects reflecting the power of duality: problems which resist a solution in one description may be easily dealt with, understood and solved when viewed from the percpective of the alternative dual description. Another important aspect of duality, is the (possible) emergence of new and unpredictable features when passing to the dual picture, properties that were not apparent in the primary picture. For example, there is a duality between commutive rings and affine schemes to every commutative ring $\mathfrak{A}$ there is an affine spectrum $\mathsf{Spec}\mathfrak{A}$, conversely, given an affine scheme S, one gets back a ring by taking global sections of the structure Sheaf. In addition, ring homorphisms are in one-to-one correspondence with morphisms of affine schemes, thereby there is an equivalence In a number of situations, the objects of two categories linked by a duality are partially ordered, i.e., there is some notion of an object "being smaller" than another one. In such a situation, a duality that respects the orderings in question is known as a Galois connection An example is the standard duality in Galois theory (Fundamental theorem) between field extensions and subgroups of the Galois group a bigger field extension under the mapping that assigns to any extension $[L : K]$ (inside some fixed bigger field Ω) the Galois group $Gal(\Omega, L)$ Pontryagin duality states that the character group is again locally compact abelian and that $G \cong \chi(\chi(G))$ Moreover, discrete groups correspond to compact abelian groups. Pontryagin duality is the background to Fourier analysis. The Tannaka-Krein dualty a non-commutative analogue of Pontryagin duality. Gelfund duality relating commutative C^* algebras and compact Hausdorff spaces. and the Pontryagin duality can be deduced and formalized in a largely formal category-theoretic way, using the so called Hom contravriant functor Hom(-C) where C is a so-called co-separotor (the dual of separator). This categorial duality also covers the Stone duality between Boolean algebras and Boolean spaces (compact Hausdorff totally disconnected spaces). The isomorphisms here induced by the Hom functor applied twice is natural in the categorial sense (like in vector spaces where a vector space V is naturally isomorphic to its double dual $(V^*)^*$; here for a vector space W, W^* is the space consisting of all linear functionals, and the co-separotar here is the field on which W is defined viwed s a vector space over itself (of dimension one). In the Gelfund duality $\mathbb{C}$ is the co-separtor, in Pontryagin duality the circle $\mathbb{R}/\mathbb{Z}$ is the co-separator, and in the Stone duality the two element Boolean algebra is the co-separator.

In algebraic geometry the concept of a *ringed space* is important. A ringed space, is basically a site (a cartesian closed category, with additional localization properties), or more concretely a topological space, where each point in the underlying set of the topology, is associated with a ring, called a germ or a stalk, such that the this topological space with an amalgamation of the local rings, form a *sheaf*. A sheaf is a tool for systematically tracking locally defined data attached to the open sets of the topological space. This data can be restricted to partitions of open sets into smaller sets, and the data assigned to an open set is equivalent to all collections of compatible data assigned to the collection of smaller sets covering the original one. For example, such data can consist of the rings defined on each such set. Sheaves are by design quite general and abstract objects, and their correct definition is rather technical. They exist in several varieties, such as sheaves of sets, or sheaves of rings, depending on the type of data assigned to open sets. We will be primarily concerned with sheaves associated to algebraizations of a multitude of predicate logics. The topological space on which our sheaves are based is the prime spectrum of the zero-dimensional subreducts of the algebra in question, endowed with the Zariski topology, and the germs will be homomorphic images of the algebra, determined by the ideals generated by the prime ideals of the zero dimensional part. In all cases we are dealing with sheaves of 'locally algebraised' topological spaces.

We will be also concerned with the notion of *pre-sheaves* on *Grothendieck sites*, but from the logical point of view and not the geometric one. In more detail, we will study *forcing in a topos*, which is an abstraction of the category of sets. On the hand, an *elementary topos* is a category that is closed under familar operations on sets (like finite products), on the other hand a general topoi can be viewed as a functor from a site to a category of pre-sheaves. Boolean algebras correspond to sets, example via Stone representation theorem, and Boolean algebras are also used in forcing by perturbing the ground model, forming Boolean valued models. Such forcing can be formulated in the context of the topos of sets, but it lends itself to many generalizations. For example one can study forcing, by viewing the comulative heirarchy of sets as pre-sheaves defined on Heyting algebras, or sheaves defined on a topological space. In algebraic geometry, affine varieties also carry a Zarski topology by declaring the closed sets to be to be precisely the affine algebraic sets. Given an affine variety A^n over a field K, the ring associated with A^n is the ring $K[x_1, \ldots x_n]$, which is the polynomial ring in n variables over the field K. If $V \subseteq {}^nA$, and $I(V)$ is the ideal of all functions vanishing on V, that is $\{f \in K[x_1, ..x_n] : f(s) = 0, \forall s \in V\}$, then for any affine algebraic set V, the coordinate ring or structure ring of V is the quotient of the polynomial ring by this ideal. $I(V)$ is a prime ideal in the polynomial ring, and the Zariski topology can be viewed as the topology based on this prime spectrum.

This phenomena has a more than one abstract setting. We mention two.

(1) This is a particular case of what is known in the literature of algebraic geometry as coherent sheaves of modules. (This is basically obtained by replacing the field K by a ring, and the affine variety by a module).

(2) If $Spec(R)$ is the prime spectrum of a commutative ring R, then the *stalk* at P equals the localization of R at P, and this is a local ring. Endowed with the Zariski topology, $Spec(R)$, commonly augmented also with a sheaf structure, makes it a locally ringed space.

A typical problem concerning ringed spaces, and in particular affine varieties is describing the polyonomial ring associated with an affine variety in terms of the local rings, or stalks, given at a point of the variety. In the first paper part, our points will be theories, or prime ideals in the quantifier free reducts of the Lindenbaum Tarski algebra, or the algebra of sentences. Following Comer, we describe the algebra of formulas, corresponding to a given theory as the algebra of continuous sections of a sheaf, that is, the continuous maps from the prime spectrum of the zero dimensional part of this algebra with the Zariski toplogy, to an amalgamation of the stalks, endowed with a natural topology. This theory corresponds to an ideal. In our case, the stalks are algebras defined locally at each point of the spectrum, and they can be amalgamated by taking ther algebraic product. In favourable circumstances, for example in the case of locally finite cylindric algebras, or quasipolyadic algebras, the stalks turn out to be simple algebras. Then in this case, what we are actually doing is taking the algebra factored out by the intersection of all maximal ideals containing the ideal we started off with. This algebra turns out *naturally isomorphic* to the algebra of formulas (in the precise categorial sense). Our situation also has affinity to the duality of Boolean algebras and Stone spaces, in fact, it is a generalization thereof. The Stone duality is obtained from Comer's duality theory for cylindric algebras, when the stalks are just the two element Boolean algebra.

Expressed in a metalogical setting, we deal with *two algebras* not just one, the former is the algebra of formulas, which will have extra operations reflecting quantifiers, the other is the Boolean algebra of sentences. The Stone space of the algebra of sentences is used to define a dual of the algebra of formulas, but this does not capture the quantifier structure; the dual we are looking for will actually be a triple, the Stone space of the algebra of sentences, a disjoint union of stalks (which are homomorphic images of the algebra of formulas, obtained by factoring out this algebra by complete extensions of the given theory; every such extension gives a stalk), endowed with a natural topology induced, by the projection map π, which projects the stalk at a completion of the theory, to the theory. Stone duality thus becomes a special case, because in this case we do not have have quantifiers so the algebra of formulas is the same as algebra of sentences. More rigorously, the stalks are just the two element Boolean algebra, and the duality in this case, reduces to the classical Stone case, implemented via applying the contravariant Hom functor, with second component a co-separator, twice. We also formulate our duality theorem concerning expansions of distributive lattices, as a double application of the Hom contravarinat functor. (This will be further elaborated upon below). In the concrete case of usual first order logic the stalks are simple algebras, so that the disjoint union can be viewed in a natural way as a semisimple algebra. Such algebras are called regular; regular algebras are semisimple, but the converse fails dramatically. On the other hand, it is not always the case that the stalks are simple algebras nor indeed subdirectly indecomposable, for other extensions of first order logic. The idea is taken from Comer, except that here we substantially generalize Comer's duality theory of cylindric algebras. Comer's results apply to first order logic, particularly, to studying interpolation theorems like Craig interpolation and Beth definabilty. Our results apply to a plathora of logics, like many valued logics, intuitionistic logic, fuzzy logic, different modifications of first order logic (like finite variable fragments) and its extensions like Keislers logics.

Since Comer dealt with classical logic, the topology used on the algebra of sentences is the Stone topology. Here we shall deal with algebraic structures, whose dual topology, is based on the spectrum of prime ideals (these do not nessecarily coincide with maximal ones), namely, the Zariski topology, or ever the weaker Priestly topology. To each such structure $\mathfrak{A}$, which will be expanded by operators reflecting quantification viewed in the modal sense, one associates a sheaf, whose first component is a Priestly topology. The latter has underlying set X, the set of prime ideals of the algebra consisting of zero dimensional elements. These are the elements that are fixed points of the operators, and indeed form a subalgebra of the reduct of the original algebra obtained by discarding the modalities. Like in the classical case, the second component is both a disjoint union of stalks $\mathcal{G}_x$, $x \in X$ that is endowed with a topology induced by certain maps, namey the σ_as defined below, and also can be formulated as an algebraic product. The third is the projection map.

If one takes the dual of this triple, more concisely the sheaf thereby obtained, one can represent the algebra he started off with as the algebra of continuous sections of this sheaf. These are maps from X to the disjoint union of stalks δ, and the continuity here is with respect to the Priestly topology on X, and the smallest topology on δ with respect to which the all maps of the form $\sigma_a : X \to \delta$, $\sigma_a(x) = a/\mathfrak{I}\mathfrak{g}^{\mathfrak{A}}x \in \mathcal{G}_x$ are continuous. So here we are infront of a very natural (natural) isomorphism, or a double dual. The representation in this geometric context is implemented by a contravariant functor, which describes this process and its inverse. One aspect of this duality is that, for example, there is an isomorphism between the set of certain ideals of $\mathfrak{A}$ called regular ones, onto the lattice of open subsets of X. (This will be proved below). A regular ideal is one that gives rise to a simple stalk. Furthermore, one can use such duality, to prove theorems formulated in categorial jargon for such algebras, by working in the dual space of sheaves. A typical example is whether epimorphisms are surjective or not in the given class, which is the equivalent (in a very broad context, including multi -dimensonal modal logics) of whether the logic in question has the Beth definability property, which in turn is equivalent to whether monomorphisms are injective in the dual space of sheaves. Sometimes, it is much easier to work in the dual world, by turning round arrows and reversing composition; this happens often in situations that involve arrows, via morphisms. A blatant example of such a phenomena is the fact that to prove that epimorphisms are surjective in Boolean algebras, then it is so much easier to prove that monomorphisms are injective in the category of Stone spaces.

Functors from sites to a category of pre-sheaves covers forcing constructions, both in the clasical and intuitionistic sense. In both cases the notion of forcing is a partially ordered set $(P, \leq)$, which gives rise to a cite. In the intuitionistic context, the Heyting algebra obtained consists of open sets of a natural topology on P, whihe in the classical sense, the Boolean algebra is based on the regular open sets on the same topology. Both are complete. Both the Heyting and Boolean valued model can be seen as a reflective subcategory of a category of pre-sheaves, consisting only of sheaves. One can do forcing in a topos, and also one can define semantics of higher order logics in topoi. These two approaches to higher order logics, which has other re-incarnations in the literature, like type theory and Lambda calculas are very much related. In fact, F is a functor from a small site C to a category of pre-sheaves, then F defines a notion of forcing C and the Yoneda image of C, namely, the set of all Hom functors on C, is the ramified language. This view can make one define fuzzy forcing by interpreting the Lukasiewicz conjunction on a monoid C, as induced by a tensor product in the target monoidal category. In Category theory a Grothendeick topology is a structure on a category C that makes the objects of C act like open sets, it need not be a real topology. In Grothendeick topology the notion of a collection of open subsets that are stable under inclusion is replaced by a *seive.* A seive on C is a sub-functor of the functor $Hom(-, C)$. Two cornerstones in category theory, are the Yoneda lemma, which says that a category can be faithfuly and fully embedded in the category of functors from C to Set, namely, the sheaves on $\mathfrak{C}$, and the Morita equivalence. In abstract forcing on topoi, these two notions are intimately related. The Yoneda lemma tell us, that for a site C with a Grothendieck topolgy J one can do the forcing in $Sh(C, J)$. The forcing conditions are elements of C, while $Sh(C, J)$ is basically the ramified language language obtained by adding names. It is also the comulative C heirarchy of sets. The formulas dealt with are usual first order formulas, and $p \models \phi$ is defined via morphisms in C, that is, elements in $Sh(C)$. The Morita equivalence, formulated essentially in the theory of Modules has a formulation on the category of sheaves, in fact, it has a formulation in any abelian category, of which modules and sheaves are the most prime example. Two Grothendieck sites are Morita equivalent, if they give the same functor category of sheaves. This translated to forcing in topoi, says that two site are Morita equivalent, if they give the same comulative heirarchy.

8.1 Sheaves, back and forth between logic and geometry

A sheaf is a central concept that occurs in algebraic geometry, and its definition is somewhat technical. But roughly a sheaf can be viewed as pre-sheaf with an additional 'glueing' condition, and it is best formulated in category theory. (A task implemented by the giant Grothendieck, in the context of algebraic geometry). One way to define a pre-sheaf is, to define it as a contravariant functor from a site X, or more concretely a topological space X to a target concrete category C. This category C can consist of just sets, or groups or rings and sets of maps or whatever. The objects of X are the open sets and the morphisms are inclusions, while in the target category the objects are the concrete objects and the morphisms are the natural ones (in algebras they are the homomorphisms, in topological spaces they are homeomorphisms, and so on). Let F be a functor defining a pre-sheaf. A *stalk* of the pre-sheaf is $i^{-}F(\{x\})$, where i is the inclusion of the one point $\{x\}$ into X. A sheaf is obtained when one requires, in addition, that the stalks can be 'glued', for example if the cite is a topological space, the stalks are algebras, which is usually the case, then a glueing, is 'continuously varying' these algebras, and this can be implemented by a subdirect product of the algebras, and the continuity is measured with respect to the smallest topology on this product that makes a given family of maps from the topological space X, to the the product, continuous. In algebraic geometry the sheaf on an affine variety glueing the local rings is extracted from the the structure of the variety using the data of the Zariski topology and (a subsheaf of the) sheaf of *germs*, which is a sheaf of local rings.

Let X be a topological space, define $O(X)$ whose objects are open sets of X and morphism are inclusions. Then a presheaf is a contravariant functor from $O(X)$ to C any category. Keeping the target category as it as, this definition can be widened when the source category does nor arise so concretely from a topological space. If F is a C valued presheaf on X, then U is an open set, then $F(U)$ is called the sections of F over U. This generalizes when the source category is known as a site. We first give the concrete definition, with minimal categorical jargon:

Definition 8.1. Let X be a topological space, and let $\mathbf{C}$ be a category. A presheaf F on X with values in $\mathbf{C}$ is given by:

- For each open set U of X there corresponds an object $F(U)$ in C.
- For each inclusion $U \subseteq V$, there corresponds a morphism $res_{U,V}$: $F(U) \to F(V)$ in the category $\mathbf{C}$.

These morphisms, are called restriction morphisms. They satisfy the following compatibilty conditons:

- $res_{U,U} : F(U) \to F(U)$ is the identity
- For $U \subseteq V \subseteq U$, we have $res_{W,V} \circ res_{V,U} = res_{W,U}$

A sheaf is a presheaf that satisfies

- if (U_i) is a open cover of an open set U and if $s, t \in F(U)$ such that $s \restriction U_i = t \restriction U_i$ for each U_i then $s = t$, and
- If (U_i) is an open covering of an open set U and for each i there is a section s_i of F over U_i such that for each pair U_i,U_j if the covering sets the restrictions of s_i and s_j agree on overlaps: $s_i :\restriction U_i \cap U_j = s_j \restriction U_i \cap U_j$, then there is a section $s \in F(U)$ such that $s \restriction U_i = s_i$ for each i.

In the above definition, we started with a topogical space. In Category theory a Grothendeick topology is a structure on a category C that makes the objects of C act like open sets, it need not be a real topology. In Grothendeick topology the notion of a collection of open subsets that are stable under inclusion is replaced by a *seive.* A seive on C is a subfunctor of the functor $Hom(-, C)$ The best example to give for sheaves and pre-sheaves, is in the context of forcing formulated in a topos, rather than in the category Set, and the above narrow context. The general definition of a topoi is a functor from a site to a category of pre-sheaves (we will use this definition later, in the context of defining a notion of forcing in cartesian categories). An *elementary topoi*, is a category that enjoys closure of certain operations performed in sets. More concretely an elementary topoi, is a category that has a subobject classifer (like the 2 element Boolean algebra) has finite limits and is cartesian closed. In Set the subobject classifier is the two element set, and it is cartesian closed because exponentials exist. Also Set is closed under finite products and it has equalizers. Topoi appeared in algebraic geometry, where they are abundant via the generalisation of a sheaf over a topological space. Categorially they they are a sheaf over a site (defined after the example). Here the notion of forcing is not a Boolean algebra in the ground model, but rather a Heyting algebra. The Heyting algebra, will be based on a notion of forcing P, the site, and the elements of the topoi will be maps from P^{op}, where we reverse the order of P, to Set, or rather to certain pre-sheaves in Set. This gives rise to intuitionistic set theory. This set theory, weaker than than that based on the topoi Set, is extremely interesting, for example in such a set theory, Zorns lemma is not equivalent to the axiom of choice.

Example 8.2. Consider the category Set^P of sets varying over a partially ordered set. Objects are functors $F : P \to Set$, that is maps F which assign to each $p \in P$ a set $F(p)$ and to each $p, q \in P$ such that $p \leq q$ a map $F_{pq} : F(p) \to F(q)$ satisfying that for $p \leq q \leq r$ $F_{qr} \circ F_{pq} = F_{pr}$. An arrow $\eta : F \to G$ is a natural transformation between F and G, in this case, an assignment of a map $\eta_p : F(p) \to G(p)$ to each $p \in P$ in such a way that

$$\eta_q \circ F_{pq} = G_{pq} \circ \eta_p.$$

A truth value object Ω^P is determined as follows. A subset U of $O_p = \{q \in P : p \leq q\}$ such that $q \in U$, $r \leq q$ implies $r \in U$ is said to be upward closed over P. Then $\Omega(p)$ is the family of all upward closed sets over P, and $\Omega_{pq}(U) = U \cap O_q$ for $p \leq q$, $U \in \Omega(p)$. The terminal object 1in Set^P is the functor P with contant value $\{0\}$, $true : 1 \to \Omega$ $true_p(0) = O_p$. In this concrete example objects in $Set^{P^{op}}$, where P^{op} is the partially ordered set obtained by reversing the order on P, are the *pre-sheaves.* If F is a presheaf, $x \in F(p)$ and $q \leq p$, we write $x \restriction_F q$ for $F_{pq}(x)$.

Now let H be a complete Heyting algeba, associated with the partially ordered set P. This has universe $O_p = \{q \in P : q \leq p\}$. and, further, P embeds into H, via $p \mapsto O_p$. The presheaf on H is a sheaf if whenever $p = \bigvee_{i\in I} p_i$ in H and $s_i \in F(p_i)$ for all $i \in I$ satisfy

$$s \restriction_F (p_i \cap p_j) = s_j \restriction_F (p_i \cap p_j)$$

for all $i, j \in I$, then there is a unique $s \in F(U)$ such that $s|p_i = s_i$ for all $i \in I$. The category $Sheave(H)$ has objects as sheaves and as arrows, the arrows between such objects viewed as pre-sheaves. We wil show that this category is equivalent to to the Heyting valued model V^L.

Let L be any algebraic structure in a category $\mathfrak{C}$. We can form a category Set_L based on L. The objects are pairs (A, α) where $\mathfrak{A} \in \mathsf{C}$, $\alpha : A \times A \to L$ is a map such that, $\alpha(x, y) \leq \alpha(x, x) \wedge \alpha(y, y)$, $\alpha(x, y) = \alpha(y, x)$ and $\alpha(x, y) \vee (\alpha(y, y) \to \alpha(y, x) \leq \alpha(x, z)$. The morphisms between the objects (A, α) (B, β) are maps $f : A \to B$ such that $(\forall x, y \in A)(\beta(f(x), f(y)) \geq \alpha(x, y)$ and $(\forall x \in A)(\alpha(x, x) = \beta(f(x), f(x)))$. Now if L is a Heyting algebra or, for that matter a Boolean algebra, one can form V^L, the universe of sets perturbed by L, the usual way. We we can obtain another (apparently different) category as follows. First we identify elements $u, v \in V^L$ for which $||u = v|| = 1$. The objects of Set^L are the identified objects and arrows are those identified $f \in V^L$ such that $||f$ is a function $|| = 1$.

All three categories are equivalent.

Passing from the concrete to the abstract, we now give a general definition, of pre-sheaves, and sheaves. We abstract away from topological space to sites. Let V be any category. A V valued pre-sheaf F on a category C is a functor. A pre-sheaf is defined to be a Set-valued pre-sheaf, but its domain is C^{op}. That is, a pre-sheave is a functor from C^{op} to Set. In our example the site was a Heyting algebra (viewed naturally as category, with arrows or morphisms reflecting the order). If C is the poset of open sets in a topological space interpreted as a category, then one recovers the usual notion of pre-sheaf on a topological space. (This will be our definition of sheaves of locally algebraised sites). The class of all such functors, for a fixed site C, is denoted by $[C^{op}, Set]$ is a category called the category of pre-sheaves, where a morphism of pre-sheaves is defined to be a natural transformation of functors. This makes the collection of all pre-sheaves into a category. The reflective subcategory consisting consisting of *pre-sheaves that can be glued* was our category $\mathsf{Set_L}$, in fact it was all three given in the last example. In particular, one can view the comulative heirarchy V_L as a functor category of *sheaves* on L by reversing the order of the latter. We formalize the notion of glueing. Our next definition tells us what will be glued.

Definition 8.3. A site is a cartesian category C with a notion of localization, i,.e for every $A \in |C|$ there are given a non empty class $Loc(A)$ of families of morphism $(A_i \to A)_{i\in I}$ of C called the localizations of A which are stable under pullbacks.

If H is a a complete Heyting algebra considered as a category Then the canonical localization is that (a_i) covers a, if $a = supa_i$, and as pointed out before $Sh(B) = V^B$, and so a sheaf is nothing more than a function from P^{op} to set, where B is based on P. Now generalizing, we get. A sheaf over C is a functor $F : C^{op} \to Sets$ satisfying the following for every $(f_i : A_i \to A)_{i\in I} \in Loc(A)$:

(i) if $\eta, \mu \in F(A)$ are such that $\eta_i = F(f_i)(\eta) = F(f_i)(\mu) = \mu_i$, for all $i \in I$, then $\eta = \mu$.

(ii) If $(\eta_i)_{i\in I}$ is a family such that $\eta_i \in F(A_i)$ for all $i \in I$ and is compatible, i.e the diagram

$$\eta_i \in F(A_i) \to F(A_{i_A} \times A_j) \to F(A_i) \in \eta_i$$

obtained via F from

$$A_i \to A_{i_A} \times A_j \to A_j$$

we have

$$F(\pi_i)(\eta_i) = F(\pi - j)(\eta_i),$$

for all $i, j \in I$, then there is $\eta \in F(A)$ such that

$$\eta_i = F(f_i)(\eta),$$

for all $i \in I$.

We call $Sh(C)$ to be the full subcategory of sheaves of the functor category $[C^{op}, Set]$. Now this time passing from the concrete to the abstract, we first make the following observation. Set_L is both a topoi and a functor category consisting of sheaves, so this prompts the following very general definition of a topoi:

Definition 8.4. A *topos* is a a functor category of sheaves. A *Grothendieck* topos is a functor category of sheaves whose domain is a site.

We give yet another narrow more concrete definition of a sheaf that suffices for our purpose. X will be the prime spectrum of some subalgebra $\mathfrak{B}$ of a reduct of a larger algebra $\mathfrak{A}$. The target category will be Set. The functor takes a basic open set $N_a \subseteq X$, $a \in B$, to its characteristic function on X.

Definition 8.5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras with $\mathfrak{B} \subseteq \mathfrak{Rd}\mathfrak{A}$. Let X be the prime spectrum of $\mathfrak{B}$. A stalk at $x \in X$ is the algebra $\mathcal{G}_x = \mathfrak{A}/\mathfrak{Ig}^{\mathfrak{A}}x$. A sheaf is a triple (X, δ, π), where δ is a topological space with underlying set $\bigcup \mathcal{G}_x$, a disjoint union of stalks, and $\pi : \delta \to X$ is defined by $\pi(s) = x$, where $s \in G_x$. For $a \in A$, let $\sigma_a : X \to \delta$ by $\sigma_a(x) = a/\mathfrak{Ig}^{\mathfrak{A}}x \in \mathcal{G}_x$, then the topology on δ is the smallest topology for which all these functions are open.

Here, δ glues the stalks, via a disjoint union, and the topology it gets varies them continously. The information coded in the stalks, lifts to a global dimension via this glueing. We start by concrete example addressing variants and extension first order logics. The following discussion applies to L_n (first order logic with n variables), $L_{\omega,\omega}$ (usual first order logic), rich logics, Keislers logics with and without equality, finitray logics of infinitary relations; the latter three logics are infinitary extensions of first order logic, though the former and the latter have a finitary flavour, because quantification is taken only on finitely many variables. These logics have an extensive literature in algebraic logic.

Example 8.6. Let L is a multi-dimensional modal logic, then a theory T of this logic can be represented as the continuous sections of a sheaf. More precisely, a theory is determined by all complete theories containing it, that is by the Stone space, X_T of $\mathfrak{Zd}(\mathfrak{Fm}_T)$. One takes δT to be the following disjoint union

$$\delta = \bigcup_{\Delta\in X_T} \{\Delta\} \times \mathfrak{Fm}/\Delta.$$

On δT, one takes the product topology with basic open sets

$$B_{\psi,\phi} = \{(\Delta, [\phi]_\Delta), \psi \in \Delta, \Delta \in X_\Gamma\}.$$

Then $(X_T, \delta T, \pi)$ is a *sheaf*, where $\pi : \delta \to X_T$ is defined for $s \in \delta$, via

$$\Delta \times \phi/\Delta \mapsto \Delta.$$

Furthermore, the set $\Gamma(X_T, \delta)$ of continous maps, with operations defined pointwise, is isomorphic to $\mathfrak{Fm}_T$, via

$$\eta(\phi_T) \mapsto \sigma_\phi,$$

where

$$\sigma_\phi(\Delta) = \phi_\Delta.$$

The glueing of the G_x's amounts to taking the product $\prod_\Delta \mathfrak{Fm}/\Delta$ which is a quotient of $\mathfrak{Fm}$ by $\bigcap T_i$ where each T_i is a complete extension of T. So each stalk, gives some information about T, all together gives an exact information about T.

Example 8.7. Let $\mathfrak{A} = \prod_{i\in I} \mathfrak{B}_i$, where $\mathfrak{B}_i$ are directly indecomposable BAOs. Then $\mathfrak{Zd}\mathfrak{A} = {}^I2$ and $X(\mathfrak{A})$ is the Stone space of this algebra. The stalk $\delta_M(\mathfrak{A})$ of $\mathfrak{A}^\delta$ over $M \in X(\mathfrak{A})$ is the ultraproduct $\prod_{i\in I} \mathfrak{B}_i/F$ where F is the ultrafilter on $\wp(I)$ corresponding to M.

Definition 8.8. Let $\mathfrak{C}$ be a locally finite CA_ω. A Boolean ultrafliter F of $\mathfrak{C}$ is called *Henkin* if for all $x \in \mathfrak{A}$, and $k < \omega$, whenever $\mathsf{c}_k x \in F$, then there a $i \in \omega \sim \Delta x$ such $\mathsf{s}_k^l x \in F$.

Theorem 8.9. *(1) Let $\mathfrak{C}$ be a locally finite simple cylindric algebra of dimension ω, and let P, denoted by $\mathcal{H}(\mathfrak{C})$ in Theorem 5.1, be its Polish space of Henkin ultrafilters. Then P is homeomorphic to the category of functors from $[C, Cset]$ consisting of representations of C via set algebras. Any representation $f : \mathfrak{A} \to \mathfrak{C}$ is a geometric morphism in $Sh(C)$, and so it inherits the Grothendieck topology on $Sh(C)$. Here a geometric morphism $f : \mathfrak{A} \to \mathfrak{C}$ is equivalent a pair (p, p^*), where p^* is a left exact functor having p as right adjoint.*

(2) The dual of the Polish space of Henkin ultrafilters, can be defined using Comer's duality, and this is isomorphic to $\mathfrak{A}$, In other words, this dual is isomorphic to $\cong Hom_{top}(Hom(\mathfrak{Zd}(C), 2), \bigcup_{x\in X} \mathcal{G}_x)$, and so the Polish space of Henkin ultrafilters is homeomorphic to $(Hom(\mathfrak{Zd}(C), 2), \bigcup_{x\in X} \mathcal{G}_x)$ endowed with the smallest topology such that the maps $\sigma_a(x) = C/\mathfrak{Ig}^C x \in G_x$ are continuous. Here G_x is the stalk at x.

8.2 Weaker structures

Now we define certain topologies, that give rise to classes of topological spaces, that are duals to various algebraic structures all having a distributive lattice reduct.

Definition 8.10. Let $(X, \leq)$ is a partially ordered set, and τ be a topology on X. (X, τ) is called a Priestly space if

(a) τ is a Stone space,

(b) For any $x, y \in X$ such that $x \not\leq y$ there is a downward clopen set U such that $y \in U$ and $x \notin U$. (Downward here, means that when $u \in U$ and $v \leq u$, then $v \in U$.

Definition 8.11. (1) A non-empty subset I of a partially ordered set $(P, \leq)$ is an ideal if the following conditions hold:

(a) For every $x \in I$, $y \leq x$ implies that $y \in I$ (I is a lower set).

(b) For every $x, y \in L$ there is a $z \in I$ such that $x \leq z$ and $y \leq z$ (I is a directed set).

(2) I as above is a prime ideal if for every elements x and y in P, $x \wedge y \in P$ implies $x \in P$ or $y \in I$. Here $x \wedge y$ denotes $inf\{x, y\}$; it is maximal if it is not properly contained in any proper ideal.

(3) A lattice is simple if has only the universal congruence and the identity one.

Definition 8.12. Let V be the class of bounded distributive lattices, and let $L \in V$. We consider lattices as algebraic structurses $(L, \wedge, \vee, 0, 1)$. Then

$Spec(L)$, the set of prime ideals, endowed with the the topology whose base is of the form $N_a = \{P \in Spec(L) : a \notin P\}$ and their complements is called the Priestly space corresponding to L, or simply, the Priestly space of L.

We can describe the Priestly topology using Sheaves:

Theorem 8.13. *Let D be a distrubitive lattice, regarded as a coherent category. Then one can recover the Priestly topology using sheaves as follows. Let $E = Sh(D, J_{coh})$ whete J_{coh}, is a topology on D that has a localization via sieves. Then E can be identified with the prime filters and $Sh(D, J_{coh})$ has a topology which is the Priestly topology.*

We now deal with much weaker algebraic structures, namely, bounded distributive lattices with operators (reflecting quantifiers), denoted by BLOs. This notion covers a plathora of logics starting from many valued logic, fuzzy logic, intuitionistic logic, multi-modal logic, different versions (like extensions and reducts) of first order logic.

Example 8.14. Let $\mathfrak{L}$ be the predicate language for BL algebras, $\mathfrak{Fm}$ denotes the set of L formulas, and $\mathfrak{Sn}$ denotes the set of all sentences (formulas with no free variables). This for example includes MV algebras; that are, in turn, algebraisations of many valued logics. Let X_T be the Zarski (equivalently the Priestly) topology on $\mathfrak{Sn}/T$ based on $\{\Delta \in Spec(\mathfrak{Sn}) : a \notin \Delta\}$. Let $\delta T = \bigcup_{\Delta \in X_T} \{\Delta\} \times \mathfrak{Fm}_\Delta$. Then again, we have $(X_T, \delta T)$ is a *sheaf*, and its dual consisting of the continuous sections with operations defined pointwise, $\Gamma(T, \Delta)$ is actually isomorphic to $\mathfrak{Fm}_T$.

Definition 8.15. A BLO is an algebra of the form $(L, f_i)_{i \in I}$ where L is a distributive bounded lattice, I is a set (could be infinite) and the f_i's are unary operators that preserve order, and joins, and are idempotent $f_i f_i(x) = f_i(x)$, on L, such that $f_i(0) = 0$, $f_1(1) = 1$, and if $x \in L$, and $\Delta x = \{i \in I : f_i(x) \neq x\}$, then $\Delta(x \vee y) \subseteq \Delta x \cup \Delta y$ and same for meets.

Definition 8.16. Let $\mathfrak{A} = (L, f_i)_{i \in I}$ be a BLO. Then a subset I of $\mathfrak{A}$ is an ideal of $\mathfrak{A}$, if I is an ideal of L and for all $i \in I$, and all $x \in L$, if $x \in I$, then $f_i(x) \in L$

What distinguishes the algebraic treatment of logics corresponding to such BLOs, is their propositional part; it can be a BL algebra, an MV algebra, a Heyting algebra, a Boolean algebra and so forth. Now our desired end, is to represent such structures as the continuous sections of sheaves; the representation in this geometric will be implemented by a contravariant functor. Let us formalize the above concrete examples in an abstract more general setting, that allows further applications. Let $\mathfrak{A}$ be a bounded distributive lattice with extra operations $(f_i : i \in I)$. $\mathfrak{Id}\mathfrak{A}$ denotes the distributive bounded lattice

$\mathfrak{Zd}\mathfrak{A} = \{x \in \mathfrak{A} : f_i x = x, \ \forall i \in I\}$, where the operations are the natural restrictions.(Idempotency of the f_is guarantees that this is well defined). If $\mathfrak{A}$ is a locally finite algebra of formulas of first order logic or predicate modal logic or intiutionistic logic, or any predicate logic where the f_is are interpreted as the existential quantifiers, then $\mathfrak{Zd}\mathfrak{A}$ is the Boolean algebra of sentences. Let K be class of bounded distributive lattices with extra operations $(f_i : i \in I)$. We describe a functor that associates to each $\mathfrak{A} \in \mathsf{K}$, and $J \subseteq I$, a pair of topological spaces $(X(\mathfrak{A}, J), \delta(\mathfrak{A})) = \mathfrak{A}^d$, where $\delta(\mathfrak{A})$ has an algebraic structure, as well; in fact it is a subdirect product of distributive lattices. This pair is called the dual space of $\mathfrak{A}$. For $J \subseteq I$, let $\mathsf{Nr}_J\mathfrak{A} = \{x \in A : f_i x = x \forall i \notin j\}$, with operations $f_i : i \in J$. $X(\mathfrak{A}, J)$ is the usual dual space of $\mathsf{Nr}_J\mathfrak{A}$, that is, the set of all prime ideals of the lattice $\mathsf{Nr}_J\mathfrak{A}$, this becomes a Priestly space (compact, Hausdorff and totally disconnected), when we take the collection of all sets $N_a = \{x \in X(\mathfrak{A}, J) : a \notin x\}$, and their complements, as a base for the topology. For a subset Y of an algebra $\mathfrak{A}$ we let $\mathsf{Co}^{\mathfrak{A}}\mathsf{Y}$ denote the congruence relation generated by Y (in the universal algebraic sense). This is defined as the intersection of all congruence relations that have Y as an equivalence class. Now we turn to defining the second component; this is more involved. For $x \in X(\mathfrak{A}, J)$, let $\mathcal{G}_x = \mathfrak{A}/\mathsf{Co}^{\mathfrak{A}}\mathsf{x}$ and $\delta(\mathfrak{A}) = \bigcup\{\mathcal{G}_x : x \in X(\mathfrak{A})\}$. This is clearly a disjoint union, and hence it can also be looked upon as the following product $\prod_{x \in \mathfrak{A}} \mathcal{G}_x$ of algebras. This is not semi-simple, because x is only prime, least maximal in $\mathsf{Nr}_J\mathfrak{A}$, and even if it was, there is no guarantee that the congruence it generates in the big algebra is maximal. But, when it is, that is when $\prod_{x \in \mathfrak{A}} \mathcal{G}_x$ is semi-simple case will deserve special attention. The projection $\pi : \delta(\mathfrak{A}) \to X(\mathfrak{A})$ is defined for $s \in \mathcal{G}_x$ by $\pi(s) = x$. Here $\mathcal{G}_x = \pi^{-1}x$ is the stalk over x. For $a \in A$, we define a function $\sigma_a : X(\mathfrak{A}) \to \delta(\mathfrak{A})$ by $\sigma_a(x) = a/\mathfrak{Ig}^{\mathfrak{A}}x \in \mathcal{G}_x$. Now we define the topology on $\delta(\mathfrak{A})$. It is the smallest topology for which all these functions are open, so $\delta(\mathfrak{A})$ has both an algebraic structure and a topological one, and they are compatible.

We can turn the glass around. Having such a space we associate a bounded distributive lattice in K. Let $\pi : \mathcal{G} \to X$ denote the projection associated with the space $(X, \mathcal{G})$, built on $\mathfrak{A}$. A function $\sigma : X \to \mathcal{G}$ is a section of $(X, \mathcal{G})$ if $\pi \circ \sigma$ is the identity on X. Dually, the inverse construction uses the sectional functor. The set $\Gamma(X, \mathcal{G})$ of all continuous sections of $(X, \mathcal{G})$ becomes a BLO by defining the operations pointwise, recall that $\mathcal{G} = \prod \mathcal{G}_x$ is a product of bounded distributive lattices. The mapping $\eta : \mathfrak{A} \to \Gamma(X(\mathfrak{A}, J), \delta(\mathfrak{A}))$ defined by $\eta(a) = \sigma_a$ is as easily checked an isomorphism.

To complete the definition of the contravariant functor we need to define the dual of morphisms. These are natural transformations corresponding to the defining functors of the sheaves, but more concretely: Given two spaces $(Y, \mathcal{G})$ and $(X, \mathfrak{L})$ a sheaf morphism $H : (Y, \mathcal{G}) \to (X, \mathfrak{L})$ is a pair (λ, μ) where $\lambda : Y \to X$ is a continous map and μ is a continous map $Y +_\lambda \mathfrak{L} \to \mathcal{G}$ such that $\mu_y = \mu(y, -)$ is a homomorphism of $\mathfrak{L}_{\lambda(y)}$ into $\mathcal{G}_y$. We consider

$Y +_\lambda \mathfrak{L} = \{(y,t) \in Y \times \mathfrak{L} : \lambda(y) = \pi(t)\}$ as a subspace of $Y \times \mathfrak{L}$. That is, it inherits its topology from the product topology on $Y \times \mathfrak{L}$. A sheaf morphism $(\lambda, \mu) = H : (Y, \mathcal{G}) \to (X, \mathfrak{L})$ produces a homomorphism of lattices $\Gamma(H) : \Gamma(X, \mathfrak{L}) \to \Gamma(Y, \mathcal{G})$ the natural way: for $\sigma \in \Gamma(X, \mathfrak{L})$ define $\Gamma(H)\sigma$ by $(\Gamma(H)\sigma)(y) = \mu(y, \sigma(\lambda y))$ for all $y \in Y$. A sheaf morphism $h^d : \mathfrak{B}^d \to \mathfrak{A}^d$ can also be asociated with a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$. Define $h^d = (h^*, h^o)$ where for $y \in X(\mathfrak{B})$, $h^*(y) = h^{-1} \cap Zd\mathfrak{A}$ and for $y \in X(\mathfrak{B})$ and $a \in A$

$$h^0(h, a/\mathfrak{Ig}^{\mathfrak{A}} h^*(y)) = h(a)/\mathfrak{Ig}^{\mathfrak{B}} y.$$

This is indeed a generalization of the Stone duality. Indeed, given a Boolean algebra $\mathfrak{A}$, then in this case the stalks are just the 2 element Boolean algebra, and the dual space is just $Hom_{Bool}(X, 2)$ the Stone space (here the topology is the subspace topology of the Cantor set), and the double dual, are the continuous functions from the Stone space to the two element discrete space, with operations defined pointwise, namely $Hom_{top}(Hom_{bool}(X, 2), 2)$, where the last is the 2 element discrete space and the first is the 2 element Boolean algebra. Both are co-separators in their category, and the existence of a coseparator C defines a natural isomorphism via the contravariant Hom functor $Hom(-, C)$, applied twice.This is naturally isomorphic to A. (Other similar contexts, mentioned above, are the duality between C star algebras and Compact Hausdorf spaces (via the Gelfund Hom functor; here $\mathbb{C}$ is the co separator), and the category of abelian groups and locally compact abelian groups, via the Pontreyagen Hom functor; here $\mathbb{R}/\mathbb{Z}$ is the co separator.) We can also formulate the above natural isomorphism, in terms of Hom functors, namely:. For a variety V of BLO's we denote by $\mathfrak{r}dV$, the class obtained by discarding the extra operators.

Theorem 8.17. *Let V be an algebraic category whose objects have a distributive lattice structure. Assume that there exists $C \in V$, such that the contravariant hom functor $Hom(-, C)$ is an equivalence between $\mathfrak{R}dV$ and $Priest(\mathfrak{R}dV)$. Let $\mathfrak{A} \in V$, let $\mathfrak{Zd}\mathfrak{A}$ be its zero-dimensional part, and X be the Priestly space of $Zd\mathfrak{A}$. Let δ be the disjoint union of stalks with the topology as defined above, that is $\delta = \bigcup_{x \in X} \mathcal{G}_x$, with $\mathcal{G}_x = \mathfrak{A}/\mathfrak{Ig}^{\mathfrak{A}}(x)$, with topology induced by the maps $\sigma_a : X \to \delta$ via $x \mapsto a/\mathfrak{Ig}^{\mathfrak{A}} x$, for every $a \in \mathfrak{Zd}\mathfrak{A}$. Then*

$$\mathfrak{A} \cong Hom_{top}(Hom(\mathfrak{Zd}(\mathfrak{A}), 2), \bigcup_{x \in X} \mathcal{G}_x),$$

and the isomorphism is natural.

Let $\mathfrak{A}$ be a bounded distributive lattice with extra operations $(f_i : i \in I)$. $\mathfrak{Zd}\mathfrak{A}$ denotes the distributive bounded lattice $\mathfrak{Zd}\mathfrak{A} = \{x \in \mathfrak{A} : f_i x = x, \ \forall i \in I\}$, where the operations are the natural restrictions.(Idempotency of the f_is guarantees that this is well defined). The next theorem uses the duality just described. In the statement of the theorem ES abreviates that epimorphisms (in the categorial sense) are surjective. Such abstract property is equivalent to the well-known Beth definability property for many abstract logics, including fragments of first order logic, and multi-modal logics.

Theorem 8.18. *Let V be a class of distributive bounded lattices such that the simple lattices in V have the amalgamation property (AP). Assume that there exist strongly regular lattices $\mathfrak{A}, \mathfrak{B} \in V$ and an epimorphism $f : \mathfrak{A} \to \mathfrak{B}$ that is not onto. Then ES fails in the class of simple lattices.*

Proof. Suppose, to the contrary that ES holds for simple algebras. Let $f^* : \mathfrak{A} \to \mathfrak{B}$ be the given epimorphism that is not onto. We assume that $\mathfrak{A}^d = (X, \mathfrak{L})$ and $\mathfrak{B}^d = (Y, \mathcal{G})$ are the corresponding dual sheaves over the Priestly spaces X and Y and by duality that $(h, k) = H : (Y, \mathcal{G}) \to (X, \mathfrak{L})$ is a monomorphism. Recall that X is the set of prime ideals in $Zd\mathfrak{A}$, and similarly for Y. We shall first prove

(i) h is one to one

(ii) for each y a maximal ideal in $\mathfrak{Zd}\mathfrak{B}$, $k(y, -)$ is a surjection of the stalk over $h(y)$ onto the stalk over y.

Suppose that $h(x) = h(y)$ for some $x, y \in Y$. Then $\mathcal{G}_x$, $\mathcal{G}_y$ and $\mathfrak{L}_{hx}$ are simple algebra, so there exists a simple $\mathfrak{D} \in V$ and monomorphism $f_x : \mathcal{G}_x \to \mathfrak{D}$ and $f_y : \mathcal{G}_y \to \mathfrak{D}$ such that

$$f_x \circ k_x = f_y \circ k_y.$$

Here we are using that the algebras considered are strongly regular, and that the simple algebras have AP. Consider the sheaf $(1, D)$ over the one point space $\{0\} = 1$ and sheaf morphisms $H_x : (\lambda_x, \mu) : (1, D) \to (Y, \mathcal{G})$ and $H_y = (\lambda_y, v) : (1, D) \to (Y, \mathcal{G})$ where $\lambda_x(0) = x$ $\lambda_y(0) = y$ $\mu_0 = f_x$ and $v_0 = f_y$. The sheaf $(1, \mathfrak{D})$ is the space dual to $\mathfrak{D} \in V$ and we have $H \circ H_x = H \circ H_y$. Since H is a monomorphism $H_x = H_y$ that is $x = y$. We have shown that h is one to one. Fix $x \in Y$. Since, we are assuming that ES holds for simple algebras of V, in order to show that $k_x : \mathfrak{L}_{hx} \to \mathcal{G}_x$ is onto, it suffices to show that k_x is an epimorphism. Hence suppose that $f_0 : \mathcal{G}_x \to \mathfrak{D}$ and $f_1 : \mathcal{G}_x \to \mathfrak{D}$ for some simple $\mathfrak{D}$ such that $f_0 \circ k_x = f_1 \circ k_x$. Introduce sheaf morphisms $H_0 : (\lambda, \mu) : (1, \mathfrak{D}) \to (Y, \mathcal{G})$ and $H_1 = (\lambda, v) : (1, \mathfrak{D}) \to (Y, \mathcal{G})$ where $\lambda(0) = x$, $\mu_0 = f_0$ and $v_0 = f_1$. Then $H \circ H_0 = H \circ H_1$, but H is a monomorphism, so we have $H_0 = H_1$ from which we infer that $f_0 = f_1$.

We now show that (i) and (ii) implies that f^* is onto, which is a contradiction. Let $\mathfrak{A}^d = (X, \mathfrak{L})$ and $\mathfrak{B}^d = (Y, \mathcal{G})$. It suffices to show that $\Gamma((f^*)^d)$ is onto (Here we are taking a double dual) . So suppose $\sigma \in \Gamma(Y, \mathcal{G})$. For each $x \in Y$, $k(x, -)$ is onto so $k(x, t) = \sigma(x)$ for some $t \in \mathfrak{L}_{h(x)}$. That is $t = \tau_x(h(x))$ for some $\tau_x \in \Gamma(X, \mathcal{G})$. Hence there is a clopen neighborhood N_x of x such that $\Gamma(f^*)^d)(\tau_x)(y) = \sigma(y)$ for all $y \in N_x$. Since h is one to one and X, Y are Boolean spaces, we get that $h(N_x)$ is clopen in $h(Y)$ and there is a clopen set M_x in X such that $h(N_x) = M_x \cap h(Y)$. Using compactness, there exists a partition of X into clopen subsets $M_0 \ldots M_{k-1}$ and sections $\tau_i \in \Gamma(M_i, L)$ such that

$$k(y, \tau_i(h(y)) = \sigma(y)$$

wherever $h(x) \in M_i$ for $i < k$. Defining τ by $\tau(z) = \tau_i(z)$ whenever $z \in M_i$ $i < k$, it follows that $\tau \in \Gamma(X, \mathfrak{L})$ and $\Gamma((f^*)^d)\tau = \sigma$. Thus $\Gamma((f^*)^d)$ is onto $\Gamma(\mathfrak{B}^d)$, and we are done. ■

And as an application, using known results, we readily obtain:

Corollary 8.19. *(1) Epimorphisms are not surjective in simple cylindric algebras, quasipolyadic algebras and Pinters algebras of infinite dimension*

(2) Epimorphisms are not surjecive in simple cylindric lattices of infinite dimension

Proof. (1) In [39, 40] two strongly regular algebras $\mathfrak{A} \subseteq \mathfrak{B}$ are constructed such that the inclusion is an epimorphism that is not surjective. (2) Using essentially the same ideas in the first item two strongly regular algebras $\mathfrak{A} \subseteq \mathfrak{B}$ can be constructed, and the inclusion is not an epimorphism □

There is a very thin line between superamalgamation ($SUPAP$) and strong amalgamation (SAP). However, Maksimova and Shelah constructed varieties of BAOs with SAP but not $SUPAP$, the latter is a variety of representable cylindric algebras. The second item of the next corollary makes one cross this line.

Corollary 8.20. *(1) Let V be a variety of BAOs such that every semisimple algebra is regular. Then if ES holds for simple algebras, the it holds for semisimple algebras.*

(2) Let V be a variety that has the strong amalgamation property, such that the simple algebras have ES. Then V has the superamalgamation property.

Proof. We only prove the second part. If $SUPAP$ fails in V, then ES does, because V has SAP and both together are equivalent to $SUPAP$, but then ES fails in simple algebras and this is a contradiction. □

It is known that ES fails for semisimple cylindric algebras of infinite dimension. In view of the first part of the previous corollary, the next example gives a necessary condition for this. But first a definition. An epimorphism $f : \mathfrak{A} \to \mathfrak{B}$ is said to be conformal if $f(\mathfrak{Zd}\mathfrak{A}) \subseteq \mathfrak{Zd}\mathfrak{B}$.

Example 8.21. Let $\mathfrak{C}$ be a subdirectly indecomposable cylindric algebra of dimension α. Let I be the set of all finite subsets of subsets of α. Let F be an ultraflier on I such that $X_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\} \in F$ for all $\Gamma \in I$. Then the epimorphism ${}^I\mathfrak{C}/ \to {}^I\mathfrak{C}/F$ induced by F is not conformal.

The above example actually shows that semisimple algebras need not be regular, and moroever the stalks of the dual space of a semisimple algebra may not be even subdirectly indecomposable.

8.3 Forcing in topoi

Topoi first appeared in algebraic geometry, where they are abundant via the generalisation of a sheaf over a topological space. Categorially they they are a sheaves over a site. Let H be a small category whose members have a distributive lattice reduct. Then one can form the universe of sets based on H and V the usual way, by taking at step α the set of all functions for V_β^H to 2. The question is can we capture such a notion of generic extensions using a forcing relation between elements in H and formulas in the logic, so that p forces ϕ iff for every generic filter G of H, with $p \in G$, we have $V[G] \models \phi^G$ where the G interpretation of an H term is defined the usual way. That is is $a^G = \{y : \exists p \in P : (p, y) \in a\}$. The Heyting example we did forcing with respect to the topology have open sets O_p, where $p \in P$. The set $\{O_p \in P\}$ forms a Heyting algebra, and is an instance of a site. The connection between logic and topos, or rather elementary topos, is somewhat deep. Topoi is a categorial reflection of Set, as much as abelian categories are catgorial reflection of Ab. So different forms of forcing, like Cohen's, Robinson's and Kripke, can be seen as *forcing with sheaves on different sites*, as indicated in the previous example. Another rich source is the interpretation of higher order languages, via Henkin semantics as many sorted languages, viewed as type theory or logics of partial elements in topoi [21]. Here we have a *completeness theorem*, indicating that topoi are *just the right abstraction*. Our next theorem which is an abstraction of forcing using topoi instead of sets, relies heaviy on some categorial concepts. We will show that given a small category that happens to be a site, then one can look at this category as forcing conditions, and the ramified language as its Yoneda image.

Complete Heyting algebras are the objects of three different categories; the category $CHey$, the category Loc of locales, and its opposite the category Frm of frames. Although these three categories contain the same objects, they differ in their morshisms , and thus get distinct names. Only the morphisms of $CHey$ are homorphisms of complete Heyting algebras. Locales and frames form the foundation of pointless topology which, instead of building on pount set toplogy, recasts the ideas of general topology in categorical terms, as statements on frames and locales. Let Ω be a sub -object classifier in some topos.

A point of a locale X is defined to be a local morphsim $p : 1 \to \Omega$, equivalently a frame homomorphism $p^* : O(\Omega) \to O(1) = \{0, 1\}$. This is the truth values Ω. The pair (p, p^*) form a geometric morphism. Let X be a locale. The space of points of Ω is the set X_p of all points of Ω (i.e a set of locale morphisms), equipped with the topology given by the image of the frame homomorphism $\phi_X : O(\Omega) \to \wp(X_p)$ defined via $\phi_X(U) \mapsto \{p \in X_p : p^*(U) = 1\}$ for any $U \in O(\Omega)$. The set X_p corresponds bijectively with the set P of points of the topos of sheaves $Sh(\Omega)$. Now let E be a co-complete topos, Γ a subframe of $Sub_E(1)$ and $i : \Omega \to P$ be an indexing function of a set P of points of E by a set X. The Γ sub-terminal topology $\tau^E_{\Gamma,i}$ on the set Ω is the image of the functions

$$\phi_{\Gamma,E(u)} = \{x \in \Omega : \eta(x)^*(u) \cong 1_{Set}\}.$$

If $\Gamma = Sub_E(1)$ then we denote the topology by τ^E_i. The topology on the prime spectrum of A, is $X^{Sh(A)}_\tau$. This is a form of the more general the categorial equivalence,

$$Sh(C) = Hom_{Functor}(C^{op}, E) \cong Geo(Hom_{Top}(Set, Sh(C))).$$

which is basically a Yoneda lemma. *This is the categorial essence of the formulation of forcing.*

Now a locally small category C (where $Hom(A, -)$ are sets, embeds fully and faithfully into the category of set-valued pre-sheaves via the Yoneda embdding. The presheaf category is (up to equivalence of categories) the free co-limit completion of the category C. This is indeed a generalization of Boolean and Heyting forcing. For the latter we started with a Heyting algebra H, Set_H, the perturbed universe of sets by formulas taking value in H, *is* its the reflective subcategory of its *Yoneda image*, namely, it is the topos of sheaves on H. The Yoneda lemma says that instead of studying the locally small (that is hom-sets are sets) C, one should study the category of all functors of C into Set, that is the pre-sheaves on C. Each object A of C gives rise a hom functor $h^A = Hom(A, -)$. For an arbitray functor F from C to set, for each object A of C, the natural transformations from h^A to F are in one to one correspondence with the elements of $F(A)$, that is $Nat(h^A, F) \cong F(A)$, and contravariantly $Nat(h^A, G) \cong G(A)$. Furthermore, this isomorphism is natural in A and F, when both sides are regarded as functors from ${}^C\mathsf{Set} \times C$ to Set.

Theorem 8.22. *Assume that C is an algebraic site or a topos. Then C can be used to define a notion of forcing; the forcing conditions will be members of the category.*

Proof. Let $\bar{C}$ be the set of all functors from C^{op} to Set, that is sheaves on C Write $y : C \to \bar{C}$ to be the Yoneda embedding. The comulative heirarchy of sets will be formed in $\bar{C}$, elements of the site C will be the forcing conditions, and a C-term will be represented by a functor from C to Set, that is an element in $\bar{C}$. So intuitively C is the forcing notion and $\bar{C}$ is the ramified language.

0 is the initial object in C, and we have the operation $P(A)$ on objects abstracting the operation of power set. The maps $i_{\alpha,\beta} : P^{\alpha}(0) \to \wp^{\beta}(0)$, for all ordinals α, β are all inclusions, therefore we may construct $colimit_{\alpha\in On}P^{\alpha}(0)$ as the union

$$V^{(\bar{C})} = \bigcup_{\alpha\in On} V_{\alpha}^{\bar{C}}.$$

Using the fact that in $\bar{C}$ power objects are constructed for $A \in \bar{C}$ as

$$P(A)(I) = Sub_{\bar{C}}(y(I) \times A),$$

where Sub denotes subobjects and $I \in C$, one may consider $V^{\bar{C}}$ as defined inductively by the rules

$$a \in V^{\bar{C}}(I) \text{ iff } a \text{ is a set valued sub-presheaf (sub-functor) of } y(I) \times V^{\bar{C}}$$

for objects I of C, this step corresponds to the inductive step in usual set theory, when we obtain $V_{\alpha+1}$ from V_{α}.

If $a \in V^{\bar{C}}(I)$ and $u : I \to J$ is a morphism in C, then we write $a.u$ for the subobject of $y(J) \times V^{(\bar{C})}$ where $(v, c) \in a.u$ iff $(uv, c) \in a$. This allows one to interpret the membership relation as:

$$\epsilon \mapsto V^{\bar{C}} \times P(V^{(\bar{C})}).$$

Equality is defined as usual in presheaves. The Kripke-Joyal semantics of this interpretation in pre-sheaves over $\bar{C}$ give rise to the following forcing clauses, where $I \in C$, and $a, b \in V^{\bar{C}}$; and the language is the usual first order language for set theory:

$$I \models a \in b \Longleftrightarrow (Id_I, a) \in b$$

$$I \models a = b \Longleftrightarrow a = b$$

$$I \models \phi \wedge \psi(c) \Longleftrightarrow I \models \phi(c) \text{ and } I \models \psi(c)$$

$$I \models \phi \to \psi(c) \text{ iff for all } u : J \to I \text{ from } J \models \phi(c.u) \text{ it follows that } J \models \psi(c.u)$$

$$I \models \phi \vee \psi(c) \Longleftrightarrow J \models \phi(c) \text{ or } I \models \psi(c)$$

$$I \models \forall x\phi(x, c) \Longleftrightarrow J \models \phi(a, c.u), \forall u : J \to I, a \in V^{\bar{C}}(J)$$

$$I \models \exists x\phi(x.c) \Longleftrightarrow I \models \phi(a, c).$$

Note that when C is just a Heyting algebra then $\mathcal{C}$ is the set of all presheaves, that is functions from C to Set. And the above construction gives exactly intuitionistic forcing. This is an interpretation of set theory in the sheaf topos $E = Sh(\bar{C}, J)$ where J is a Grothendieck topology on small category C.

If C is not small, one can also interpret set theory in the same Sheaf topoi, with some modifications The main obstacle here is that colimits of transfinite chains of inclusions are not simply union but rather union followed by sheafification $a : \bar{C} \to Sh(C, J)$ that is a left adjoint to $i : Sh(\mathfrak{C}, J) \to \bar{C}$ and that the reflection maps $\eta_X : X \to a(X)$ in general cannot be viewed as inclusions. This can be overcome by obtaining the model for E as a quotient of the model in $\bar{C}$. Construct by transfinite recursion a family of morphisms $e_\alpha : V_\alpha^{C)} \to V_{\underline{\alpha}}^E$ where V^C and V^E refer to cumulative heirachies in the Gothendieck toposes $\bar{C}$ and E. These (e_α) satisfy

$$i_{\alpha,\beta} \circ e_\alpha = e_\beta \circ i_{\alpha,\beta}$$

and they are all dense w.r.t to the topology and for successors they are epics in $\mathfrak{C}$. Then $e : V^C \to V^E$ the unique mediating arrow between them is an epimorphism. □

In recent years, several algebras have appeared in the framework of fuzzy logics; some can be readily seen as extensions of intuitionistic logic. All those algebras are based on the notion of residuated lattices. Examples include MV algebras, BL algebras. What distinguishes, these algebras is the presence of two distinct conjunctions, $\wedge$ and $\otimes$, the latter the Lukasiewicz conjunction. There has been work in interpreting many sorted theories in quasi-topos based on MV algebras, but the best that was achieved, to the best of our knowlege, is that the interpretation is only faithfully represented in the Heyting algebra obtained from the MV by identifying the two conjunctions. For the Heyting case, a completeness theorem is obtained for such theories using the topoi Set_H, where H is a sub-object classifier, in the category $E(T)$ of definable terms and total functions in the many sorted language. This category which happens to be a topos, is a reflective subactegory of the category of fuzzy sets corresponding to T [21]. Also, forcing in Heyting valued models is known, it is natural to ask about forcing in such new algebras. The crucial missing link here, is the interpretation of both conjunctions, particulary, the Lukasiewicz conjunction. In the latter case, it is not obvious how to define p forces $\phi \otimes \psi$, and in the second case, it is not clear how to interpret the truth values of the MV algebra. First thing to notice is that the first conjunction has to do with the lattice structure, but the second lends itself to a monoidal structure. In fact, the class of MV algebras form *an abelian category*, which a generalization of Ab and BL algebras form a *monoidal* category an extension of Monoid There is a topos theory for such algebras, where one can form Set_L similar to Heyting algebars, in fact one can construct all three equivalent categories, that are the qusi topoi of the comulative heriarchy. But this process has to do with the first part of forcing constructions namely, perturbing the the universe of sets via an algebra. The V_βs are just functions from V_α to the algebra, and the interaction of functions is defined pontwise. In translating the forcing in the ground model, to capture all generic extensions, one has to define clauses that involve the non-classical conjunction. In usual forcing the inductive clauses are defined by the 'meta meaning' of the connectives, namely in English, they are not defined intrinsically; this is the case for both classical forcing and intuitionistic forcing. However, what can guide us here is the duality between the bi- functors tensor products and Hom. On the very basic level, namely in the algebra, this is expresed by $a \otimes b \leq c$ iff $a \leq (b \to c)$.

To conclude, one can extend results of topoi of sheaves, which forms a reflective subcategory of the quasi-topos of pre-sheaves. In this context, we have been forcing with *two sites*, the first forcing is ordinary forcing in the sense that the meta language is usual first order thery. This was implemented using the Yoneda lemma in category theory, which says that you can look at a site, in the more concrete case a Heyting algebra as a category of functors on this site, to Set.This is a natural isomorphism (in the categorial sense, in more than one parameter). This is also a natural generalization of ordinary forcing where the site represents the forcing conditions and the set of all such functors represents the ramified language. The advantage to work on the level of sheaves and topoi, is that this view lends itself to other generalization, like Kripke forcing, and intuitionistic forcing. This also enabled us to view the comulative heirarchy of sets V^H, based on a Heyting algebra, as a topos of sheaves, and this is precisely the image of the Yoneda lemma applied to the forcing conditions, namely, the site, which is H. From the different view of interpreting many sorted higher order logic in topoi, we ended up at the same point. If we restrict to set theory (formulated as a higher order logic), then the completeness theorem, is provided by giving truth values to the sub-object classifier of the resulting topos of definable sets and total functions. This is the same topos, as above, namely V^H. Now, the forcing done above was implemented by forcing (on sheaves) on different cites, but *on sheaves*. It is very natural therefore, to ask whether the forcing can be implemented by *pre-sheaves*, which is a larger functor category. Many valued logic gives this forcing. Indeed, in this case, we have a new connective, namely Lukasiewicz conjunction, so we are actually working outside first order logic. Furthermore, the universe of sets Set_L defined above for an MV algebra L consists of all presheaves on L, in particular, it is only a quasi-topoi. In usual forcing, even in topoi, other complex clauses are defined from the primitive ones using the usual meaning of $\wedge$ $\vee$,etc. But when we have a fuzzy connective, we have to think diferently. So we observed two things. An MV algebra has a monodial facet. Furthermore, in abelian categories, we have the bifunctors tensor and Hom are dual, expressed on the algebra level we have $a \otimes b \leq c$ iff $a \leq b \to c$. In usual topoi forcing, the Yoneda lemma, takes us to sheaves, that is functors from the cite to Set. The natural thing to do is to replace Set, by Monoid, which has a tensor product, and define the semantics of $\otimes$ using this tensor product. Starting from a higher order many sorted many valued theory T, one can implement the above strategy, to define a quasi-topoi, which only contains only an almost all subobject classifier [21]. But we not need a full sub-object classifier, to interpret T in quasi-topoi, in fact we do not have one. So the semantics of the Lukasiewicz conjunction is defined like in the case of forcing. In the special case when T is set theory, then the semantics defined by forcing, will provide a complete semantics for fuzzy set theory.

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