GGP-Injective Rings

Zhanmin Zhu

Department of Mathematics
Jiaxing University
Jiaxing, Zhejiang Province, 314001, China

Qiaoling Guo

Department of Mathematics
Jiaxing University
Jiaxing, Zhejiang Province, 314001, China

Abstract

A ring $R$ is called right generalized GP-injective (or right GGP-injective for short) if, for any $0 \neq a \in R$, there exists an element $b \in R$ such that $ab \neq 0$ and any right $R$-homomorphism from $abR$ to $R$ extends to an endomorphism of $R$. In this paper, several properties of this class of rings are given, some interesting results are obtained. Using the concept of right GGP-injective rings, we present some new characterizations of QF-rings.

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1 Introduction

Throughout this paper, $R$ is an associative ring with identity, and all modules are unitary. As usual, $J = J(R)$, $Z_l (Z_r)$ and $S_l (S_r)$ denote respectively the Jacobson radical, the left (right) singular ideal and the left (right) socle of $R$. The left (respectively, right) annihilators of a subset $X$ of $R$ is denoted by $l(X)$ (respectively, $r(X)$).
Recall that a ring $R$ is right $P$-injective [7] if every $R$-homomorphism from a principal right ideal of $R$ to $R$ extends to an endomorphism of $R$. A ring $R$ is right generalized principally injective (briefly right GP-injective) [6] if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and any right $R$-homomorphism from $a^n R$ to $R$ extends to an endomorphism of $R$. It is easy to see that a ring $R$ is right GP-injective if and only if for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and $\text{lr}(a^n) = Ra^n$. GP-injective rings are studied in papers [6, 2, 3, 5, 11]. In [11], GP-injective rings are called $YJ$-injective rings. It is easy to see that right P-injective rings are right GP-injective, but right GP-injective rings need not be right P-injective by [5, Example 1]. In this paper, we shall generalize the concept of right GP-injective rings to right GGP-injective rings, some properties of these rings will be given, conditions under which right GGP-injective rings are QF-rings will be given.

## 2 GGP-injective rings

**Definition 2.1** A ring $R$ is called right generalized GP-injective (or GGP-injective for short) if, for any $0 \neq a \in R$, there exists an element $b \in R$ such that $ab \neq 0$ and any right $R$-homomorphism from $ab R$ to $R$ extends to an endomorphism of $R$.

**Theorem 2.2** For a ring $R$, the following conditions are equivalent:

1. $R$ is right GGP-injective;
2. for any $0 \neq a \in R$, there exists $b \in R$ such that $ab \neq 0$ and $\text{lr}(ab) = Rab$.

**Proof.** $(1) \Rightarrow (2)$. For any $0 \neq a \in R$, since $R$ is right GGP-injective, there exists an element $b \in R$, such that $ab \neq 0$ and any $R$-homomorphism from $ab R$ to $R$ extends to $R$. Now let $x \in \text{lr}(ab)$, we define $f : ab R \to R$ by $abr \mapsto xr$, then $f$ is a well defined right $R$-homomorphism and hence $f$ extends to an endomorphism $g$ of $R$. Take $c = g(1)$, then $x = cab \in Rab$. This shows that $\text{lr}(ab) = Rab$.

$(2) \Rightarrow (1)$. For any $0 \neq a \in R$, by (2), there exists $b \in R$ such that $ab \neq 0$ and $\text{lr}(ab) = Rab$. Suppose $f \in \text{Hom}_R(ab R, R)$, then $f(ab) \in \text{lr}(ab)$, and so there exists $c \in R$ such that $f(ab) = cab$. Let $g : R \to R; x \mapsto cx$. Then $g$ extends $f$.

It is easy to see that right GP-injective rings are right GGP-injective. Our next example shows that a right GGP-injective ring need not be right GP-injective.
**Example 2.3** Let \( M = \bigoplus_{i=1}^\infty \mathbb{Z}_{p_i} \), where \( p_i \) is the \( i \)th prime number, and let

\[
R = \left\{ \begin{bmatrix} n & x \\ 0 & n \end{bmatrix} \mid n \in \mathbb{Z}, x \in M \right\}.
\]

Then \( R \) is not right GP-injective, but \( R \) is right GGP-injective.

**Proof.** Suppose that \( R \) is right GP-injective. Take \( a = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \). Then by [6, Lemma 1], there exists a positive integer \( k \), such that \( \operatorname{lr}(a^{2k}) = R(a^{2k}) \).

Since \( \operatorname{lr}(a^{2k-1}) = \operatorname{lr}(a^{2k}) \), we have \( a^{2k-1} \in \operatorname{lr}(a^{2k}) = R(a^{2k}) = Ra^{2k} \). But \( a^{2k-1} \notin Ra^{2k} \), a contradiction. So, \( R \) is not right GP-injective. On the other hand, for any \( 0 \neq a = \begin{bmatrix} n & x \\ 0 & n \end{bmatrix} \in R \). If \( n \neq 0 \), then there exists \( y \in M \) such that \( ny \neq 0 \). Now let \( b = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \), then \( 0 \neq ab = \begin{bmatrix} 0 & ny \\ 0 & 0 \end{bmatrix} \in J(R) \).

If \( n = 0 \), then \( a \in J(R) \). Thus, by the proof of [10, Example 3.1], for any \( 0 \neq a \in R \), there is a \( b \in R \), such that \( ab \neq 0 \) and \( \operatorname{lr}(ab) = R(ab) \), and so \( R \) is right GGP-injective by Theorem 2.2.

Recall that a ring \( R \) is called right mininjective \([8]\) if every \( R \)-homomorphism from a minimal right ideal of \( R \) into \( R \) extends to \( R \).

**Theorem 2.4** Let \( R \) be right GGP-injective. Then

1. \( R \) is right mininjective.
2. \( J(R) \subseteq \mathbb{Z}_r \).

**Proof.** (1). It is obvious.

(2). Let \( a \in J(R) \), then we will show that \( a \in \mathbb{Z}_r \). If not, then there exists \( 0 \neq b \in R \) such that \( \operatorname{r}(a) \cap bR = 0 \). Clearly \( ab \neq 0 \). Since \( R \) is right GGP-injective, there exists \( c \in R \) such that \( abc \neq 0 \) and \( \operatorname{lr}(abc) = Rabc \), so \( u \in Rabc \) for every \( u \in R \) with \( \operatorname{r}(abc) = \operatorname{r}(u) \). Note that \( \operatorname{r}(abc) = \operatorname{r}(bc) \), we have \( bc = dabc \) for some \( d \in R \). Thus \( (1 - da)bc = 0 \). Since \( a \in J(R) \), \( 1 - da \) is invertible, and so \( bc = 0 \), a contradiction.

Observing that the ring \( \mathbb{Z} \) of integers is right mininjective but not right GGP-injective, so right mininjective rings need not be right GGP-injective.

**Corollary 2.5** Let \( R \) be a right GGP-injective ring. Suppose that, for any sequence \( \{a_1, a_2, \cdots \} \subseteq R \), the chain \( \operatorname{r}(a_1) \subseteq \operatorname{r}(a_2 a_1) \subseteq \cdots \) terminates. Then \( J(R) = \mathbb{Z}_r \).
Proof. Since $R$ is right GGP-injective, by Theorem 2.4, $J(R) \subseteq Z_r$. Since the chain $r(a_1) \subseteq r(a_2a_1) \subseteq \ldots$ terminates for any sequence $\{a_1, a_2, \ldots\} \subseteq R$, by [12, Lemma 3.10], $Z_r$ is right T-nilpotent, and so $Z_r$ is nil. It follows that $Z_r \subseteq J(R)$, and hence $J(R) = Z_r$. □

Lemma 2.6 Let $R$ be a right GGP-injective ring. If $a \notin Z_r$, then the inclusion $r(a) \subset r(a - ca)$ is strict for some $c \in R$.

Proof. Since $r(a)$ is not an essential right ideal, there exists a nonzero right ideal $I$ of $R$ such that $r(a) \oplus I$ is essential in $R$. Take $0 \neq b \in I$, then $ab \neq 0$. By the right GGP-injectivity of $R$, there is an element $c_1$ in $R$ such that $abc_1 \neq 0$ and any right $R$-homomorphism from $abc_1$ to $R$ extends to an endomorphism of $R$. Observing that $bR \cap r(a) = 0$, we have a right $R$-homomorphism $g : abc_1 R \to R$ given by $g(abc_1 r) = bc_1 r$. Thus $bc_1 = cabc_1$ for some $c \in R$, and so $bc_1 \in r(1-ca)$, whence $bc_1 \in r(a-ca)$. Noting that $bc_1 \notin r(a)$, we have that the inclusion $r(a) \subset r(a - ca)$ is strict.

Theorem 2.7 If $R$ is right GGP-injective, then the following statements are equivalent.

1. $R$ is right perfect;
2. the ascending chain $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots$ of $R$.

Proof. By Corollary 2.5, Lemma 2.6 and [12, Lemma 3.10], we can complete the proof in a similar way to that of [12, Theorem 3.11].

Recall that a ring $R$ is called right Kasch [9] if every simple right $R$-module embeds in $R$, equivalently if $l(T) \neq 0$ for every maximal right ideal $T$ of $R$. Left Kasch rings can be defined similarly; a ring $R$ is called right minfull [8] if it is semiperfect, right mininjective, and $Soc(eR) \neq 0$ for each local idempotent $e \in R$.

Corollary 2.8 If $R$ is a right GGP-injective ring with ACC on right annihilators, then

1. $R$ is semiprimary.
2. $R$ is left and right Kasch.

Proof. (1) It is well known that $Z_r$ is nilpotent for any ring $R$ with ACC on right annihilators. By Theorem 2.7 and Theorem 2.4(2), $R$ is semiprimary.

(2) By (1), $R$ is semiprimary, so $R$ is semiperfect with essential right socle. Since $R$ is right mininjective by Theorem 2.4(1), it is right minfull, and thus (2) follows from [9, Theorem 3.12(1)]. □

Corollary 2.9 Let $R$ be a right GGP-injective ring. Then $R$ is right noetherian if and only if $R$ is right artinian.
Proof. Let $R$ be a right noetherian right GGP-injective ring. Then by Corollary 2.8, $R$ is a right noetherian semiprimary ring, and so $R$ is right artinian.

Corollary 2.10 Let $R$ be a right GGP-injective ring with ACC on right annihilators and $S_t \subseteq S_r$. Then $R$ is left artinian if and only if $S_t$ is a finitely generated left ideal.

Proof. By Corollary 2.8, $R$ is semiprimary. By Theorem 2.4 and [9, Theorem 1.14(4)], $S_r \subseteq S_t$, and so $S_t = S_r$ by the hypothesis. Now the result follows from [11, Lemma 6].

Recall that a ring $R$ is called a left minannihilator ring [8], if every minimal left ideal $K$ is a left annihilator, equivalently, if $\text{lr}(K) = K$. It is easy to see that a right GP-injective ring is a left minannihilator ring.

Corollary 2.11 Let $R$ be a right GGP-injective ring with ACC on right annihilators. If $R$ is a left minannihilator ring, then:

1. $R$ is left artinian;
2. $R$ is right artinian if and only if $S_r$ is finitely generated as a right ideal of $R$.

Proof. (1). By Corollary 2.8, $R$ is semiprimary. By [8, Corollary 3.13], $S_t = S_r$ is finite dimensional as a left $R$-module. Now, by [1, Lemma 6], $R$ is left artinian.

(2). The assertion follows from (1) and [1, Lemma 6].

Corollary 2.12 [2, Theorem 3.7]. Let $R$ be a right GP-injective ring with ACC on right annihilators. Then:

1. $R$ is left artinian;
2. $R$ is right artinian if and only if $S_r$ is finitely generated as a right ideal of $R$.

Recall that a ring $R$ is QF if it is right or left self-injective and right or left artinian; a ring $R$ is semiregular if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$; a ring $R$ is right $CF$ if every cyclic right $R$-module embeds in a free module; a ring $R$ is called right (left) min-CS if every minimal right (left) ideal of $R$ is essential in a direct summand of $R_R$ ($_R R$); a ring $R$ is called right min-PF ring if $R$ is a semiperfect, right mininjective ring in which $S_r \subseteq^\text{ess} R_R$ and $\text{lr}(K) = K$ for every simple left ideal $K \subseteq Re$, where $e^2 = e$ is local. These concepts can be found in [9]. It is well known that right CF-rings are left P-injective [9, Lemma 7.2 (1)]; and a ring $R$ is QF if and only if $R$ is right artinian and right and left mininjective [8, Corollary 4.8]. According to [13], a ring $R$ is right 2-simple injective if every $R$-homomorphism from a 2-generated right ideal of $R$ to $R$ with simple image extends to an endomorphism of $R$. 
**Theorem 2.13** Let $R$ be a right GGP-injective ring. Then the following are equivalent:

1. $R$ is a QF-ring;
2. $R$ is a left mininjective ring with ACC on right annihilators;
3. $R$ is right min-CS, left minanilator ring with ACC on right annihilators;
4. $R$ is a two-sided min-CS ring with ACC on right annihilators;
5. $R$ is a right 2-simple injective ring with ACC on right annihilators;
6. $R$ is right CF-ring and the ascending chain $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \cdots$ terminates for every sequence $\{a_1, a_2, \cdots\} \subseteq R$;
7. $R$ is a semiregular right CF-ring.

**Proof.** It is obvious that (1) implies (2) through (5).

(2) $\Rightarrow$ (1). By Corollary 2.8(1), $R$ is semiprimary, so it is a semilocal, left and right mininjective ring with ACC on right annihilators in which $S_r \subseteq^{\text{ess}} R_R$. By [9, Theorem 3.31], $R$ is a QF-ring.

(3) $\Rightarrow$ (1). Since $R$ is a semiprimary left minanilator ring, it is a right min-PF ring by [10, Corollary 3.25], and so $S_r = S_l$ by [9, Theorem 3.24]. Then $R$ is a right minanilator ring by [9, Lemma 4.4] because it is right min-CS. Hence $R$ is left min-PF, again by [9, Corollary 3.25]. Now [9, Theorem 3.38] shows that $R$ is QF.

(4) $\Rightarrow$ (1). By Corollary 2.8(2), $R$ is left and right Kasch, and hence $S_r = S_l$ by [10, Lemma 4.5(2)] because $R$ is left and right min-CS. Thus $R$ is a left and right min-PF ring by [9, Corollary 4.6], so $R$ is QF, again by [9, Theorem 3.38].

(5) $\Rightarrow$ (1). Suppose (5) holds. Then since $R$ is a right GGP-injective ring with ACC on right annihilators, by Corollary 2,8(1), $R$ is semiprimary. Noting that $R$ is right 2-simple injective, by [13, Theorem 17(17)], $R$ is a QF-ring.

(1) $\Rightarrow$ (6). Assume (1). Then since every injective module over a QF-ring is projective, so every right $R$-module embeds in a free module, and hence $R$ is a right CF-ring. Note that a QF-ring is right noetherian, the last assertion of (6) is clear.

(6) $\Rightarrow$ (7). By Theorem 2.7, $R$ is right perfect, so that it is semiregular.

(7) $\Rightarrow$ (1). Note that the right GGP-injectivity implies that $J(R) \subseteq Z_r$ by Theorem 2.4(2). Thus, $R$ is right artinian by [4, Corollary 2.9]. Since $R$ is right and left mininjective, by [8, Corollary 4.8], $R$ is QF.

Observing that a right GP-injective ring is a left minanilator ring, by Theorem 2.13, we have the following corollary.

**Corollary 2.14** The following are equivalent for a ring $R$:

1. $R$ is a QF-ring;
2. $R$ is right min-CS, right GP-injective ring with ACC on right annihilators.
Theorem 2.15 The following are equivalent for a ring $R$:

1. $R$ is a QF-ring;
2. $R$ is a right noetherian, right GGP-injective right min annihilator ring.

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$. Since $R$ is right noetherian, right GGP-injective, by Corollary 2.8, $R$ is semiprimary, and so it is right artinian. Let $K = Ra$ be a minimal left ideal. Since $R$ is right artinian, $aR$ contains a minimal right ideal $I = bR$. Clearly, $l(a) \subseteq l(b)$. Since $l(a)$ is a maximal left ideal, $l(a) = l(b)$. Now $aR \subseteq rl(a) = rl(b) = rl(bR) = bR$ because $R$ is a right min annihilator ring, so $aR = bR$, which shows that $rl(a) = aR$. By [8, Lemma 1.1], $R$ is left mininjective. Thus, $R$ is a two-sided mininjective right artinian ring, and therefore it a Quasi-Frobenius ring by [8, Corollary 4.8].

Corollary 2.16 The following are equivalent for a ring $R$:

1. $R$ is a QF-ring;
2. $R$ is a right noetherian, left GP-injective right GGP-injective ring.

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References


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