A Remark on the Ideals of BCK-Algebras and Lattices

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Abstract

In this paper, we shall show that the Iséki’s ideals of a BCK-algebra agree with the ideals as lattice in a bounded, commutative BCK-algebra.

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1 Introduction

BCK-algebras were introduced by Kiyoshi Iséki into the study of general algebra. In the 1960s, Iséki led the citizen’s group to study mathematics: Suugaku-kisoron-no-kai (Meeting for the study of foundations of mathematics). The members of this group have studied mainly propositional calculi. They published their research papers in Proceedings of the Japan Academy Ser.A Mathematical Sciences. In 1966, Y. Imai and K. Iséki defined BCK-algebras in the article [2] entitled “On axiom system of propositional calculi XIV”. This algebra was generalized in one from the notion of the algebra of sets with the set-difference as the only fundamental operation, and was generalized in the other from the notion of propositional calculi which contain the only implication functor among the logical functors.

Here, we give one point of attention. Letters B, C, K are compatible with the next inference rules:

(B) $CC_{q}rCC_{pq}C_{pr}$ \[ ((q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))) \],

(C) $CC_{p}C_{q}rCC_{q}C_{pr}$ \[ (p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r))) \],
(K) \( C_p C_q p \) \( (p \Rightarrow (q \Rightarrow p)) \).

Well, I was one of the last members of Suugaku-kisoron-no-kai. In 2011, K. Iséki died and we lost a superior leader. And unfortunately, this group has disappeared. In this 2019, what is more, we lost an irreplaceable teacher S. Tanaka.

Now, we recall the definition of a BCK-algebra.

**Definition 1.1.** A BCK-algebra is an algebra \( X = \langle X; *, 0 \rangle \) of type \( < 2, 0 > \), provided with the following conditions (I) \( \sim \) (V) for any \( x, y, z \in X \).

1. \( \{(x * y) * (x * z)\} * (z * y) = 0 \),
2. \( \{x * (x * y)\} * y = 0 \),
3. \( x * x = 0 \),
4. \( 0 * x = 0 \),
5. \( x * y = 0, y * x = 0 \) imply \( x = y \),

In the following, we denote \( x \leq y \) when \( x * y = 0 \) for \( x, y \) in \( X \).

## 2 BCK-algebras and lattices

Here, we will present that an important special class in BCK-algebras becomes a lattice.

In 1975, S. Tanaka [6] studied a special class in BCK-algebras, which is the algebra satisfying the condition (VI) besides,

\[ x \wedge y = y \wedge x \text{ with } x \wedge y = y * (y * x). \]

This algebra is called a *commutative BCK-algebra*. He showed that any commutative BCK-algebra \( X = \langle X; *, 0 \rangle \) is a lower semi-lattice with respect to the operation \( \wedge \), and further that the greatest lower bound of \( x \) and \( y \) in \( X \) is given by \( x \wedge y \).

In 1974, K. Iséki [3] defined a special class in BCK-algebras. If there is a element 1 in a BCK-algebra \( X = \langle X; *, 0 \rangle \) satisfying \( x \leq 1 \) for any \( x \in X \), then a BCK-algebra with unit 1 is called a *bounded BCK-algebra*.

In 1978, K. Iséki and S. Tanaka [5] proved that any bounded, commutative BCK-algebra \( X = \langle X; *, 0 \rangle \) is a lattice with respect to the operations \( \wedge \) and \( \vee \) given by \( x \wedge y = y * (y * x) \) and \( x \vee y = 1 * \{(1 * x) \wedge (1 * y)\} \).

Moreover, this algebra satisfies the identities like the de Morgan law;

\[ N x \vee N y = N(x \wedge y), N x \wedge N y = N(x \vee y) \] with \( N x = 1 * x \) for any \( x, y \in X \).
In 1979, T. Trazyk [7] proved that a bound, commutative BCK-algebra satisfies the distributive law with respect to the operations $\land$ and $\lor$;

$$z \land (x \lor y) = (z \land x) \lor (z \land y), z \lor (x \land y) = (z \lor x) \land (z \lor y)$$

for any $x, y, z \in X$.

Then, a bounded, commutative BCK-algebra $X =< X; \land, \lor, 0, 1 >$ is a de Morgan algebra, but of course this algebra is a lattice.

3 The ideals of BCK-algebras and lattices

In this section, we will consider the next question.

Question Can we say that the Iséki’s ideals of a BCK-algebra agree with the ideals as lattice?

At first, an ideal of a BCK-algebra (see K. Iséki [4]) is defined as following.

Definition 3.1 A subset $A$ of a BCK-algebra $X$ is called an Iséki’s ideal, provided with the following conditions (i) and (ii) for any $x, y \in X$;

(i) $0 \in A$,
(ii) $y \ast x \in A$, $x \in A$ imply $y \in A$.

On the other hand, an ideal of a lattice that is defined in the next.

Definition 3.2 (G. Birkoff [1]) An ideal is a subset $I$ of a lattice $L$ with the following properties (1) and (2) for any $x, y \in L$;

(1) $a \in I$, $x \in L$, $x \leq a$ imply $x \in I$,
(2) $a \in I$, $b \in I$ imply $a \lor b \in I$.

Noting that a bounded, commutative BCK-algebra is also a lattice with respect to the operations to $\land$ and $\lor$.

Now, we will show that an Iséki’s ideal of a bounded, commutative BCK-algebras is none other than an ideal in the sense of de Morgan algebras.

Theorem 3.3 Let $X =< X; \ast, 0 >$ be a bounded, commutative BCK-algebra. Then an Iséki’s Ideal $I$ of $X$ for the operation $\ast$ is an ideal of a de Morgan algebra $X$ for the operation $\land$ and $\lor$.

Proof: Let $X =< X; \ast, 0 >$ and $I$ be a bounded, commutative BCK-algebra and an Iséki’s ideal of $X$, respectively. As are stated in Sec.2, $X =< X; \ast, 0 >$ is a distributive lattice $X =< X; \land, \lor, 0, 1 >$. Namely, we will show that the next condition (1) and (2) holds;

(1) $x \in I$, $z \in X$, $z \leq x$ imply $z \in I$,
(2) $x \in I$, $y \in I$ imply $x \lor y \in I$. 
Take \( x \in I, \ z \in X, \) and let be \( z \leq x. \) Since \( I \) is an Iséki's ideal of the BCK-algebra \( X, \ x \in I \) and \( z \ast x = 0 \in I, \) by (ii), then \( z \) belongs to \( I. \)

This shows the condition (1) holds.

In a bounded, commutative BCK-algebras \( X, \) the equality

\[
\{(x \lor y) \ast y\} \ast x = 0
\]

holds for any \( x, y \in X, \) which is shown in the lemma 3.4 below.

Let \( x \) and \( y \) be in \( I. \) Since \( I \) is an Iséki's ideal of the BCK-algebra \( X, \ x \in I \) and \( \{(x \lor v) \ast y\} \ast x = 0 \in I, \) by (ii), then \( (x \lor y) \ast y \in I. \) Similarly, since \( (x \lor y) \ast y \in I \) and \( y \in I, \) we have \( x \lor y \in I. \)

This shows the condition (2) holds.

This completes the proof of the theorem 3.3.

**Lemma 3.4** Let \( X \) be a bounded commutative BCK-algebra.

Then we have

\[
\{(x \lor y) \ast y\} \ast x = 0
\]

for any \( x, y \in X. \)

**Proof:** In a bounded, commutative BCK-algebra \( X, \) the identities

\[
NNx = x. \tag{3.1}
\]

\[
x \ast y = Ny \ast Nx. \tag{3.2}
\]

hold for any \( x, y \in X. \)

Then, using (3.1) and (3.2), we have

\[
(x \lor y) \ast y = N(Nx \land Ny) \ast y
\]

\[
= Ny \ast NN(Nx \land Ny)
\]

\[
= Ny \ast (Nx \land Ny)
\]

\[
= Ny \ast \{Ny \ast (Ny \ast Nx)\}
\]

\[
= Ny \ast \{Ny \ast (x \ast y)\}
\]

\[
\leq x \ast y
\]

for any \( x, y \in X. \)

Hence, \( \{(x \lor y) \ast y\} \ast x = 0. \)

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References


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