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Excellent Extensions of Tilted Algebras

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Abstract

In this paper, we study the tilting property of excellent extensions. Let B be an excellent extension of an artinian algebra A such that B is centrally projective over A, we prove that A is a tilted algebra if and only if so is B.

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1 Introduction

The classical 1-tilting modules and tilted algebras are two important and fundamental object in the tilting theory. The former was introduced by Brenner and Butler in [5] and the later was introduced as endomorphism algebras of 1-tilting modules over hereditary algebras by Happel and Ringel in [11]. It is well known that tilting modules and tilted algebras play a prominent role in the representation theory of algebras (cf. [3]).

On the other hand, in studying the algebraic structure of group rings, Passman in [14] introduced the notion of the excellent extensions of rings(the name comes from [4]). Such extensions of rings are vital since they include two important classes of extensions of rings: finite matrix rings and skew group rings AG, where the finite group G satisfies the condition $|G|^{-1} \in A$. Many authors have studied the invariant properties of rings under excellent extensions ([4, 8, 10, 13, 14, 15, 18]). It is known that many important homological properties, such as the global dimension of rings, the projectivity, injectivity and flatness of modules and so on, are invariant under excellent extensions ([13, 18]).

Let B be an excellent extension of an artinian algebra A. A is a hereditary algebra if and only if so is B in [13]. On the other hand, it was proved by Reiten and Riedtmann in [17] that if A is an artinian algebra the skew group algebra AG with G a finite group and $|G|^{-1} \in A$ is tilted algebra if and only if so is A. Based on these facts, it is natural to ask the following question.

Question 1.1. Let $B \geq A$ be an excellent extension, is B a tilted algebra if and only if so is A?

In this paper, we will study the tilting properties of excellent extensions and give a partial answer to this question. This paper is organized as follows.

In Section 2, we give some notations in our terminology and some preliminary results which are often used in this paper. In Section 3, we prove the following

Theorem 1.2. *Let B be an excellent extension of an artinian algebra A such that B is centrally projective over A, then A is a tilted algebra if and only if so is B.*

As an application of theorem 1.2, we investigate tilted algebras under base field extensions.

Theorem 1.3. *Let A be a finite dimensional K-algebra, and F a finite separable field extension of K. Then A is a tilted algebra if and only if $A \otimes_K F$ is also a tilted algebra.*

2 Preliminary Notes

Throughout this paper, all modules are finitely generated right modules unless stated otherwise. Let A be an artinian algebra over a commutative artinian ring R, that is, A is an R- algebra (associative, with identity) which is finitely generated as an R-module, we denote by mod$A$ the category of finitely generated right A-modules, and by gl.dim$A$ the global dimension of A. For a module M in mod$A$, we denote by $\mathrm{pd}_A M$ the projective dimension of M, and denote by addM_A the full subcategory of mod$A$ consisting of the modules isomorphic to the direct summands of finite direct sums of finite copies of M.

We begin with the definition of excellent extensions of artinian algebras.

Definition 2.1. *Let A be a subalgebra of an artinian algebra B, such that A and B have the same identity. Then B is called an algebraic extension of A, and denoted by $A \leq B$. An algebraic extension $A \leq B$ is called an excellent extension, if*

(1) $A \leq B$ *is right* A*-projective* ([14], p.273), *that is, if* N_B *is a submodule of* M_B *and if* N_A *is a direct summand of* M_A*, denoted by* $N_A|M_A$*, then* $N_B|M_B$*.*

(2) B *is a free normalizing extension of* A *with a basis that includes 1; that is, there is a finite set* $\{b_1, b_2, \cdots, b_n\} \in B$ *such that* $b_1 = 1, B = b_1A + \cdots + b_nA$ *and* $b_iA = Ab_i$*, for each* i*, and* B *is free with basis* $\{b_1, b_2, \cdots, b_n\} \in B$ *as both a right and left* A*-module.*

Lemma 2.2. (See[[14], p.275, Lemma 2.3].) *Let* A *be a finite dimensional* K*-algebra, and* F *be a finite separable field extension of* K*. Then* $A \otimes_K F$ *is an excellent extension of* A*.*

We also need the following lemma, which may been found in [7, 13, 18].

Lemma 2.3. *Let* B *be an excellent extension of* A*.*

(1)(see [[18], Lemma 1.1].) *For any* B*-module* M*, we have* $M_B|(M \otimes_A B)_B$*;*

(2)(see [[18], Lemma 1.4].) *For any* B*-module* N*, we have* $\mathrm{pd}_B N = \mathrm{pd}_A N = \mathrm{pd}_A (N \otimes_A B)_B$;

(3)(see[[13], Theorem 3].) *gl.dimA= gl.dimB.*

Let A be an artinian algebra, and M a (A, A)-bimodule. Then according to K. Hirata [9] M is called *centrally projective over* A, if M is isomorphic to a direct summand of a finite direct sum of the copies of A as (A, A)-bimodule. Then next lemma is due to K. sugano [16].

Lemma 2.4.(see [[16], Lemma 3].) *Let* R *be a commutative ring. If* A *is an artinian* R*-algebra and* M *is a finitely generated projective* R*-module, then* $A \otimes_R M$ *is centrally projective over* A*.*

For any additional category $\mathcal{A}$ we denote by $(\mathcal{A}^{op}, Ab)$ the category of contravariant functors from $\mathcal{A}$ to $\mathcal{A}b$, where $\mathcal{A}b$ is the category of all abelian groups. Recall from [2] that a functor $F : \mathcal{A}^{op} \to Ab$ is called *coherent* if there is an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-, A_1) \to \mathrm{Hom}_{\mathcal{A}}(-, A_0) \to F \to 0$$

in $(\mathcal{A}^{op}, Ab)$ with $A_i \in \mathcal{A}$ for $i = 0, 1$. We denoted by $\widehat{\mathcal{A}}$ the full subcategory of the functor category consisting of all coherent functors. By Yoneda's lemma, the projective object in $\widehat{A}$ is of the form $\mathrm{Hom}_{\mathcal{A}}(-, X)$ with X an object in $\mathcal{A}$, and each coherent functor F can be determined by morphism $f : A_1 \to A_0$, that is, there is an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-, A_1) \xrightarrow{(-,f)} \mathrm{Hom}_{\mathcal{A}}(-, A_0) \to F \to 0$$

in $\widehat{\mathcal{A}}$. As in the case of a module category, we may define the global dimension of the category $\widehat{\mathcal{A}}$ to be the supremum of the projective dimensions of all functors in $\widehat{\mathcal{A}}$. The precise connection between the global dimension of an artinian algebra and that of the coherent functor category is recorded in the following lemma due to Auslander in [1].

Lemma 2.5 *Let M be an A-module. Then the category $\widehat{\operatorname{add} M}$ and* $\operatorname{mod}(\operatorname{End}({}_A M))$ *are equvalent. In particular, gl.dim*$(\operatorname{End}_A M)$=*gl.dim*$(\widehat{\operatorname{add} M})$.

Recall from [12] that a module $T \in \operatorname{mod} A$ is called an *m-tilting module* if the following conditions are satisfied: (1) $\operatorname{pd}_A T \leq m$, (2) $\operatorname{Ext}^i_A(T,T)$=0 for any $i \geq 1$, and (3) there exists an exact sequence $0 \to A_A \to T_1 \to T_2 \cdots \to T_m \to 0$ in $\operatorname{mod} A$ with all $T_i \in \operatorname{add}_A T$. Recall from [5] that an artinian algebra A is called a *tilted algebra*, if there exists 1-tilting module ${}_B T$ over a hereditary algebra such that $A = \operatorname{End}_B T$. It is easy to see that A is tilted if and only if there exists a 1-tilting module T_A such that the endomorphism algebra of T_A is a hereditary algebra.

3 Main Results

To characterize the tilting property of excellent extensions, we need the following lemma, which may be of independent interest.

Lemma 3.1. *Let A be an Artin algebra, and T_A a finitely generated A-module with* $\operatorname{Ext}^1_A(T,T) = 0$. *Then gl.dim* $\operatorname{End}_A T \leq 1$ *if and only if* $\operatorname{add} T_A$ *is closed under submodules.*

Proof. We first prove the necessity. In order to prove that $\operatorname{add} T_A$ is closed under submodules, it suffices to show that every submodule of finite copies of T_A is in $\operatorname{add}_A T$. Let M be any nonzero submodule of T^n for some $n \geq 1$ and $i : 0 \to M \to T^n$ the inclusion homomorphism. Then we have an exact sequence: $0 \to \operatorname{Hom}_A(-,M) \xrightarrow{(-,i)} \operatorname{Hom}_A(-,T^n) \to H \to 0$, where $H \in ((\operatorname{add} T)^{op}, \mathcal{A}b)$ with $H = \operatorname{Coker}(-,i)$. It follows from ([12], Lemma 1.3) that $\operatorname{Hom}_A(-,M) \in \widehat{\operatorname{add} T}$, so $H \in \widehat{\operatorname{add} T}$. Because gl.dimEnd $T \leq 1$ by assumption, and $\operatorname{pd}_{\widehat{\operatorname{add} T}} H \leq 1$ by Lemma 2.5. Hence, $\operatorname{Hom}_A(-,M)$ is projective in $\widehat{\operatorname{add} T}$, and so $M \in \operatorname{add} T$ by Yoneda's lemma.

We next prove the sufficiency. For any $F \in \widehat{\operatorname{add} T}$, there exists a homomorphism $f : T_1 \to T_0$ in $\operatorname{mod} A$ such that

$$\operatorname{Hom}_A(-,T_1) \xrightarrow{(-,f)} \operatorname{Hom}_A(-,T_0) \to F \to 0$$

is exact in $\widehat{\operatorname{add}_A T}$. Put $T_2 = \operatorname{Ker} f$, and $Y = \operatorname{Im} f$, then $T_2, Y \in \operatorname{add}_A T$, by assumption. Because $\operatorname{Ext}^1_A(T,T) = 0$, the exact sequence $0 \to T_2 \to T_1 \to Y \to 0$ are split. So we get an exact sequence: $0 \to \operatorname{Hom}_A(-,T_2) \to \operatorname{Hom}_A(-,T_1) \to \operatorname{Hom}_A(-,Y) \to 0$ in $\widehat{\operatorname{add}_A T}$. And hence $0 \to \operatorname{Hom}_A(-,Y) \to$

$\mathrm{Hom}_A(-,T_0) \to F \to 0$ is exact in $\widehat{\mathrm{add}_A T}$, which implies $\mathrm{pd}_{\widehat{\mathrm{add}_A T}} F \le 1$. Thus gl.dim $\mathrm{End}_A(T) = \text{gl.dim}(\widehat{\mathrm{add}\, T_A}) \le 1$ by Lemma 2.5.

The following lemma is directly.

Lemma 3.2. *Let A be an artinian algebras and M, N A-modules, if $\mathrm{Ext}_A^{i\ge 1}(M,N)$=0, we have $\mathrm{Ext}_A^{i\ge 1}(M,X)$=0, for any $X \in \mathrm{add}\, N_A$.*

Lemma 3.3. *Let B be an excellent extension of an artinian algebra A, such that B is centrally projective over A.*

(1) *For any A-module M, we have* $\mathrm{add}\, M_A = \mathrm{add}(M \otimes_A B)_A$.

(2) *For any B-module N, we have* $\mathrm{add}\, N_B = \mathrm{add}(N \otimes_A B)_B$.

The proof is obvious.

Lemma 3.4. *Let B be an excellent extension of an artinian algebra A, such that B is centrally projective over A.*

(1) If T is an m-tilting A-module, we have $T \otimes_A B$ is an m-tilting B-module.

(2) If Y is an m-tilting B-module, we have Y is an m-tilting A-module.

Proof. (1) Note that B is a projective A-module, we have $\mathrm{pd}_B(T \otimes_A B) = \mathrm{pd}_A T \le m$ by Lemma 2.3.

By Lemma 3.3., we have $(T \otimes_A B)_A \in \mathrm{add}\, T_A$. Since $\mathrm{Ext}_A^i(T,T) = 0$ for any natural number i, we have $\mathrm{Ext}_B^i(T \otimes_A B, T \otimes_A B) \cong \mathrm{Ext}_A^i(T, T \otimes_A B) = 0$ by ([10], Lemma 4.7) and lemma 3.2.

Since T is a m-tilting A-module, there exists a long exact sequence in $\mathrm{mod}\, A$

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_m \to 0,$$

with $T_i \in \mathrm{add}\, T_A$, for all i.

Applying the functor $- \otimes_A B$ to the above sequence, we have the following exact sequence in $\mathrm{mod}\, B$

$$0 \to B \to T_0 \otimes_A B \to T_1 \otimes_A B \to \cdots \to T_m \otimes_A B \to 0,$$

with $T_i \otimes_A B \in \mathrm{add}(T_A \otimes_A B)_B$, for all i. So it follows that $T \otimes_A B$ is a m-tilting module from the definition of m-tilting module.

(2) Since Y is an m-tilting B-module, we have $\mathrm{pd}_A Y = \mathrm{pd}_B Y \le m$ by Lemma 2.3. Because $\mathrm{Ext}_B^i(Y,Y) = 0$, we have $\mathrm{Ext}_B^i(Y \otimes_A B, Y) = 0$ by Lemma 3.2. and Lemma 3.3. And hence, for any i we have $\mathrm{Ext}_A^i(Y,Y) \cong \mathrm{Ext}_B^i(Y \otimes_A B, Y) = 0$, by ([10], Lemma 4.7).

Since there exist an exact sequence:

$$0 \to B \to Y_1' \to Y_2' \to \cdots \to Y_m' \to 0$$

in $\mathrm{mod}\, B$ with all $Y_i' \in \mathrm{add}\, Y_B$. Note that $A_A | B_A$, so there exists an exact sequence:

$$0 \to A \to Y_1 \to Y_2 \to \cdots \to Y_m \to 0$$

in $\operatorname{mod} A$ with all $Y_i \in \operatorname{add} Y_A$. Hence we conclude that Y is an m-tilting A-module.

Theorem 3.5. *Let B be an excellent extension of an artinian algebra A such that B centrally projective over A, then A is a tilted algebra if and only if so is B.*

Proof. Assume that A is a tilted algebra. Then there exists a 1-tilting A-module T, such that $\text{gl.dimEnd}_A T \leq 1$. By Lemma 3.1, $\operatorname{add} T_A$ is closed under submodules. We claim that $\operatorname{add}(T \otimes_A B)_B$ is closed under submodules. It suffices to prove that any submodule of finite copies of $(T \otimes_A B)_B$ is in $\operatorname{add}(T \otimes_A B)_B$. Let M_B be a submodule of $(T \otimes_A B)_B^{(t)}$ for some $t \geq 1$. By Lemma 3.2.,$(T \otimes_A B)_A^{(t)} \in \operatorname{add} T_A$, and so $M_A \in \operatorname{add} T_A$. Hence, we have $(M \otimes_A B)_B \in \operatorname{add}(T \otimes_A B)_B$. By Lemma 2.3., we have $M_B | (M \otimes_A B)_B$, and therefore $M_B \in \operatorname{add}(T \otimes_A B)_B$, we obtain our claim. And by Lemma 3.1, we have $\text{gl.dimEnd}_B(T \otimes_A B) \leq 1$. On the other hand, by Lemma 3.4(1), $T \otimes_A B$ is a 1-tilting B-module, which implies that B is a tilted algebra.

Conversely, suppose that B is a tilted algebra, there exist is a 1-tilting B-module X, such that $\operatorname{End}_B X$ is a hereditary algebra. By Lemma 3.4(2), X_A is a 1-tilting A-module. We claim that $\operatorname{add} X_A$ is closed under submodules. Let N be an submodule of X_A^m for some $m \geq 1$. Because B_A is a projective A-module, $N \otimes_A B$ is isomorphic to a submodule of $X^m \otimes_A B$. By Lemma 3.2., $X^m \otimes_A B \in \operatorname{add} X_B$, and so $(N \otimes_A B)_A \in \operatorname{add} X_A$. Since A is an excellent extension, we have $N_A | (N \otimes_A B)_A$, we get our claim. And so $\operatorname{add}_R X$ is closed under submodules. By Lemma 3.1, $\operatorname{End}_A X$ is a hereditary algebra. Therefore, A is a tilted algebra.

By Lemma 2.4 and theoren 3.4, we have

Corollary 3.6 *Let A be an artinian R-algebra. If A is an excellent extension of R, then R is a tilted algebra if and only if so is A.*

We in a position to prove the tilting property of artin algebras under base field extension.

Theorem 3.7. *Let A be a finite dimensional K-algebra, and let F be a finite separable field extension of K. Then A is a tilted algebra if and only if $A \otimes_K F$ is also a tilted algebra.*

Proof. By Lemma 2.2., $A \otimes_K F$ is an excellent extension of A. And $A \otimes_K F$ is centrally projective over A, by Lemma 2.4. It follows from by Theorem 3.5.

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