

A Note on the Canonical Divisor of the Generalised Affine Stiefel Algebraic Varieties

Umar Hayat

Department of Mathematics, Quaid-i-Azam University
Islamabad, Pakistan

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2019 Hikari Ltd.

Abstract

Let X and Y be $r \times s$ and $s \times r$ matrices. In this paper we study varieties defined by the matrix equation $XY = I$. We call these varieties generalised affine Stiefel algebraic varieties. We show that the canonical divisor of these varieties is Cartier using two different methods. The main aim is to characterise the canonical divisor of these varieties in terms of group representations. Affine Stiefel algebraic variety is a special case of the generalised affine Stiefel algebraic varieties.

Mathematics Subject Classification: 14M17, 14J60, 20C15

Keywords: Stiefel manifold, Homogeneous spaces, Matrix equations, Vector bundle

1 Introduction

Let G be an algebraic group. If G acts on a space M transitively, we say that M is a homogeneous space of G . Let $x \in M$, the group elements fixing x is a subgroup of G . We call this the stabiliser or isotropy subgroup. Equivalently, a homogeneous space may be realised as a space of cosets of closed isotropy subgroup of G . The geometric aspects of homogeneous spaces are often used to study representation theory of algebraic groups. Many authors [3–8] have studied homogeneous and quasi-homogeneous spaces in different contexts. For weighted analogs of certain homogeneous spaces, see [9].

Let X be an $r \times s$ matrix where $r \leq s$. An affine Stiefel algebraic variety is defined by the following matrix equation

$$XX^{tr} = I_{r \times r}.$$

Let X and Y be $r \times s$ and $s \times r$ matrices. We define a variety $V \subset \mathbb{C}^{2(r \times s)}$ by the following matrix equation

$$XY = I_{r \times r}.$$

We call this variety generalised affine Stiefel algebraic variety and denote it by $GSV(r, s)$.

In this paper, we show that $GSV(r, s)$ is a homogeneous space, an orbit of a special vector which is not a weight vector.

In Section 2, proposition 1, we show that the canonical divisor of the generalised affine Stiefel algebraic variety $GSV(r, s)$ is Cartier. Using a representation-theoretic approach the canonical class of the $GSV(r, s)$ is calculated in Section 3, Theorem 1.

2 The variety in Equations

Let X and Y be the $r \times s$ and $s \times r$ matrices as given below:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{rs} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s1} & y_{s2} & \cdots & y_{sr} \end{pmatrix},$$

where $r \leq s$.

We define a variety $V \subset \mathbb{C}^{2(r \times s)}$ by the equation

$$XY = I_{r \times r}.$$

If both X and Y are of maximal rank then V has codimension $rs + r(s - r)$. We show in Section 3 that V is a homogeneous space, the orbit of the vector

$$\left(X_0 = (I_{r \times r} \quad 0_{r \times (s-r)}), \quad Y_0 = \begin{pmatrix} I_{r \times r} \\ 0_{(s-r) \times s} \end{pmatrix} \right),$$

under an action of $G = GL(r) \times GL(s)$. This is explained in Section 3.

If the rank of X and Y are maximal then we may suppose that the first minor X_{s_1, \dots, s_r} of X is non vanishing. This nonzero minor can be used to eliminate the first r rows of Y .

In all other cases where rank X or rank Y drops, the condition $XY = I_{r \times r}$ is not satisfied so it does not influence the calculation of the canonical class of V .

The tangent bundle and its dual, the cotangent bundle are the most important vector bundles on a smooth variety. Algebraic geometers traditionally work with the cotangent bundle, which can be identified with the sheaf of Kähler differentials. The canonical bundle is the top exterior power of the cotangent bundle and its first Chern class is called the canonical class.

2.1 The canonical class of $GSV(r, s)$

For some $r \times r$ nonzero minor of X one can explicitly write the r rows of Y and the remaining entries of Y and X are the coordinates. For example, suppose that the first minor X_{s_1, \dots, s_r} of X is nonzero. Then first r rows of Y , $y_{11}, \dots, y_{1r}, y_{21}, y_{22}, \dots, y_{2r}, \dots, y_{r1}, y_{r2}, \dots, y_{rr}$ can be written using remaining entries of X and Y . Using the same procedure for the second nonzero minor $X_{s_2, \dots, s_{r+1}}$ of X , we can solve for $y_{21}, \dots, y_{2r}, y_{31}, y_{32}, \dots, y_{3r}, \dots, y_{r+1,1}, y_{r+1,2}, \dots, y_{r+1,r}$.

Here we explicitly explain in terms of coordinates. Suppose $\zeta_1, \dots, \zeta_{rs}, \xi_{rs+1}, \dots, \xi_{rs+r(s-r)}$ and $\mu_1, \dots, \mu_{rs}, \mu_{rs+1}, \dots, \mu_{rs+r(s-r)}$ are coordinates on the two charts $U_{X_{s_1, \dots, s_r} \neq 0}$ and $U_{X_{s_2, \dots, s_{r+1}} \neq 0}$ of V . The majority of the coordinates between the two charts are common because these charts differ only by one row of the Y and there are exactly $rs + r(s - r) - r$ coordinates in common. The coordinate transformation matrix J from one chart to the other is given by

$$J = \begin{pmatrix} I_{rs+r(s-r)-r \times rs+r(s-r)-r} & 0_{rs+r(s-r)-r \times r} \\ C_{r \times rs+r(s-r)-r} & D_{r \times r} \end{pmatrix},$$

where the submatrix $C_{r \times rs+r(s-r)-r}$ is calculated by taking the first order partial derivatives of the variables which are not in the intersection of two charts with respect to the variables common in both the charts and D is the diagonal matrix with entries $\frac{X_{s_2, \dots, s_{r+1}}}{X_{s_1, \dots, s_r}}$ and the determinant of the Jacobian matrix J

is $\left(\frac{X_{s_2, \dots, s_{r+1}}}{X_{s_1, \dots, s_r}} \right)^r$.

Since the minor X_{s_1, \dots, s_r} is nonzero on the chart $U_{X_{s_1, \dots, s_r}}$ and similarly $X_{s_2, \dots, s_{r+1}}$ is invertible on the chart $U_{X_{s_2, \dots, s_{r+1}}}$. Suppose

$$\sigma_{1, \dots, r} = \frac{d\zeta_1 \wedge \dots \wedge d\zeta_{rs+r(s-r)}}{(X_{s_1, \dots, s_r})^r} \text{ and similarly } \sigma_{2, \dots, r+1} = \frac{d\mu_1 \wedge \dots \wedge d\mu_{rs+r(s-r)}}{(X_{s_2, \dots, s_{r+1}})^r}.$$

We put $(X_{s_1, \dots, s_r})^r$ in the denominator to simplify the determinant of the Jaco-

bian. The sheaf of the canonical differentials is

$$\mathcal{O}(K_V) = \bigwedge^{rs+r(s-r)} \Omega_V^1 \text{ and } \mathcal{O}(K_V) |_{U_{X_{s_1, \dots, s_r} \neq 0}} = \mathcal{O}_{U_{X_{s_1, \dots, s_r} \neq 0}} \cdot \sigma_{1, \dots, r}.$$

It is evident from calculations that $\sigma_{1, \dots, r} = \sigma_{2, \dots, r+1}$ and using the similar calculations, defines $\sigma = \sigma_{i_1, \dots, i_r}$ independently of i_1, \dots, i_r . Since σ_{i_1, \dots, i_r} is a basis of $\bigwedge^{rt+r(s-r)} \Omega_V^1$ and has no poles or zeros, because X_{i_1, \dots, i_r} is nonzero, we have

$$K_V = \text{divisor}(\sigma) = 0.$$

In the above discussion we have shown that there is an open cover $\{(U_{X_{i_1, \dots, i_r}})\}$ for the $GSV(r, s)$, with transition functions $\frac{1}{X_{i_1, \dots, i_r}} \in k(U_{X_{i_1, \dots, i_r}})^*$.

Furthermore $\frac{X_{s_1, \dots, s_r}^r}{X_{s_2, \dots, s_{r+1}}^r} \in \mathcal{O}^*(U_{X_{s_1, \dots, s_r}} \cap U_{X_{s_2, \dots, s_{r+1}}}) = \mathcal{O}^*(U_{X_{s_1, \dots, s_r} \neq 0, X_{s_2, \dots, s_{r+1}} \neq 0})$.

We summarise the discussion in the preceding section in the following proposition.

Proposition 1. *The canonical divisor of the generalised affine Stiefel algebraic variety $GSV(r, s)$ is Cartier.*

3 The generalised affine Stiefel Variety $GSV(r, s)$ as a homogeneous space

In this section we understand the variety V as an orbit of a certain vector, which is not a weight vector. We fix $G = GL(r) \times GL(s)$. Let W_r and W_s be the natural representations of $GL(r)$ and $GL(s)$ of dimension r and s respectively.

We wish to define an action of $G = GL(r) \times GL(s)$ on the representation $W = Hom(W_r, W_s) \oplus Hom(W_s, W_r)$ that keeps V invariant. In coordinate-free terms, $X \in Hom(W_r, W_s)$ and $Y \in Hom(W_s, W_r)$ and the action of $(A, B) \in G$ with $A \in GL(r)$, $B \in GL(s)$ is defined as follows,

$$\begin{aligned} X &\longmapsto AXB^{-1} \\ Y &\longmapsto BYA^{-1}. \end{aligned}$$

If X and Y are full rank matrices then both X and Y can be written the following form

$$\left(X_0 = \begin{pmatrix} I_{r \times r} & 0_{r \times s-r} \end{pmatrix}, \quad Y_0 = \begin{pmatrix} I_{r \times r} \\ 0_{s-r \times r} \end{pmatrix} \right).$$

The isotropy subgroup of G that stabilises the vector $v = (X_0, Y_0)$ is given by

$$H = Stab(v) = \left\{ (A, B) \mid B = \begin{pmatrix} A & 0 \\ 0 & *_{s-r} \end{pmatrix} \right\},$$

where no restriction on this block represented (*). It can be noticed that v is not a weight vector. Here G acts transitively and

$$V = G/H \simeq G \cdot v \hookrightarrow W.$$

The generalised affine Stiefel algebraic variety $GSV(r, s)$ is a homogeneous space under this action of G .

3.1 The Weyl group

To understand representations of the group $G = GL(r) \times GL(s)$ one can use the Weyl group $W(G) \cong S_r \times S_s$. It is obvious from Section 3 that $W = Hom(W_r, W_s) \oplus Hom(W_s, W_r)$ is a representation of G . Here we explain action of the Weyl group. The groups S_r and S_s act on any $X \in Hom(W_r, W_s)$ from the left and right respectively and permute the rows and the columns of X . Similarly, S_s permutes the rows of $Y \in Hom(W_s, W_r)$ when acts on Y from the left and the columns are permuted when S_r acts from right.

3.2 The Weyl group and torus action

The maximal torus $\mathbb{T} \subset G$ given below

$$\mathbb{T} = \left[T_A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{rr} \end{pmatrix}, T_B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{ss} \end{pmatrix} \right]$$

acts on

$$X_{11} = \begin{pmatrix} x_{11} & 0_{1 \times s-1} \\ 0_{r-1 \times 1} & 0_{r-1 \times s-1} \end{pmatrix} \quad \text{and} \quad Y_{11} = \begin{pmatrix} y_{11} & 0_{1 \times r-1} \\ 0_{s-1 \times 1} & 0_{s-1 \times r-1} \end{pmatrix}$$

as explained in section 3. Under this action X_{11} is a weight vector with weight $\frac{a_{11}}{b_{11}}$ and Y_{11} has weight $\frac{b_{11}}{a_{11}}$.

3.3 A representation theoretic approach to calculate the canonical class

For the maximal torus $\mathbb{T} \subset G$, the canonical differential $\sigma_{1, \dots, r} = \frac{d\zeta_1 \wedge \dots \wedge d\zeta_{rs+r(s-r)}}{(X_{s_1, \dots, s_r})^r}$ is a weight vector with weight 1. Similarly all the $\sigma = \sigma_{i_1, \dots, i_r}$ are weight vectors with unit weight. But the stabiliser H is not normalised by the maximal torus $\mathbb{T} \subset G$. If we take $\mathbb{T}_H = \mathbb{T} \cap N_H$, where N_H is the normaliser of H then \mathbb{T}_H

$$\left\{ T_A, T_B = \begin{pmatrix} T_A & 0_{r \times s} \\ 0_{s \times r} & *_{s-r \times s-r} \end{pmatrix} \right\}$$

acts on $\sigma_{1, \dots, r} = \frac{d\zeta_1 \wedge \dots \wedge d\zeta_{rs+r(s-r)}}{(X_{s_1, \dots, s_r})^r}$ and $\sigma_{1, \dots, r} = \frac{d\zeta_1 \wedge \dots \wedge d\zeta_{rs+r(s-r)}}{(X_{s_1, \dots, s_r})^r}$ is a weight vector having weight equal to 1.

Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H respectively. The tangent space $T_{G/H}$ to G/H at the identity H can be identified with the quotient vector space $\mathfrak{g}/\mathfrak{h}$, and the tangent space to any other coset $gH \in G/H$ is obtained by $\mathfrak{g}/g\mathfrak{h}g^{-1}$. The canonical class of the variety G/H is

$$K_{G/H} = \text{divisor} \left(\bigwedge^{rs+s(s-r)} T_{G/H}^\vee \right).$$

We write the weight of the canonical differential $K_{G/H}$ as a product of those weights of G those are not weights of H .

Theorem 1. *The weight of the canonical differential $K_{G/H}$ is 1.*

Proof. In total, there are $rs + s(s - r)$ weights that are weights of G but not of H . We show that their product is 1.

Each asterisk block in the following matrix corresponds to weight spaces of G that are not the weight spaces for H ,

$$\left(\begin{array}{c|c} *_{r \times r} & *_{r \times s-r} \\ \hline *_{s-r \times r} & 0_{s-r \times s-r} \end{array} \right).$$

The product of weights coming from the top left blok is equal to 1. Also for every weight corresponding to a weight space in the top right block there is the reciprocal of that coming from the left bottom block. Hence the product of all the weights is 1.

This shows when \mathbb{T}_H acts on $\mathfrak{g}/\mathfrak{h}$ the weight of the canonical differential 1. ■

4 Special cases of $GSV(r, s)$

Stiefel Variety: Let X be a $r \times s$ matrix and Y be the transpose of X given by

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{rs} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{r1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1s} & x_{2s} & \cdots & x_{rs} \end{pmatrix}.$$

The variety defined by $XY = XX^{tr} = I_{r \times r}$ is a Stiefel manifold.

5 Future Applications of Generalised Stiefel Varieties

Many authors have proposed suitable algebraic structures to obtain good packings on a complex Stiefel manifolds by numerical methods, see [2] for further details. In future one can try to understand what happens in case of generalised Stiefel manifolds.

In [1] authors have developed various algorithms on the Stiefel and Grassmann manifolds. The above mentioned methods may be developed on the generalised Stiefel manifolds as well.

Acknowledgements. This work is mainly carried out during author's visit to the International Center for Theoretical Physics(ICTP), Trieste, Italy. The author is very thankful to the Mathematics Section of the ICTP for awarding short term fellowship. This research is also partially funded through Quaid-i-Azam University grant URF-2015 .

References

- [1] A. Edelman, T. A. Arias, and S. T. Smith, Geometry of algorithms with orthogonality constraints, *SIAM J. Matrix. Anal. Appl.*, **20** (2), 303-353. <https://doi.org/10.1137/s0895479895290954>
- [2] G. Han, J. Rosenthal, Good packings in the complex Stiefel manifold using numerical methods. Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, July 24-28, 2006.

- [3] U. Hayat, The Cramer varieties $\text{Cr}(r,r+s,s)$, *Journal of Geometry and Physics*, **79** (2014), 53-58. <https://doi.org/10.1016/j.geomphys.2014.02.001>
- [4] U. Hayat, A note on the canonical divisor of quasi-homogeneous affine algebraic varieties, *Communications in Algebra*, **44** (3) (2016), 1275-1279. <https://doi.org/10.1080/00927872.2015.1027347>
- [5] M. Hochster, Topics in the homological theory of modules over commutative rings. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975, expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24-28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24.
- [6] A. R. Kustin, M. Miller, Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four, *Math. Z.*, **173** (2) (1980), 171-184. <https://doi.org/10.1007/bf01159957>
- [7] S. A. Papadakis, Kustin-Miller unprojection with complexes, *J. Algebraic Geom.* **13** (2) (2004), 249-268. <https://doi.org/10.1090/s1056-3911-03-00350-3>
- [8] S. A. Papadakis, M. Reid, Kustin-Miller unprojection without complexes, *J. Algebraic Geom.*, **13** (3) (2004), 563-577. <https://doi.org/10.1090/s1056-3911-04-00343-1>
- [9] M. Reid, A. Corti, Weighted Grassmannians, in: *Algebraic geometry*, de Gruyter, Berlin, 2002, pp. 141-163.

Received: December 5, 2019; Published: December 18, 2019