Lie-Rinehart Algebra of a $r$-Jacobi Algebra

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Abstract

We denote by $A$ a commutative and unitary algebra over a commutative field $K$ of characteristic zero and an integer $r$ greater than or equal to 1. In studying the concept of $r$-Jacobi algebra on $A$: we construct an ordinary Lie algebra from which we deduce a structure of Lie-Rinehart algebra. We define the 1-canonical form of a $r$-Jacobi algebra.

Keywords: Module of Kähler Differential, Lie algebra of Order $r$, Lie-Rinehart algebra, Jacobi algebra of Order $r$

1 Introduction

The concept of $n$-Lie algebra over a commutative field $K$, $n$ an integer $\geq 2$, introduced by Fillipov [5], is a generalization of concept of Lie algebra over a commutative field $K$, which corresponds to the case where $n = 2$. A structure of $n$-Lie algebra over a $K$-vector space $W$ is the given of an alternating multilinear mapping of degree $n$

$$\{,\cdots,\} : W^n = W \times \cdots \times W \longrightarrow W, (x_1, x_2, \cdots, x_n) \longmapsto \{x_1, x_2, \cdots, x_n\}$$

verifying the identity

$$\{x_1, x_2, \cdots, x_{n-1}, \{y_1, y_2, \cdots, y_n\}\}$$

$$= \sum_{i=1}^{n} \{y_1, y_2, \cdots, y_{i-1}, \{x_1, x_2, \cdots, x_{n-1}, y_i\}, y_{i+1}, \cdots, y_n\}$$
for all \( x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n \in W \). This identity is called Jacobi identity of \( n \)-Lie algebra \( W \) [5], [1].

A derivation of a \( n \)-Lie algebra \( (W, \{ \cdot, \cdot, \cdot \}) \) is a \( K \)-linear map 
\[
D : W \longrightarrow W
\]
such that 
\[
D \{ x_1, x_2, \ldots, x_n \} = \sum_{i=1}^{n} \{ x_1, \ldots, D(x_i), \ldots, x_n \}
\]
for all \( x_1, x_2, \ldots, x_n \in W \).

The set of all derivations of a \( n \)-Lie algebra \( W \) is a \( K \)-Lie algebra denoted by \( \text{Der}_K(W) \).

If \( (W, \{ \cdot, \cdot, \cdot \}) \) is a \( n \)-Lie algebra, then for all \( x_1, x_2, \ldots, x_{n-1} \in W \), the map 
\[
ad(x_1, x_2, \ldots, x_{n-1}) : W \longrightarrow W, y \longmapsto \{ x_1, x_2, \ldots, x_{n-1}, y \}
\]
is a derivation of \( (W, \{ \cdot, \cdot, \cdot \}) \).

When \( A \) is a commutative algebra, with unit 1\(_A\) over a commutative field \( K \) of characteristic zero, and when \( M \) is a \( A \)-module, a linear map 
\[
\partial : A \longrightarrow M
\]
is a differential operator of order \( \leq 1 \) if, for all \( a \) and \( b \) belonging to \( A \),
\[
\partial(ab) = \partial(a) \cdot b + a \cdot \partial(b) - a \cdot b \cdot \partial(1_A).
\]
When \( \partial(1_A) = 0 \), we have the usual notion of derivation from \( A \) into \( M \).

We denote by \( \text{Diff}_K(A, M) \), the \( A \)-module of differential operator of order \( \leq 1 \) from \( A \) into \( M \) and by \( \text{Diff}_K(A) \) the \( A \)-module of differential operator of order \( \leq 1 \) on \( A \) \( (M = A) \).

Let \( A \) be a commutative algebra with unit 1\(_A\) over a commutative field \( K \) of characteristic zero [7]. A structure of Lie-Rinehart algebra on \( A \) is a pair \( (\mathcal{G}, \rho) \) where \( \mathcal{G} \) is simultaneously a \( K \)-Lie algebra of bracket \([\cdot, \cdot]\) and a \( A \)-module, and,

\[
\rho : \mathcal{G} \longrightarrow \text{Diff}_K(A)
\]
is simultaneously a morphism of \( K \)-Lie algebras and \( A \)-modules satifying 
\[
[x, a \cdot y] = [\rho(x)(a) - a \cdot \rho(x)(1_A)] + a \cdot [x, y]
\]
for all \( x, y \in \mathcal{G}, a \in A \).

We recall the notions of a \( r \)-Jacobi algebra and an canonical form associated with an \( r \)-Jacobi algebra [2].

The aim of this work is to show the existence of a structure of Lie-Rinehart algebra from a \( r \)-Jacobi algebra.
In the following, \( A \) denotes a unitary commutative algebra over a commutative field \( K \) of characteristic zero with unit \( 1_A \) and \( \Omega_K (A) \) the module of Kähler differential of \( A \) and
\[
d_{A/K} : A \rightarrow \Omega_K (A), \ a \mapsto d_{A/K} (a)
\]
the canonical derivation [6], [7].

2 Structure of Jacobi algebra of order \( r \geq 1 \)

2.1 \( A \)-module \( A \times \Omega_K (A) \)

In this part, we recall some properties of \( A \)-module \( A \times \Omega_K (A) \).

**Proposition 2.1** [6] The map
\[
D_{A/K} : A \rightarrow A \times \Omega_K (A), \ a \mapsto (a, d_{A/K} (a))
\]
is a differential operator of order \( \leq 1 \). Moreover the image of \( D_{A/K} \) generates the \( A \)-module \( A \times \Omega_K (A) \).

**Theorem 2.2** [6] The pair \( (A \times \Omega_K (A), D_{A/K}) \) has the following universal property: for all \( A \)-module \( M \) and for all differential operator of order \( \leq 1 \)
\[
\varphi : A \rightarrow M
\]
there exists an unique \( A \)-linear map
\[
\tilde{\varphi} : A \times \Omega_K (A) \rightarrow M
\]
such that
\[
\tilde{\varphi} \circ D_{A/K} = \varphi.
\]
Moreover, the map
\[
\text{Hom}_A (A \times \Omega_K (A) , M) \rightarrow \text{Diff}_K (A , M) , \psi \mapsto \psi \circ D_{A/K}
\]
is an isomorphism of \( A \)-modules.

For all integer \( p \geq 1 \), we say that an alternating \( K \)-multilinear map
\[
\varphi : A^p = A \times A \times \cdots \times A \rightarrow M
\]
is an alternating \( p \)-differential operator if for all \( a_1, a_2, \cdots , a_p \), the map
\[
\varphi^i : A \rightarrow M, a_i \mapsto \varphi (a_1, a_2, \cdots , a_{i-1}, a_i, a_{i+1}, \cdots , a_p)
\]
is a differential operator of order $\leq 1$ for all $i = 1, 2, \ldots, p$.

We denote by $\mathcal{L}^p_{alt} (A \times \Omega_K (A), M)$ the $A$-module of alternating $A$-multilinear maps of degree $p$ from $A \times \Omega_K (A)$ into $M$ and $\text{Diff}^p_{alt} (A, M)$, the $A$-module of alternating $p$-differential operators from $A$ into $M$.

One notes

$$D_{A/K}^{(p)} = D_{A/K} \times D_{A/K} \times \cdots \times D_{A/K} : A^p \longrightarrow [A \times \Omega_K (A)]^p,$$

such that

$$D_{A/K}^{(p)} (a_1, a_2, \cdots, a_p) = (D_{A/K} (a_1), D_{A/K} (a_2), \cdots, D_{A/K} (a_p))$$

for all $a_1, a_2, \cdots, a_p \in A$.

**Corollary 2.3** For all $A$-module $M$ and for all alternating $p$-differential operator

$$\varphi : A^p \longrightarrow M,$$

there exists an unique alternating $A$-multilinear map of degree $p$

$$\tilde{\varphi} : [A \times \Omega_K (A)]^p \longrightarrow M$$

such that

$$\tilde{\varphi} \circ D_{A/K}^{(p)} = \varphi.$$

Thus, the existence of an unique $A$-linear map

$$\varphi : A^p[A \times \Omega_K (A)] \longrightarrow M$$

such that

$$\varphi (D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda \cdots \Lambda D_{A/K} (a_p)) = \varphi (a_1, a_2, \cdots, a_p)$$

for all $a_1, a_2, \cdots, a_p$ elements of $A$ when the map

$$\varphi : A^p \longrightarrow M,$$

is a alternating $p$-differential operator. Moreover, the map

$$\mathcal{L}^p_{alt} (A \times \Omega_K (A), M) \longrightarrow \text{Diff}^p_{alt} (A, M), \ f \longmapsto f \circ D_{A/K}^{(p)}$$

is an isomorphism of $A$-modules $[3]$.

When $\Lambda_A [A \times \Omega_K (A)] = \bigoplus_{n \in \mathbb{N}} \Lambda^n_A [A \times \Omega_K (A)]$ is the $A$-exterior algebra of the $A$-module $A \times \Omega_K (A)$, the differential operator

$$D_{A/K} : A \longrightarrow A \times \Omega_K (A)$$
can be extended into a differential operator again noted
\[ D_{A/K} : \Lambda_A [A \times \Omega_K (A)] \to \Lambda_A [A \times \Omega_K (A)] \]
of degree +1 and of square 0. Thus, the pair \((\Lambda_A [A \times \Omega_K (A)], D_{A/K})\) is a
differential complex [6].

When \(\varphi \in \text{Diff}_K (A), p \in \mathbb{N}\) the map
\[ \sigma_\varphi : [A \times \Omega_K (A)]^p \to \Lambda^{p-1}_A [A \times \Omega_K (A)] \]
defined by
\[ \sigma_\varphi (x_1, \cdots, x_p) = \sum_{i=1}^p (-1)^i \varphi_i (x_i) \cdot x_1 \Lambda \cdots \Lambda \hat{x}_1 \Lambda \cdots \Lambda x_p \]
is alternating \(A\)-multilinear and induced an unique \(A\)-linear map of degree −1
\[ i_\varphi : \Lambda_A [A \times \Omega_K (A)] \to \Lambda_A [A \times \Omega_K (A)] \]
such that
\[ i_\varphi (x_1 \Lambda \cdots \Lambda x_p) = \sigma_\varphi (x_1, \cdots, x_p) . \]
The map
\[ i_\varphi : \Lambda_A [A \times \Omega_K (A)] \to \Lambda_A [A \times \Omega_K (A)] \]
is a dérivation of degree −1.
The map
\[ \mathcal{L}_\varphi = i_\varphi \circ D_{A/K} + D_{A/K} \circ i_\varphi : \Lambda_A [A \times \Omega_K (A)] \to \Lambda_A [A \times \Omega_K (A)] \]
is a differential operator of degree zéro and is the Lie derivate with respect to
a differential operator \(\varphi\).

**Proposition 2.4** For all \(\varphi \in \text{Diff}_K (A), a \in A, \eta \in \Lambda_A [A \times \Omega_K (A)]\) then
\[ \mathcal{L}_{a \cdot \varphi} (\eta) = a \cdot \mathcal{L}_\varphi (\eta) + [D_{A/K} (a) - a \cdot D_{A/K} (1_A)] \Lambda i_\varphi \eta \]
\[ \mathcal{L}_\varphi (a \eta) = [\varphi (a) - a \cdot \varphi (1_A)] \cdot \eta + a \cdot \mathcal{L}_\varphi (\eta) \]
\[ \mathcal{L}_\varphi [D_{A/K} (a)] = D_{A/K} [\varphi (a)] . \]

**Proof** The demonstration presents no difficulty.
2.2 Structure of r-Jacobi algebra

[2] We say that a commutative algebra with unit $1_A$ on a commutative field $K$ of characteristic zero, is a $r$-Jacobi algebra, $r \geq 1$ an integer, if $A$ is provided with a structure of $2r$-Lie algebra over $K$ of bracket $\{,\ldots,\}$, such that for all $(a_1, a_2, \ldots, a_{2r-1}) \in A^{2r-1}$, the map

$$ad(a_1, a_2, \ldots, a_{2r-1}) : A \longrightarrow A, b \longmapsto \{a_1, a_2, \ldots, a_{2r-1}, b\}$$

is a differential operator of order $\leq 1$.

**Proposition 2.5** When $A$ is a $r$-Jacobi algebra, then there exist an unique $A$-linear map

$$\overline{ad} : \Lambda_A^{2r-1} [A \times \Omega_K (A)] \longrightarrow \text{Diff}_K (A)$$

such that, for all $(a_1, a_2, \ldots, a_{2r-1}) \in A$

$$\overline{ad} (D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda \cdots \Lambda D_{A/K} (a_{2r-1})) = ad(a_1, a_2, \ldots, a_{2r-1}).$$

**Proof** The map

$$ad : A^{2r-1} \longrightarrow \text{Diff}_K (A), (a_1, a_2, \ldots, a_{2r-1}) \longmapsto ad(a_1, a_2, \ldots, a_{2r-1})$$

is an alternating $(2r - 1)$ - differential operator. Thus deduced the existence and the uniqueness of the $A$-linear map

$$\overline{ad} : \Lambda_A^{2r-1} [A \times \Omega_K (A)] \longrightarrow \text{Diff}_K (A)$$

such that

$$\overline{ad} (D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda \cdots \Lambda D_{A/K} (a_{2r-1})) = ad(a_1, a_2, \ldots, a_{2r-1}).$$

That ends the proof.

When $A$ is a $r$-Jacobi algebra of bracket $\{,\ldots,\}$, for all $(a_1, a_2, \ldots, a_{2r}) \in A^{2r}$, we verify that

$$\mathcal{L}_{ad} (D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda \cdots \Lambda D_{A/K} (a_{2r-1}) \Lambda \cdots \Lambda D_{A/K} (a_{2r})) D_{A/K} (a_i)$$

$$= (-1)^i D_{A/K} \{a_1, a_2, \ldots, a_{2r} \}.$$

2.3 The associated canonical form of a Jacobi $r$-algebra

In what follows, $A$ is a $r$-Jacobi algebra.
Proposition 2.6 [2] The map
\[ A^{2r} \longrightarrow A, (a_1, a_2, \ldots, a_{2r-1}, a_{2r}) \longmapsto (1 - 2r) \cdot \{a_1, a_2, \ldots, a_{2r-1}, a_{2r}\} \]
is an alternating 2r-differential operator and induces an alternating A-multilinear mapping and only one of degree 2r
\[ \omega_{2r} : [A \times \Omega_K(A)]^{2r} \longrightarrow A \]
such that
\[ \omega_{2r} \left( D_{A/K}(a_1), \ldots, D_{A/K}(a_{2r-1}), D_{A/K}(a_{2r}) \right) = (1 - 2r) \cdot \{a_1, a_2, \ldots, a_{2r-1}, a_{2r}\}. \]

Proof As the map
\[ ad : A^{2r-1} \longrightarrow \text{Diff}_K(A), (a_1, a_2, \ldots, a_{2r-1}) \longmapsto ad(a_1, a_2, \ldots, a_{2r-1}) \]
is an alternating (2r - 1) - differential operator.

Corollary 2.7 The unique A-alternating multinear map of degree 2r
\[ \omega_{2r} : [A \times \Omega_K(A)]^{2r} \longrightarrow A \]
induce an unique A-linear map
\[ \omega : \Lambda_A^{2r} [A \times \Omega_K(A)] \longrightarrow A \]
such that
\[ \omega \left( D_{A/K}(a_1) \Lambda \cdots \Lambda D_{A/K}(a_{2r-1}) \Lambda D_{A/K}(a_{2r}) \right) = (1 - 2r) \cdot \{a_1, a_2, \ldots, a_{2r-1}\} \]
for all \(a_1, a_2, \ldots, a_{2r} \in A\).

We say that \(\omega\) is the canonical form associated with the \(r\)-Jacobi algebra \(A\). [2],

Corollary 2.8 For all \(u \in \Lambda_A^{2r-1} [A \times \Omega_K(A)]\),
\[ \left[ \overline{ad}(u) \right] (a) = (1 - 2r)^{-1} \cdot \omega \left( u \Lambda D_{A/K}(a) \right) \]
for any \(a \in A\).
3 Main Results

3.1 Structure of $2r$-Lie algebra on the $A$-module $A \times \Omega_K(A)$ when $A$ is a Jacobi algebra of order $r$

In what follows, we construct a structure of $K$-Lie algebra of order $2r$ on the $A$-module $A \times \Omega_K(A)$ when $A$ is a Jacobi algebra of order $r$.

**Theorem 3.1.** If $A$ is a $r$-Jacobi algebra of associated canonical form $\omega$ then the map

$$[\cdot, \cdots, \cdot] : [A \times \Omega_K(A)]^{2r} \rightarrow [A \times \Omega_K(A)]$$

defined, for all $x_1, x_2, \cdots, x_{2r} \in A \times \Omega_K(A)$, by

$$[x_1, x_2, \cdots, x_{2r}] = D_{A/K} \left[ \omega(x_1 \Lambda x_2 \Lambda \cdots \Lambda x_{2r}) \right] + \sum_{i=1}^{2r} (-1)^i L_{a \alpha(x_1 \Lambda x_2 \Lambda \cdots \Lambda x_{2r})} x_i$$

defines a structure of $2r$-Lie algebra on the $A$-module $A \times \Omega_K(A)$. Moreover, for all $(a_1, a_2, \cdots, a_{2r}) \in A^{2r}$, $(x_1, x_2, \cdots, x_{2r}) \in [A \times \Omega_K(A)]^{2r}$ and for all $a \in A$ we have:

$$D_{A/K} \{a_1, a_2, \cdots, a_{2r}\} = \left[ D_{A/K}(a_1), D_{A/K}(a_2), \cdots, D_{A/K}(a_{2r}) \right]$$

$$[x_1, x_2, \cdots, x_{i-1}, a \cdot x_i, x_{i+1}, \cdots, x_{2r}]$$

$$= (-1)^i \left( L_{a \alpha(x_1 \Lambda x_2 \Lambda \cdots \Lambda x_{2r})} (a) - a \cdot L_{a \alpha(x_1 \Lambda x_2 \Lambda \cdots \Lambda x_{2r})} (1_A) \right) \cdot x_i$$

$$+ a \cdot [x_1, x_2, \cdots, x_i, \cdots, x_{2r}].$$

**Proof** We verify that the map

$$[\cdot, \cdots, \cdot] : [A \times \Omega_K(A)]^{2r} \rightarrow [A \times \Omega_K(A)], (x_1, x_2, \cdots, x_{2r}) \mapsto [x_1, x_2, \cdots, x_{2r}]$$

is $K$-alternating multilinear.

For all $(a_1, a_2, \cdots, a_{2r}) \in A^{2r}$, as

$$L_{a \alpha(a_1, a_2, \cdots, a_{2r})} D_{A/K}(a_i)$$

$$= (-1)^i D_{A/K} \{a_1, a_2, \cdots, a_{2r}\},$$
then
\[
\begin{align*}
[D_{A/K}(a_1), D_{A/K}(a_2), \ldots, D_{A/K}(a_{2r})] &= D_{A/K} [\omega (D_{A/K}(a_1) \Lambda D_{A/K}(a_2) \Lambda \cdots \Lambda D_{A/K}(a_{2r}))] \\
+ \sum_{i=1}^{2r} (-1)^i \mathcal{L}_{ad}(D_{A/K}(a_1), D_{A/K}(a_2), \ldots, D_{A/K}(a_{2r})) D_{A/K}(a_i) &= D_{A/K} [\omega (D_{A/K}(a_1) \Lambda D_{A/K}(a_2) \Lambda \cdots \Lambda D_{A/K}(a_{2r}))] \\
+ \sum_{i=1}^{2r} (-1)^i (-1)^i D_{A/K} \{a_1, a_2, \ldots, a_{2r}\} &= D_{A/K} [\{1 - 2r\} \cdot \{a_1, a_2, \ldots, a_{2r}\}] \\
+ 2r \cdot D_{A/K} \{a_1, a_2, \ldots, a_{2r}\} &= D_{A/K} \{a_1, a_2, \ldots, a_{2r}\}.
\end{align*}
\]

The second equality requires a simple check.

We verify that the map
\[
[, \ldots, :] : [A \times \Omega_K(A)]^{2r} \longrightarrow [A \times \Omega_K(A)], (x_1, x_2, \ldots, x_{2r}) \longmapsto [x_1, x_2, \ldots, x_{2r}]
\]
defines a structure of 2r-Lie algebra on the A-module A × \Omega_K(A).

### 3.2 Structure of a Lie-Rinehart algebra deduced from a r-Jacobi algebra A

Since the map
\[
\overline{ad} (u) : A \longrightarrow A
\]
is a differential operator for any \( u \in A^{2r-1}_A [A \times \Omega_K(A)] \), so there exists an unique A-linear map
\[
\varphi_u : A \times \Omega_K(A) \longrightarrow A
\]
such that
\[
\varphi_u \circ D_{A/K} = \overline{ad} (u).
\]

It is recalled that [3], when
\[
f : W \longrightarrow W
\]
is an endomorphism of a \( K \)-vector space \( W \) and \( \Lambda_K(W) \) is the \( K \)-exterior algebra of \( W \), then there exists an unique derivation of degree zero
\[
D_f : \Lambda_K(W) \longrightarrow \Lambda_K(W)
\]
such that for all $p \in \mathbb{N}$

$$D_f (w_1 \Lambda w_2 \Lambda \cdots \Lambda w_p) = \sum_{i=1}^{p} w_1 \Lambda w_2 \Lambda \cdots \Lambda w_{i-1} \Lambda f (w_i) \Lambda w_{i+1} \cdots \Lambda w_p$$

for all $w_1, w_2, \cdots, w_p \in W$. When

$$f : W \longrightarrow W$$
$$g : W \longrightarrow W$$

are endomorphisms of the $K$-vector space $W$ then

$$[D_f, D_g] = D_{[f,g]},$$

where the bracket $[,]$ is the usual bracket of endomorphisms.

**Proposition 3.2** For all $u, w \in \Lambda^{2r-1}_A [A \times \Omega_K (A)]$

$$\left[ \overline{ad} (u), \overline{ad} (u') \right] = \overline{ad} \left( D_{\Lambda D_{A/K \circ \varphi_u}} u' \right) = \overline{ad} (-D_{\Lambda D_{A/K \circ \varphi_{u'}}} u').$$

**Proof** We prove for the unde decomposable elements $u, u'$ of $\Lambda^{2r-1}_A [A \times \Omega_K (A)]$. We have

$$u = D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda \cdots \Lambda D_{A/K} (a_{2r-1});$$
$$u' = D_{A/K} (b_1) \Lambda D_{A/K} (b_2) \Lambda \cdots \Lambda D_{A/K} (b_{2r-1}).$$

Let $c$ be in $A$, we have :

$$\left[ \overline{ad} (u), \overline{ad} (u') \right] (c)$$
$$= \{a_1, a_2, \cdots, a_{n-1}, \{b_1, b_2, \cdots, b_{n-1}, c\} \} - \{b_1, b_2, \cdots, b_{n-1}, \{a_1, a_2, \cdots, a_{n-1}, c\} \}$$
$$= \sum_{i=1}^{2r-1} \{b_1, b_2, \cdots, b_{i-1}, \{a_1, a_2, \cdots, a_{2r-1}, b_i\}, b_{i+1}, \cdots, b_{2r-1}, c\}$$
$$+ \{b_1, b_2, \cdots, b_{2r-1}, \{a_1, a_2, \cdots, a_{2r-1}, c\} \}$$
$$- \{b_1, b_2, \cdots, b_{2r-1}, \{a_1, a_2, \cdots, a_{2r-1}, c\} \}$$
$$= \sum_{i=1}^{2r-1} \overline{ad}(D_{A/K} (b_1) \Lambda D_{A/K} (b_2) \Lambda \cdots \Lambda D_{A/K} (b_{i-1}))$$
$$\Lambda D_{A/K} (\overline{ad} (u) (b_i)) \Lambda D_{A/K} (b_{i+1}) \Lambda \cdots \Lambda D_{A/K} (b_{2r-1})] (c)$$
$$= \sum_{i=1}^{2r-1} \overline{ad}(D_{A/K} (b_1) \Lambda D_{A/K} (b_2) \Lambda \cdots \Lambda D_{A/K} (b_{i-1}))$$
$$\Lambda D_{A/K} (\varphi_u \circ D_{A/K} (b_i)) \Lambda D_{A/K} (b_{i+1}) \Lambda \cdots \Lambda D_{A/K} (b_{2r-1}) \overline{ad} (D_{A/K \circ \varphi_u} w) (c).$$
On the other hand

\[
\left[ \text{ad} (u), \overline{\text{ad}} (w) \right] (c) \\
= \{ a_1, a_2, ..., a_{2r-1}, \{ b_1, b_2, ..., b_{2r-1}, c \} \} \\
- \{ b_1, b_2, ..., b_{2r-1}, \{ a_1, a_2, ..., a_{2r-1}, c \} \} \\
= \{ a_1, a_2, ..., a_{2r-1}, \{ b_1, b_2, ..., b_{2r-1}, c \} \} \\
- \{ a_1, a_2, ..., a_{2r-1}, \{ b_1, b_2, ..., b_{2r-1}, c \} \} \\
- \sum_{i=1}^{2r-1} \{ a_1, a_2, ..., a_{i-1}, \{ b_1, b_2, ..., b_{2r-1}, a_i \}, a_{i+1}, ..., a_{2r-1}, c \} \\
= - \sum_{i=1}^{2r-1} \text{ad} \left( D_{A/K} (a_1) \Lambda D_{A/K} (a_2) \Lambda ... \Lambda D_{A/K} (a_{i-1}) \right) \\
\Lambda D_{A/K} \{ b_1, b_2, ..., b_{2r-1}, a_i \} \Lambda D_{A/K} (a_{i+1}) \Lambda ... \Lambda D_{A/K} (a_{2r-1}) \} (c) \\
= \overline{\text{ad}} \left( -D_{A/K} \circ \phi, u \right) (c) .
\]

Since \( c \) is any in \( A \), hence the result.

**Corollary 3.3** For all \( u, u' \in \Lambda_A^{2r-1} [A \times \Omega_K (A)] , \)

\[
\overline{\text{ad}} \left( D_{A/K} \circ \phi, u' + D_{A/K} \circ \phi, u \right) = 0 .
\]

We denote by \( v [A \times \Omega_K (A)] \) the \( A \)-sub module of \( \Lambda_A^{2r-1} [A \times \Omega_K (A)] \) generated by the elements of the form

\[
\sum_{i,j} a_i b_j \cdot \left( D_{D_{A/K} \circ \phi, u} u'_j + D_{D_{A/K} \circ \phi, u} u_i \right)
\]

where

\[
\begin{align*}
  u_i &= D_{A/K} (a'_i) \Lambda D_{A/K} (a'_i) \Lambda ... \Lambda D_{A/K} (a'_{2r-1}) ; \\
  u'_j &= D_{A/K} (b'_j) \Lambda D_{A/K} (b'_j) \Lambda ... \Lambda D_{A/K} (b'_j)
\end{align*}
\]

are the indecomposable elements of \( \Lambda_A^{2r-1} [A \times \Omega_K (A)] , \) with \( i \in I : \) finished, \( j \in J : \) finished and \( a_i, b_j \in A . \) From the above, we deduce that \( \overline{\text{ad}} \) is canceled on \( v [A \times \Omega_K (A)] . \)

So there exists an unique \( A \)-linear map

\[
\overline{\text{ad}} : \Lambda_A^{2r-1} [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]) \rightarrow \text{Diff}_K (A) , u \rightarrow \overline{\text{ad}} (u) .
\]
Theorem 3.4 For \( u = \sum_{i \in I; \text{fini}} a_i \cdot u_i \) and \( v = \sum_{j \in J; \text{fini}} b_j \cdot v_j \) in \( \Lambda^{2r-1}_A [A \times \Omega_K (A)] \), with \( u_i = D_{A/K} (a_i^1) \wedge \cdots \wedge D_{A/K} (a_i^{2r-1}) \), \( v_j = D_{A/K} (b_j^1) \wedge \cdots \wedge D_{A/K} (b_j^{2r-1}) \), the map

\[
[\cdot , \cdot ] : [\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)])]^2 \to \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)])
\]

defined by

\[
\left[ u, v \right] = \left[ \sum_{j \in J; \text{fini}} [\overline{ad} (u) (b_j) - b_j \cdot \overline{ad} (u) (1_A)] \cdot v_j \right] - \sum_{i \in I; \text{fini}} [\overline{ad} (v) (a_i) - a_i \cdot \overline{ad} (v) (1_A)] \cdot u_i + \sum_{i, j} a_i b_j \cdot D_{D_{A/K} \omega e u} v_j,
\]

is a structure of \( K \)-Lie algebra on \( \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]) \). Moreover, the map

\[
\overline{ad} : \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]) \to \text{Diff}_K (A), u \mapsto \overline{ad} (u)
\]

is a representation.

If \( A \) is a \( r \)-Jacobi algebra, we verify that the pair

\[
\left( \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), \overline{ad} \right)
\]

is a Lie-Rinehart algebra.

When \( \omega \) is associated canonical form with an \( r \)-Jacobi algebra \( A \), for all \( u \in v [A \times \Omega_K (A)] \) and for all \( x \in A \times \Omega_K (A) \), we have:

\[
\omega (u \Delta x) = 0
\]

since \( \overline{ad} (u) = 0 \).

Let \( A \) be a \( r \)-Jacobi algebra,

\[
\mathfrak{L}_{alt} (\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), A) = \bigoplus_{p \in \mathbb{N}} \mathfrak{L}^p_{alt} (\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), A),
\]

where \( \mathfrak{L}^p_{alt} (\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), A) \) is the \( A \)-module alternating multilinear \( p \)-forms on the \( A \)-module \( \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]) \).

We denote

\[
d_{\overline{ad}} : \mathfrak{L}_{alt} (\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), A) \to \mathfrak{L}_{alt} (\Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]), A)
\]

the cohomology operator associated with the representation \( \overline{ad} \).
3.3 1-canonical form of a Jacobi r-algebra $A$

We verify that the map

$$\Lambda^{2r-1}_A [A \times \Omega_K (A)] \longrightarrow A, u \longmapsto \omega \left( D_{A/K} (1_A) \Lambda u \right)$$

is canceled on $v [A \times \Omega_K (A)]$. We denote

$$\omega_1 : \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)]) \longrightarrow A$$

the unique $A$-linear map such that

$$\omega_1 \left( \overline{u} \right) = (2r - 1)^{-1} \cdot \omega \left( D_{A/K} (1_A) \Lambda u \right)$$

for all $u \in \Lambda^{2r-1}_A [A \times \Omega_K (A)]$.

We say that $\omega_1$ is the 1-canonical form of the $r$-Jacobi algebra $A$.

**Proposition 3.5** If $A$ is a $r$- Jacobi algebra of associated canonical form $\omega$, the 1-canonical form $\omega_1$ is $d_{\tilde{\text{ad}}}^{-1}$-exact with

$$\omega_1 = d_{\tilde{\text{ad}}}^{-1} 1_A.$$

**Proof** For $\overline{u} \in \Lambda^{2r-1}_A [A \times \Omega_K (A)] / (v [A \times \Omega_K (A)])$, we have:

$$\omega_1 \left( \overline{u} \right) = (2r - 1)^{-1} \cdot \omega \left( D_{A/K} (1_A) \Lambda u \right)$$

$$= (-1)^{2r-1} \cdot (2r - 1)^{-1} \cdot \omega \left( u \Lambda D_{A/K} (1_A) \right)$$

$$\omega_1 \left( \overline{u} \right) = - (2r - 1)^{-1} \cdot \omega \left( u \Lambda D_{A/K} (1_A) \right).$$

As for all $u \in \Lambda^{2r-1}_A [A \times \Omega_K (A)]$ and for all $a \in A$, we have:

$$\left[ \overline{\text{ad}} (u) \right] (a) = (1 - 2r)^{-1} \cdot \omega \left( u \Lambda D_{A/K} (a) \right),$$

we deduce that

$$\omega_1 \left( \overline{u} \right) = - (2r - 1)^{-1} \cdot (1 - 2r) \cdot \left[ \overline{\text{ad}} (u) \right] (1_A)$$

$$= \left[ \overline{\text{ad}} \left( \overline{u} \right) \right] (1_A)$$

$$\omega_1 \left( \overline{u} \right) = \left[ d_{\tilde{\text{ad}}}^{-1} (1_A) \right] (\overline{u}).$$

From where

$$\omega_1 = d_{\tilde{\text{ad}}}^{-1} 1_A.$$

What completes the proof.

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References


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