Permutability of Subgroups of Semidirect Product of $p$-Groups

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Abstract

Two subgroups $H$ and $K$ of a group $G$ are called permutable if $HK = KH$. Furthermore, $H$ is permutable subgroup (quasinormal subgroup) if $HK = KH$ for any subgroup $K$ of $G$. Certainly, every normal subgroup is permutable. Permutability does not imply normality [2]. Permutability of subgroups of $G \times H$ that are direct products of subgroups of the direct factors were investigated in [6]. In this article, the classification of all permutable subgroups of the group $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ will be investigated for any odd prime $p$ and for any integer $k \geq 2$. On the other hand, many properties of the group $G$ were studied and configured.

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1 Introduction

The classification of all 2-generator $p$-group of nilpotency class 2 was given in [1]. The authors classified these groups using the fact that they are all central extensions of a cyclic $p$-group by an abelian $p$-group of rank 2.

For groups $G \cong \mathbb{Z}_n = \langle a \rangle$ and $H \cong \mathbb{Z}_m = \langle b \rangle$, the semidirect product $G \rtimes H$ is completely determined by how $a$ acts on $b$. In this case, the action is
given by conjugation. So, the action of \( a \) on \( b \) is \( b^{-1}ab \). Clearly, \( \langle a \rangle \triangleleft G \rtimes H \).

Thus, this action is a power of \( a \). That is \( b^{-1}ab = a^i \), which indicates that \( [a, b] = a^{i-1} \), where \( a^i \) and \( n \) are coprime, and \( im \equiv 1 \pmod{m} \).

In this article, we give a complete classification of the group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) (for \( p \) odd prime and \( 2 \leq k \in \mathbb{N} \)). Significantly, these groups have always \( p \) subgroups which are permutable and not normal. In addition to the other distinctive properties the group has. All of these properties concerning group \( G \) will be introduced and studied.

In [2], the authors determined all permutable subgroups of groups of order 16, and determined which subgroup is permutable and not normal. In this article, we will extend this result, and describe all subgroups of the group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) (for \( p \) odd prime and \( 2 \leq k \in \mathbb{N} \)), in which permutability does not coincide with normality.

### 2 Notation and general setup

The cyclic group of order \( n \) will be denoted by \( \mathbb{Z}_n \). The identity element of a group \( G \) is \( e \). The order of an element \( x \in G \) is the least positive integer \( k \) such that \( x^k = e \) and denoted by \( o(x) = k \). The cyclic subgroup generated by \( x \) is \( \langle x \rangle = \{x^k \mid k = 1, 2, \cdots, o(x)\} \). The centre of a group \( G \) is denoted by \( Z(G) \), and \( Z(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\} \).

A subgroup \( H \) of a group \( G \) is called normal subgroup of \( G \) \( (H \triangleleft G) \) if and only if \( xH = Hx \) for all \( x \in G \). Clearly, \( Z(G) \triangleleft G \). This implies that, if \( x \in Z(G) \), then \( \langle x \rangle \leq G \).

The next results will be useful in our investigation.

**Theorem 2.1.** [7] (Cauchy’s theorem) Given a finite group \( G \) and a prime number \( p \) dividing the order of \( G \), then there exists an element (and hence a cyclic subgroup) of order \( p \) in \( G \).

**Theorem 2.2.** [7] [First Sylow theorem] For every prime factor \( p \) with multiplicity \( n \) of the order of a finite group \( G \), there exists a Sylow \( p \)-subgroup of \( G \) of order \( p^n \).

**Lemma 2.1.** [4] If \( G \) is a group and \( N \) is a normal subgroup of \( G \), then \( G/N \) is abelian if and only if \( N \) contains the commutator subgroup of \( G \).

### 3 Main results

The main aim of this article is to classifying the subgroups structure of a group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \). These groups have certain properties, in which there were some subgroups which permutable and not normal, and all other subgroups are
normal. On the other hand, this article will discuss some of main topics in group theory for the groups \( \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \), where \( p \) is an odd prime and \( k \) is an integer with \( k \geq 2 \).

**Corollary 3.1.** The derived subgroup of \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) is \( G' = \langle a^{p^{k-1}} \rangle \cong \mathbb{Z}_p \) and \( G' \triangleleft G \).

**Proof.** Let \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \). Then \( G \) has two generators \( a \) and \( b \) of orders \( p^k \) and \( p \), respectively. The commutator of \( a \) and \( b \) is \([a, b] = a^{-1}b^{-1}ab = a^{-p^{k-1}}\), which implies that \( \langle a^{p^{k-1}} \rangle = \langle a^{-p^{k-1}} \rangle \subseteq G' \). Note that \(|G'| = p\) and since \( o(a^{p^{k-1}}) = p\), then \( G' \subseteq \langle a^{p^{k-1}} \rangle \). Thus \( G' = \langle a^{p^{k-1}} \rangle \cong \mathbb{Z}_p \). For \( Z(G) \trianglelefteq G \) and \( G/Z(G) \) is abelian. Using Lemma 2.1, implies that \( Z(G) \) contains \( G' \). Hence, \( G' \triangleleft G \), and the claim.

**Remark 3.1.** The lower central series of a group \( G \) is:

\[
G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \geq \gamma_c(G) \geq \cdots
\]

where \( \gamma_i(G) = [\gamma_{i-1}(G), G] \) for \( i = 1, 2, 3, \cdots \). The group \( G \) is called nilpotent if there exists \( c \) such that \( \gamma_c(G) = \{e\} \), and the smallest such \( c \) is the class of nilpotency.

**Corollary 3.2.** The group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) is nilpotent group of class 2, and solvable.

**Proof.** The group \( G \) is a 2-generator \( p \)-group. Let \( G = \langle a, b \rangle \) where \( o(a) = p^k \) and \( o(b) = p \). Note that, \([x, y] = a^{m p^{k-1}}\), for \( m \in \{1, 2, \cdots, p\} \) and \( a^{p^{k-1}} \in Z(G) \) (Corollary 3.1). Then, \( \gamma_1(G) = [G, G] = G' = \langle a^{p^{k-1}} \rangle \), and so \( \gamma_2(G) = [\gamma_1(G), G] = [G', G] = \{e\} \). Hence, the nilpotency class of \( G \) is \( c = 2 \). Which also implies that, \( G \) is solvable.

The next theorem has been proved in [1] is used here to describe all finite 2-generator \( p \)-group of nilpotency class 2.

**Theorem 3.1.** [1] Let \( G \) be a finite 2-generator \( p \)-group of class 2 (\( p \) an odd prime). Consider the integers \( \alpha, \beta, \gamma \) and \( \sigma \) with \( \alpha \geq \beta \geq \gamma \geq \sigma \geq 0 \). Then \( G \) is isomorphic to exactly one group of the following three types:

1. \( G \cong G_1 = \langle a, b \mid a^{\alpha} = b^{\beta} = [a, b, a] = [a, b, b] = e \rangle \) where \( \alpha, \beta, \gamma \) are integers and \( \gamma \geq 1 \).

2. \( G \cong G_2 = \langle a, b \mid a^{\alpha} = b^{\beta} = [a, b, a] = [a, b, b] = e, a^{\alpha - \gamma} = [a, b] \rangle \) where \( \alpha \geq 2\gamma \) and \( \gamma \geq 1 \).
Let \( G \equiv G_3 = \langle a, b \mid a^{p^\alpha} = b^{p^\beta} = [a, b, a] = [a, b, b] = e, \ a^{p^\alpha+\sigma-\gamma} = [a, b]^{p^\sigma} \rangle \), where \( \alpha + \sigma \geq 2\gamma \) and \( \gamma > \sigma \geq 1 \).

The next lemma gives the orders of the groups listed in the previous theorem.

**Lemma 3.1.** \([1]\) Let \( G \) be a 2-generator \( p \)-group of class 2 (\( p \) an odd prime).

1. If \( G \equiv G_1 \), then \( |G| = p^{\alpha+\beta+\gamma} \).
2. If \( G \equiv G_2 \), then \( |G| = p^{\alpha+\beta} \).
3. If \( G \equiv G_3 \), then \( |G| = p^{\alpha+\beta+\sigma} \).

Using Theorem 3.1 and the above lemma, we give the representation of \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) in the following theorem.

**Theorem 3.2.** For any odd prime \( p \) and any integer \( k \geq 2 \), the group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) is non-abelian 2-generators \( p \)-group, and it can be represented by

\[
G = \langle a, b \mid a^{p^k} = b^p = e, \ a^{p^{k-1}} = [b, a] \rangle
\]  \( (1) \)

**Proof.** From the previous results, the group \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \) is a 2-generator \( p \)-group of class 2. Therefore, \( G \) is isomorphic to one of the groups listed in Theorem 3.1, by setting \( \alpha = k \), \( \beta = 1 \) and \( \gamma = 1 \). Since, \( [a, b] = a^{-1}b^{-1}ab \), so \( [a, b, a] = (ab)^{-1}b(ba)^{-1}aba = (ba^{1-p^k-1})^{-1}ba^{-p^k-1}b^{-1}a^{-1}aba = (ba^{1-p^k-1})^{-1}(ba^{1-p^k-1}) = e \). Similarly for \( [a, b, b] = e \). Also, \( |G| = o(a)o(b) = p^{k+1} = p^{\alpha+\beta} \), and since \( k \geq 2 \) then \( \alpha \geq 2\gamma \). This implies that, Equation 1 satisfies exactly the axioms of the group \( G_2 \) in Theorem 3.1. Moreover, \( |G'| = p^1 = p^\gamma \). \( \Box \)

**Corollary 3.3.** Let \( G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \), for an odd prime \( p \) and an integer \( k \geq 2 \). Then, the centre of \( G \), \( Z(G) = \langle a^p \rangle \cong \mathbb{Z}_{p^{k-1}} \).

**Proof.** Lemma 2.1 shows that \( Z(G) \) is non-trivial, and since \( G \) is non-abelian group, then \( p \leq |Z(G)| < p^{k+1} \), and so \( G/Z(G) \) is abelian group and not cyclic. Thus, \( G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \), implies that \( |Z(G)| = p^{k-1} \).

For any \( x \in G \) and since \( [x, a] \in Z(G) \), one has,

\[
(xa)^p = x^pa^p[x, a^{-1}]^{p(p-1)/2} = x^pa^p ([x, a^{-1}]p)^{(p-1)/2} = x^pa^p
\]

This implies that the \( p^{th} \) map \( \phi : G \rightarrow G \), defined by \( \phi(x) = x^p \), is a homomorphism. Thus

\[
1 = [x, a]^p = (x^{-1}a^{-1}xa)^p = (x^{-1}a^{-1}x)^p a^p,
\]
so that $a^p$ commutes with all $x \in G$. Therefore, $\langle a^p \rangle \subseteq Z(G)$, and since $o(a^p) = p^{k-1} = |Z(G)|$. Then, it follows that $Z(G) = \langle a^p \rangle$. □

**Corollary 3.4.** Let $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$, for an odd prime $p$ and an integer $k \geq 2$. Then $|G : Z(G)| = |G'|^2$.

**Proof.** The proof is an immediate consequence of Corollary 3.3 and Corollary 3.1, as

$$|G : Z(G)| = \frac{|G|}{|Z(G)|} = \frac{p^{k+1}}{p^{k-1}} = p^2 = |G'|^2$$

□

**Corollary 3.5.** For an odd prime $p$ and an integer $k \geq 2$, the group $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ is non-simple group.

**Proof.** It is clear, as $Z(G) \triangleleft G$. □

**Theorem 3.3.** Let $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$, $p$ odd prime and $2 \leq k \in \mathbb{N}$. Then, the subgroups $H_i = \langle a^{ipk-1}b \rangle$, $i = 1, 2, \ldots, p$ are permutable and not normal, and $|H_i| = p$ for all $i$.

**Proof.** To show that $H_i$ is not a normal subgroup of $G$ for all $i = 1, 2, \ldots, p$, let $q = p^{k-1}$, and so $pq = p^k$. Recall that, $Z(G) = \langle a^q \rangle$, thus $a^q \in Z(G)$. For $i = 1, 2, \ldots, p$, we have $H = H_i = \langle a^{ipk-1}b \rangle = \{a^{iq}b, a^{2iq}b^2, \ldots, a^{pq}b^p = e\}$. Let $x = a \not\in H$ and $h_1 = a^{iq}b \in H$. Then $xh_1x^{-1} = ah_1a^{-1} = a^{iq+1}ba^{-1}$. Note that $aba^q = ba$, then

$$ba^{-1} = aba^{q-2} = a^{q+1}ba^{-2} \quad (2)$$

Using Equation 2, implies $ah_1a^{-1} = a^{(i-1)q}(a^{q+1}ba^{-2})a = a^{(i-1)q}ba^{-1}a = a^{(i-1)q}b$. If $ah_1a^{-1} \in H$, then $(i - 1)q = imq$ for $m = 1, 2, \ldots, p$, implies that $q = 0$, which is a contradiction. Therefore, $ah_1a^{-1} \not\in H$. Hence, $H$ is not normal subgroup of $G$.

To show that $H_i$ is permutable, it is enough to show $H_iH_j = H_jH_i$ for all $i, j = 1, 2, \ldots, p$. Let $i, j \in \{1, 2, \ldots, p\}$ and so $H_i = \langle a^{iq}b \rangle = \{(a^{iq}b)^m \mid m = 1, 2, \ldots, p\}$ and $H_j = \langle a^{jq}b \rangle = \{(a^{jq}b)^n \mid n = 1, 2, \ldots, p\}$. Since $a^q \in Z(G)$. Then,

$$(a^{iq}b)^m (a^{jq}b)^n = a^{imq}b^m a^{jq}b^n = a^{jq}a^{imq}b^{m+n} = (a^{jq}b)^n (a^{iq}b)^m, \forall n, m,$$

which implies that $H_iH_j = H_jH_i$. □

**Observation 1.** The group $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$, for odd prime $p$ and $2 \leq k \in \mathbb{N}$ has $k(p+1) + 2$ subgroups, which are: The group itself, the trivial subgroup
and for each $i = 1, 2, \cdots, k$ there are $p + 1$ $p$-subgroups of order $p^i$. The normal subgroups of $G$ are: The group itself, the trivial subgroup, the derived subgroup and for each $i = 2, 3, \cdots, k$ there are $p + 1$ $p$-subgroups of order $p^i$. That is to say, the group $G$ has $k(p + 1) + 2$ subgroups, $p$ subgroups of these subgroups are permutable not normal, and the other $(k-1)(p+1)+3$ subgroups are normal. See Table A for a complete classification of these subgroups.

**Remark 3.2.** For the case $k = 2$, which gives a group $G$ of order $p^3$. All obtained results in this article coincide with results found in [4].

Next, we discuss a certain notion in finite group theory, that is capability. A finite group $G$ is said to be capable if there exists a finite group $H$ such that $H/Z(H) \cong G$. The author in [8] proved that if $G$ is finite and capable, then the index of the centre $Z(G)$ in $G$ is bounded above by some function of the order of the derived subgroup $G'$. This was indicated by the next theorem.

**Theorem 3.4.** (Theorem C,[8]) Let $G$ be a finite and capable group, and suppose that $G'$ is cyclic and that all elements of order 4 in $G'$ are central in $G$. Then $|G : Z(G)| \leq |G'|^2$, and equality holds if $G$ is nilpotent.

Recall that, the epicenter of a group $G$ denoted by $Z^*(G)$ is

$$Z^*(G) = \bigcap \{\phi Z(E) \mid (E, \phi) \text{ is a central extension of } G\}$$

**Theorem 3.5.** [5] The group $G$ is capable if and only if its epicenter $Z^*(G)$ is trivial.

**Theorem 3.6.** [3] Let $G = \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $o(a) = p^\alpha$ and $o(b) = p^\beta$, $\alpha, \beta, \gamma \in \mathbb{N}$ with $\alpha \geq 2$, $\beta \geq \gamma \geq 1$. Then $Z^*(G) = \langle a^{p^\beta}, b^{p^\alpha} \rangle$.

**Theorem 3.7.** The group $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ for an odd prime $p$ and an integer $k \geq 2$ is non-capable.

*Proof.* The proof follows by Theorem 3.5, by showing $Z^*(G) \neq \{e\}$. Since $G$ is a 2-generator $p$-group of class 2, setting $\alpha = k$, $\beta = \gamma = 1$. Then $G$ satisfies the axioms of Theorem 3.6. Therefore, $Z^*(G) = \langle a^{p^1}, b^{p^k} \rangle = \langle a^p \rangle$, implies that $|Z^*(G)| = p^{k-1}$, in addition $Z^*(G) = Z(G)$. Thus, $Z^*(G) \neq \{e\}$. $\square$

The presentation of $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ (for an odd prime $p$ and an integer $k \geq 2$) which produced in this article, shows that capability does not coincide on $|G : Z(G)| \leq |G'|^2$. This leads that the following conjecture (the converse of Theorem 3.4) is false.

**Conjecture 1.** Let $G$ be a finite nilpotent group in which $|G : Z(G)| = |G'|^2$. Then $G$ is capable group.
The group $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$ (for an odd prime $p$ and an integer $k \geq 2$) is a counterexample of this conjecture. Since, $G$ is nilpotent group of class 2 (Corollary 3.2), the index of the centre in $G$ is $|G'|^2$ (Corollary 3.4) and $G$ itself is non-capable group (Remark 3.7).

4 Conclusions

The classification of the semidirect product of two finite groups was not clearly achieved, in which a certain group action should be identified. The classification given in this article, makes all calculations on these groups very simple and explicit. Also, the algorithm shown in B, presents unrivaled method to define such groups using Groups, Algorithms and programming (GAP). In addition, it can be used to define groups of extremely large size (corresponding to the used device memory size).

References


A  Subgroups classification of \( \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p \)

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<td>( \mathbb{Z}_p \times \mathbb{Z}_p )</td>
<td>( p^2 )</td>
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B GAP Algorithm

Algorithm B.1 Representing $G = \mathbb{Z}_{p^k} \rtimes \mathbb{Z}_p$, corresponding to Theorem 3.2

```gap
gap> p:= ;;k:= ;;# Setting the group factors (p and k).
> parms:= function(k)
> return (k>=2);
> end;
> T:= function(k,p)
> local F, # Free group
> a,b, # Free Generators
> R; # Relations
> if not parms(k) then
> Print("Parameter error 
")
> return fail;
> fi;
> F:=FreeGroup("a","b"); a:=F.1; b:=F.2;
> R:=\([a^\}p^\}k),b^p,Comm(b,a)*a^\}(-p^\}(k-1))\];
> return Image(EpimorphismQuotientSystem(PQuotient(F/R,p)));
> end;
> G:=T(k,p);;g:=MinimalGeneratingSet(G);;a:=g[1];;b:=g[2];;
> StructureDescription(G);
```

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