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Birkhoff's HSP Theorem for Varieties of (n, m)-Semigroups

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Abstract

The work concerns varieties of (n, m)-semigroups in particular vector varieties of (n, m)-semigroups. We give a direct proof of Birkhoff's HSP theorem for varieties of (n, m)-semigroups and we find a corresponding HSP Theorem for vector varieties of (n, m)-semigroups (when $m \geq 2$).

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1 Introduction

This work is a continuation of our results presented in [6] and [7]. An overview on vector valued structures and the development of the combinatorial theory within is given in [5], [6], [7]. Here we give definitions and basic results necessary for the rest of the text. For entering into details, refer to [5], [6] and [7].

Let $Q \neq \emptyset$, $n, m \in \mathbb{N}$ and $n - m = k \geq 1$. We denote by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{N}_t$ and $\mathbb{N}_{t,0}$ the sets $\{1, 2, \ldots, t\}$ and $\{0, 1, 2, \ldots, t\}$, $t \in \mathbb{N}$. If $\mathbf{x} = (a_1, a_2, \ldots, a_t) \in Q^t$, we write $\mathbf{x} = a_1^t$, and we identify $\mathbf{x}$ with the word $a_1 a_2 \ldots a_t$. For such an $\mathbf{x}$ we say that its length $|\mathbf{x}|$ is t. The notation a_r^t where $r > t$ is identified with the empty word, denoted by 1. Let Q^+ be the union of all the cartesian products Q^t, $t \in \mathbb{N}$, which is the free semigroup generated by Q. We denote the set $\{\mathbf{x}|\mathbf{x} \in Q^+, |\mathbf{x}| = m + sk, s \in \mathbb{N}\}$ by $Q^{m,k}$.

A map $f : Q^n \to Q^m$ is called an (n, m)-operation on Q, and (Q, f) is called an (n, m)-groupoid. An (n, m)-groupoid (Q, f) is called an (n, m)-semigroup, if the (n, m)-operation is associative, i.e. if $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$, for any $\mathbf{xyz} = \mathbf{uvw} \in Q^{n+k}$, $\mathbf{y}, \mathbf{v} \in Q^n$ and $\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{w} \in Q^* = Q^+ \cup \{1\}$. Let $m \geq 2$.

An (n, m)-groupoid (Q, f) can be considered as an algebra with m n-ary operations $f_1, f_2, \ldots, f_m : Q^n \to Q$, such that $f(\mathbf{x}) = f_1(\mathbf{x})f_2(\mathbf{x}) \ldots f_m(\mathbf{x})$ and $f_1, f_2, \ldots, f_m$ are called component operations for f. In general, for an associative (n, m)-operation f, the corresponding component operations do not have to be associative. A map $g : Q^{m,k} \to Q^m$ is called a poly-(n, m)-operation and the structure $\boldsymbol{Q} = (Q, g)$ is called a poly-(n, m)-groupoid. A poly-(n, m)-groupoid $\boldsymbol{Q} = (Q, g)$ is called a poly-(n, m)-semigroup if $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{xyz})$, for each $\mathbf{xyz} \in Q^{m,k}$, $\mathbf{y} \in Q^{m,k}$ and $\mathbf{x}, \mathbf{z} \in Q^*, \mathbf{xz} \neq 1$. There is no essential difference between the notions of (n, m)-semigroups and poly-(n, m)-semigroups, due to the General Associative Law which holds in (n, m)-semigroups. (This is explained in [3], [4]). An (n, m)-semigroup $\boldsymbol{G}$ is trivial if $|G| = 1$. A class of (n, m)-semigroups is non-trivial iff it contains non-trivial (n, m)-semigroup(s). Bellow we recall some preliminary results from universal algebra, related to a class of (n, m)-semigroups.

Theorem 1.1 ([1])**.** If $\boldsymbol{G}$ and $\boldsymbol{G'}$ are (n, m)-semigroups and $\varphi : G \to G'$ is an epimorphism then $\boldsymbol{G}/\ker\varphi \cong \boldsymbol{G'}$.

Theorem 1.2 ([1])**.** Let $\boldsymbol{G}$ and $\boldsymbol{G'}$ are (n, m)-semigroups and $\varphi : G \to G'$ is a homomorphism. If α is a congruence on $\boldsymbol{G}$ such that $\alpha \subseteq ker\varphi$, then the map $\psi : G/\alpha \to G'$ defined by $\psi(x^\alpha) = \varphi(x)$ is a homomorphism and $\psi\, nat\alpha = \varphi$. If φ is an epimorphism, then ψ is epimorphism.

Definition 1.3 ([2])**.** Let $\mathcal{K}$ be a nonempty class of (n, m)-semigroups and let $B \neq \emptyset$. We say that $\boldsymbol{F} \in \mathcal{K}$ is a free (n, m)-semigroup in $\mathcal{K}$ freely generated by B ($\boldsymbol{F}$ is a free object in $\mathcal{K}$ with basis B) if the following conditions stand:
1) there exists a map $i : B \to F$
2) $\boldsymbol{F} =< i(B) >$ i.e. $\boldsymbol{F}$ is generated by the set $i(B)$
3) for every $\boldsymbol{G} \in \mathcal{K}$ and $\xi : B \to G$, there exists a homomorphism $\overline{\xi} : \boldsymbol{F} \to \boldsymbol{G}$ such that $\overline{\xi}i = \xi$.

Proposition 1.4 ([2])**.** If $\boldsymbol{F}$ is a free object in an non-trivial class of (n, m)-semigroups $\mathcal{K}$, then the map i in Definition 1.3 is embedding.

Proposition 1.5 ([2])**.** The homomorphism $\overline{\xi}$ in Definition 1.3 is unique.

Theorem 1.6 ([2])**.** Let $B \neq \emptyset$. If $\mathcal{K}$ is a nonempty class of (n, m)-semigroups closed under the operations of taking (n, m)-subsemigroups and direct products, then $\mathcal{K}$ contains a free (n, m)-semigroup with basis B.

The construction of a canonical form of a free poly-(n, m)-groupoid $\boldsymbol{F(B)} = (F(B), f)$ with a basis $B \neq \emptyset$ is given in [3], [4]. The construction of a canonical form of a free (n, m)-semigroup $\psi_0(\boldsymbol{F(B)}) = (\psi_0(F(B)), [\ ])$ generated by a nonempty set B is given in [4]. These constructions are presented in details in [5], [6], [7] and shall be consulted for deeper reading.

Given an (n, m)-semigroup (G, g) and a map $\xi : B \to G$, the unique homomorphism $\overline{\xi} : (\psi_0(F(B), [\ ]) \to (G, g)$ is defined (in [6]) by:

$\overline{\xi}(u) = \xi(u)$, for $u \in B$

$\overline{\xi}(i, u_1^{m+sk}) = g_i(\overline{\xi}(u_1) \dots \overline{\xi}(u_{m+sk}))$ for $(i, u_1^{m+sk}) \in \psi_0(F(B)) \backslash B$, assuming that $\overline{\xi}(u_1), \dots, \overline{\xi}(u_{m+sk})$ are already defined.

Proposition 1.7. ([3]) Every (n, m)-semigroup $\boldsymbol{H}$ is a homomorphic image of a free (n, m)-semigroup $\psi_0(\boldsymbol{F(X)})$, for an appropriately selected set X.

2 Varieties of (n, m)-semigroups

Varieties of (n, m)-semigroups were defined in [3]. Significant results were given in [6] and [7]. Without entering into details, we recall the ones we need.

If $\boldsymbol{F(}\mathbb{N}\boldsymbol{)}$ is a free poly-(n, m)-groupoid with a basis $\mathbb{N}$ and $\boldsymbol{Q} = (Q, h)$ is a poly-(n, m)-groupoid, for each $\tau \in F(\mathbb{N})$ there exists a smallest $t \in \mathbb{N}$ such that $\tau \in F(\mathbb{N}_t)$ and τ defines a t-ary operation on Q as follows:

i) If $\tau = j \in \mathbb{N}_t$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = a_j$

ii) If $\tau = (i, \tau_1^{m+sk})$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = h_i(\tau_1(\mathbf{a}) \dots \tau_{m+sk}(\mathbf{a}))$, assuming that $\tau_\nu(\mathbf{a})$ are already defined. Let $\tau, \omega \in F(\mathbb{N})$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$. A poly-(n, m)-groupoid $\boldsymbol{Q}$ satisfies the (n, m)-identity (τ, ω) if $\tau(\mathbf{a}) = \omega(\mathbf{a})$ for an arbitrary $\mathbf{a} = a_1^t \in Q^t$. We denote this as $\boldsymbol{Q} \models (\tau, \omega)$. A class of (n, m)-semigroups $\mathcal{V}$ is a variety iff there exists a set of (n, m)-identities Θ such that $\boldsymbol{G} \models \Theta$ for every every $\boldsymbol{G} \in \mathcal{V}$ i.e. $\boldsymbol{G} \models (\tau, \omega)$ for every $(\tau, \omega) \in \Theta$ and every $\boldsymbol{G} \in \mathcal{V}$. We use the notation $\mathcal{V} = Var\Theta$. For an (n, m)-identity (τ, ω) we can always assume that $\tau, \omega \in \psi_0(F(\mathbb{N}_t))$ for some $t \in \mathbb{N}$ (see [3], [6]). The following result is proved in [6].

If $\boldsymbol{H}$ is an (n, m)-semigroup, $\tau \in \psi_0(F(\mathbb{N}_t))$, $a_1^t \in H^t$, $\xi : \mathbb{N} \to H$ is a map such that $a_j = \xi(j)$, $j \in \mathbb{N}_t$, and $\overline{\xi} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{H}$ is the homomorphic extension of ξ, then the t-ary operation on H defined by τ satisfies

$$\tau(\mathbf{a}) = \overline{\xi}(\tau) \tag{1}$$

Lemma 2.1 ([6])**.** If $\mathbf{a} = a_1^t \in H^t$ and $\overline{\xi} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{H}$ is the (unique) homomorphic extension of a map $\xi : \mathbb{N} \to H$ such that $a_j = \xi(j)$, $j \in \mathbb{N}_t$, then $\tau(\mathbf{a}) = \omega(\mathbf{a})$ iff $\overline{\xi}(\tau) = \overline{\xi}(\omega)$.

Proposition 2.2 ([6])**.** If Θ is a set of (n, m)-identities, $\boldsymbol{H} \models \Theta$ and $\xi : \mathbb{N} \to H$ is a map, then $\Theta \subseteq \ker \overline{\xi}$, where $\overline{\xi} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{H}$ is the homomorphic extension of ξ.

Lemma 2.3 ([6]). Let $\boldsymbol{H}$ and $\boldsymbol{H}'$ are (n,m)-semigroups and let $\varphi : \boldsymbol{H} \to \boldsymbol{H}'$ is a homomorphism. Then for every $\tau \in \psi_0(F(\mathbb{N}_t))$ and $a_1^t \in H^t$

$$\varphi(\tau(a_1 \dots a_t)) = \tau(\varphi(a_1) \dots \varphi(a_t)).$$

Lemma 2.4 ([1]). Let $\boldsymbol{H}$ and $\boldsymbol{H'}$ are (n,m)–semigroups and $\boldsymbol{G} = \boldsymbol{H} \times \boldsymbol{H'}$. Then, for every $\tau \in \psi_0(F(\mathbb{N}_t))$ and a sequence $(a_1, a_1') \dots (a_t, a_t') \in G^t$

$$\tau((a_1, a_1') \dots (a_t, a_t')) = (\tau(a_1 \dots a_t), \tau(a_1' \dots a_t')).$$

In [6] we gave a description of the complete system of (n,m)-identities for a variety $Var\Theta$ which we denote by $\widehat{\Theta}$, and we have showed the following

Theorem 2.5 ([6]). $\widehat{\Theta}$ is a complete system of (n,m)-identities for $Var\Theta$, consisting of all (n,m)-identities satisfied by all (n,m)-semigroups in $Var\Theta$.

Proposition 2.6 ([6]). $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta}$ is a free object in $Var\Theta$ with basis $\mathbb{N}$.

Corollary 2.7 ([6]). Let $Var\Theta$ be a variety of (n,m)–semigroups. If $\boldsymbol{H} \in Var\Theta$ and $\xi : \mathbb{N} \to H$ is a map, then $\widehat{\Theta} \subseteq \ker \overline{\xi}$, where $\overline{\xi} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{H}$ is the homomorphic extension of ξ.

Recall a definition of vector (n,m)-identities ([3], [7]):
Let $p = m + sk$, $q = m + rk$, where $s, r \geq 0$ and let $(i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+$. An (n,m)-semigroup $\boldsymbol{G} = (G; g)$ satisfies the vector (n,m)-identity (i_1^p, j_1^q) (i.e. $\boldsymbol{G} \models (i_1^p, j_1^q)$), if $g(a_{i_1} \dots a_{i_p}) = g(a_{j_1} \dots a_{j_q})$ for $a_1^t \in G^t$ where $t = \max\limits_{\mu,\nu}\{i_\mu, j_\nu\}$. Every vector (n,m)-identity (i_1^p, j_1^q) induces a set of (n,m)-identities $(i_1^p, j_1^q)_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))$ defined by $(i_1^p, j_1^q)_\# = \{((i, i_1^p), (i, j_1^q)) \,|\, i = \overline{1,m}\}$. For an (n,m)-semigroup $\boldsymbol{G}$ we have that $\boldsymbol{G} \models (i_1^p, j_1^q) \iff \boldsymbol{G} \models (i_1^p, j_1^q)_\#$. Consequently, if $\Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ is a set of vector (n,m)-identities (i.e. $\Theta' = \{(i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+ \,|\, p = m + sk,\ q = m + rk, s, r \geq 0\}$), then it induces a set of (n,m)-identities $\Theta'_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))$. Moreover, an arbitrary (n,m)-semigroup $\boldsymbol{G}$ satisfies: $\boldsymbol{G} \models \Theta' \iff \boldsymbol{G} \models \Theta'_\#$.

Definition 2.8 ([7]). A variety of (n,m)-semigroups $\mathcal{V}$ is said to be a vector variety of (n,m)-semigroups, if there exists a set of vector (n,m)-identities $\Theta'_\#$ such that $\mathcal{V} = Var\Theta'_\#$.

Remark 2.9 ([7]). $Var(i_1^m, j_1^m)_\#$ where $i_1^m \neq j_1^m$ is the variety of trivial (n,m)–semigroups and $Var(i_1^m, i_1^m)_\#$ is the variety of all (n,m)–semigroups.

Proposition 2.10 ([7]). A variety of (n,m)-semigroups $Var\Theta$ is a vector variety of (n,m)-semigroups iff there exists a set of vector (n,m)-identities $\Theta'_\#$ such that $\widehat{\Theta} = \widehat{\Theta'_\#}$.

In [7] we showed that the class of vector varieties of (n,m)-semigroups is a proper subset of the class of varieties of (n,m)-semigroups (when $m \geq 2$).

Theorem 2.11 ([7])**.** A variety of (n, m)-semigroups $Var\Theta$ is vector variety of (n, m)-semigroups iff there exists a nonempty set $M \subseteq \widehat{\Theta}$ satisfying:

1) If $(\tau, \omega) \in M$, there exist some $j \in \mathbb{N}_m$ and $\mathbf{w}, \mathbf{z} \in \mathbb{N}^m \cup \mathbb{N}^{m,k}$ such that $\tau = (j, \mathbf{w})$, $\omega = (j, \mathbf{z})$ and $((i, \mathbf{w}), (i, \mathbf{z})) \in M$ for all $i \in \mathbb{N}_m$;
2) $\widehat{\Theta} \subseteq \widehat{M}$.

Theorem 2.12 ([7])**.** Necessary and sufficient conditions for a variety of (n, m)-semigroups $Var\Theta$ to be a vector variety $Var\Theta'_{\#}$ such that all nontrivial $(i_1^p, j_1^q) \in \Theta'$ have lengths $p, q > m$, are:

1) If $(\tau, \omega) \in \widehat{\Theta} \cap \mathbb{N}^2_{(1)}$ then $\tau = (j, \mathbf{w})$ and $\omega = (j, \mathbf{z})$ for some $j \in \mathbb{N}_m$, $\mathbf{w}, \mathbf{z} \in \mathbb{N}^m \cup \mathbb{N}^{m,k}$, and $((i, \mathbf{w}), (i, \mathbf{z})) \in \widehat{\Theta} \cap \mathbb{N}^2_{(1)}$ for all $i \in \mathbb{N}_m$;
2) $\widehat{\Theta} \subseteq \widehat{\widehat{\Theta} \cap \mathbb{N}^2_{(1)}}$.

3 Birkhoff's HSP theorem for varieties of (n, m)-semigroups

The problem of finding a corresponding Birkhoff's - HSP theorem for vector varieties of (n, m)-semigroups (when $m \geq 2$), was introduced in 1980, by acad. Ǵ. Čupona. We believe that our work gives a satisfactory answer.

Theorem 3.1. A class of (n, m)-semigroups is a variety of (n, m)-semigroups iff it is non-empty and closed under the operations of taking (n, m)-subsemi - groups, homomorphic images and direct products.

Proof. ($\Longrightarrow$). Let $\mathcal{V} = Var\Theta$ be a variety of (n, m)-semigroups. Proposition 2.6 implies that $\psi_0(\boldsymbol{F}(\mathbb{N}))/\widehat{\Theta} \in \mathcal{V}$ and thus $\mathcal{V} \neq \emptyset$. If $\boldsymbol{G} \in \mathcal{V}$ then $\boldsymbol{G} \models \Theta$, and if $(\tau, \omega) \in \Theta$ we have $(\tau, \omega) \in \psi_0(F(\mathbb{N}_t))$ (for some $t \in \mathbb{N}$). Now, if $\boldsymbol{H}$ is (n, m)–subsemigroup of $\boldsymbol{G}$, we have $\tau(a_1^t) = \omega(a_1^t)$ for all $a_1^t \in H^t \subseteq G^t$. Hence, $\boldsymbol{H} \models \Theta$ and therefore $\boldsymbol{H} \in \mathcal{V}$. If $\boldsymbol{G} \in \mathcal{V}$ and $\varphi : \boldsymbol{G} \to \boldsymbol{G'}$ is a homomorphism from $\boldsymbol{G}$ to some (n, m)–semigroup $\boldsymbol{G'}$, then $\varphi(\boldsymbol{G})$ is (n, m)–subsemigroup of $\boldsymbol{G'}$. Moreover, for $(\tau, \omega) \in \Theta$ and $a'_1 \dots a'_t = \varphi(a_1) \dots \varphi(a_t) \in \varphi(G)^t$, applying Lemma 2.3 we obtain: $\tau(a'_1 \dots a'_t) = \tau(\varphi(a_1) \dots \varphi(a_t)) = \varphi(\tau(a_1 \dots a_t)) = \varphi(\omega(a_1 \dots a_t)) = \omega(\varphi(a_1) \dots \varphi(a_t)) = \omega(a'_1 \dots a'_t)$. Thus, $\varphi(\boldsymbol{G}) \models (\tau, \omega)$ and we conclude that $\varphi(\boldsymbol{G}) \models \Theta$ i.e. $\varphi(\boldsymbol{G}) \in \mathcal{V}$. Assume that $\boldsymbol{G}$, $\boldsymbol{H} \in \mathcal{V}$ and consider the direct product $\boldsymbol{P} = \boldsymbol{G} \times \boldsymbol{H}$. Since the (n, m)-operation on P is defined via corresponding operations in $\boldsymbol{G}$ and $\boldsymbol{H}$, it is clear that $\boldsymbol{P}$ is (n, m)-semigroup. Also, for $(\tau, \omega) \in \Theta$ and $(a_1, a'_1) \dots (a_t, a'_t) \in P^t$, we have $\tau(a_1 \dots a_t) = \omega(a_1 \dots a_t)$, $\tau(a'_1 \dots a'_t) = \omega(a'_1 \dots a'_t)$, and applying Lemma 2.4 we obtain: $\tau((a_1, a'_1) \dots (a_t, a'_t)) = (\tau(a_1 \dots a_t), \tau(a'_1 \dots a'_t)) = (\omega(a_1 \dots a_t), \omega(a'_1 \dots a'_t)) = \omega((a_1, a'_1) \dots (a_t, a'_t))$. Hence, $\boldsymbol{P} \models (\tau, \omega)$. This is true for all $(\tau, \omega) \in \Theta$ and therefore $\boldsymbol{P} \models \Theta$ i.e. $\boldsymbol{P} \in \mathcal{V}$.

($\Longleftarrow$). If $\mathcal{V}$ is the class of trivial (n,m)–semigroups, then $\mathcal{V} = Var\{(l,j)\}$ for $l,j \in \mathbb{N}$, $l \neq j$, and the theorem is proved. Assume that $\mathcal{V}$ is non-trivial. Then $\mathcal{V}$ contains a free (n,m)–semigroup $\boldsymbol{F}$ with a basis $\mathbb{N}$ (Theorem 1.6) and we can take $\mathbb{N} \subseteq F$ (by Proposition 1.4). On the other hand, being $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})$ the free (n,m)–semigroup with basis $\mathbb{N}$, the identity map $i : \mathbb{N} \to \mathbb{N}$, can be (uniquely) extended to a homomorphism $\bar{i} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{F}$. Define a set

$$\Sigma = \ker \bar{i} = \{(\tau,\omega) \in \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \,|\, \bar{i}(\tau) = \bar{i}(\omega)\} \tag{2}$$

Let $\mathcal{U} = Var\Sigma$. We will show that $\mathcal{V} = \mathcal{U}$. Let $\boldsymbol{A} \in \mathcal{V}$, $(\tau,\omega) \in \Sigma$, $((\tau,\omega) \in \psi_0(F(\mathbb{N}_t)), t \in \mathbb{N})$, and $a_1^t \in A^t$. Consider a map $\varsigma : \mathbb{N} \to A$ such that $\varsigma(j) = a_j$, $j = \overline{1,t}$. There exists a (unique) homomorphism $\bar{\varsigma} : \boldsymbol{F} \to \boldsymbol{A}$ such that $\bar{\varsigma}_{|\mathbb{N}} = \varsigma$, and thus $a_j = \bar{\varsigma}(j)$, $j = \overline{1,t}$. On the other hand, $\bar{i}(\tau) = \bar{i}(\omega)$ and applying (1) (i.e. Lemma 2.1) we get $\tau(1\ldots t) = \bar{i}(\tau) = \bar{i}(\omega) = \omega(1\ldots t)$. Now by Lemma 2.3 we obtain: $\tau(a_1\ldots a_t) = \tau(\bar{\varsigma}(1)\ldots\bar{\varsigma}(t)) = \bar{\varsigma}(\tau(1\ldots t)) = \bar{\varsigma}(\omega(1\ldots t)) = \omega(\bar{\varsigma}(1)\ldots\bar{\varsigma}(t)) = \omega(a_1\ldots a_t)$. Hence, $\boldsymbol{A} \models (\tau,\omega)$. The same is true for all (n,m)–identities in Σ. Thus, $\boldsymbol{A} \in \mathcal{U}$ and we conclude that $\mathcal{V} \subseteq \mathcal{U}$. Assume now that $\boldsymbol{A} \in \mathcal{U} = Var\Sigma$. Proposition 1.7 implies that $\boldsymbol{A}$ is homomorphic image of a free (n,m)–semigroup $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})$ for a corresponding set Y such that $\boldsymbol{A} =< Y >$. Let $\Pi : \psi_0(\boldsymbol{F(}Y\boldsymbol{)}) \to \boldsymbol{A}$ be the homomorphic extension of the identity map 1_Y, and let $t : \mathbb{N} \to Y$ be a map. There exist unique homomorphisms $T' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \psi_0(\boldsymbol{F(}Y\boldsymbol{)}$ and $T'' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{A}$ such that $T'_{|\mathbb{N}} = t$ and $T''_{|\mathbb{N}} = t$. The composition $\Pi T' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)} \to \boldsymbol{A}$ is a homomorphism such that $(\Pi T')_{|\mathbb{N}} = T'_{|\mathbb{N}} = t = T''_{|\mathbb{N}}$. But T'' is unique (Proposition 1.5), and thus $\Pi T' = T''$. Moreover, since $\boldsymbol{A} \models \Sigma$ and $t : \mathbb{N} \to A$, Propostition 2.2 implies that $\Sigma \subseteq \ker(\Pi T')$. Hence, $\ker \bar{i} \subseteq \ker(\Pi T')$. Let now $\boldsymbol{G}_Y$ be a free (n,m)–semigroup in $\mathcal{V}$ with basis Y. The fact that $\mathcal{V}$ is non-trivial allow us to take $Y \subseteq G_Y$ i.e. $\boldsymbol{G}_Y =< Y >$ (by Proposition 1.4). Since $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})$ is a free (n,m)–semigroup with basis Y, there exists a unique homomorphism $\Psi : \psi_0(\boldsymbol{F(}Y\boldsymbol{)}) \to \boldsymbol{G}_Y$ which is an extension of the identity map $1_Y : Y \to Y$. On the other hand, since $\boldsymbol{G}_Y$ is a free (n,m)–semigroup in $\mathcal{V}$ with basis Y, every $f : Y \to \mathbb{N} \subseteq F$ can be uniquely extended to a homomorphism $L : \boldsymbol{G}_Y \to \boldsymbol{F}$. Now, the composition $L\,\Psi\,T' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{F}$ is a homomorphism, and for a carefully chosen map f, we will obtain that $\ker(L\,\Psi\,T') \subseteq \ker(\Pi\,T')$, which implies $\ker(\Psi\,T') \subseteq \ker(\Pi\,T')$. This holds for the homomorphic extension $T' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \psi_0(\boldsymbol{F(}Y\boldsymbol{)})$ of an arbitrary map $t : \mathbb{N} \to Y$, and thus it will follow that $\ker \Psi \subseteq \ker \Pi$. (Namely, for $(x,z) \in \ker \Psi \subseteq \psi_0(F(Y)) \times \psi_0(F(Y))$, there exist an appropriate $t : \mathbb{N} \to Y$ which has a homomorphic extension $T' : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \psi_0(\boldsymbol{F(}Y\boldsymbol{)}$ such that $x = T'(u)$ and $z = T'(v)$ for some $u,v \in \psi_0(F(\mathbb{N}))$. Since $\Psi\,T'(u) = \Psi\,T'(v)$, we have $(u,v) \in \ker(\Psi\,T')$ which implies that $(u,v) \in \ker(\Pi\,T')$ i.e. $\Pi\,T'(u) = \Pi\,T'(v)$ and therefore $\Pi(x) = \Pi(z)$, so $\ker \Psi \subseteq \ker \Pi$). Being Π and Ψ homomorphic extensions of 1_Y, we have $Y \subseteq \Pi(\psi_0(\boldsymbol{F(}Y\boldsymbol{)}))$ and $Y \subseteq \Psi(\psi_0(\boldsymbol{F(}Y\boldsymbol{)}))$, and being $\boldsymbol{A} =< Y >$ and

$\boldsymbol{G}_Y =< Y >$, we obtain $\boldsymbol{A} = \Pi(\psi_0(\boldsymbol{F(}Y\boldsymbol{)}))$ and $\boldsymbol{G}_Y = \Psi(\psi_0(\boldsymbol{F(}Y\boldsymbol{)}))$. Hence, Π and Ψ are epimorphisms. Theorem 1.1 implies that $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Pi \cong \boldsymbol{A}$ and $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Psi \cong \boldsymbol{G}_Y$, and from the fact that $\boldsymbol{G}_Y \in \mathcal{V}$ we obtain $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Psi \in \mathcal{V}$. On the other hand, Theorem 1.2 implies existence of a homomorphism $\sigma : \psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Psi \to \boldsymbol{A}$, and σ will be epimorphism, since Π is an epimorphism. Therefore, $\boldsymbol{A} = \sigma(\psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Psi)$, where $\psi_0(\boldsymbol{F(}Y\boldsymbol{)})/\ker\Psi \in \mathcal{V}$ so we conclude that $\boldsymbol{A} \in \mathcal{V}$. Hence, $\mathcal{U} \subseteq \mathcal{V}$. □

Remark 3.2. The set $\Sigma = \ker\bar{i}$ defined by (2) is a complete system of (n, m)–identities for $\mathcal{V} = Var\Sigma$: By Corollary 2.7 we have $\widehat{\Sigma} \subseteq \ker\bar{i}$, and thus $\widehat{\Sigma} \subseteq \Sigma$, which implies $\widehat{\Sigma} = \Sigma$. Also, $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\ker\bar{i}$ is a free (n, m)-semigroup in $\mathcal{V}$ with basis $\mathbb{N}$ (by Proposition 2.6).

We are looking for a corresponding Birkhoff's - HSP Theorem' for vector varieties of (n, m)-semigroups. For that purpose, we consider non-trivial classes of (n, m)–semigroups (Remark 2.9).

Theorem 3.3. A non-trivial class $\mathcal{V}$ of (n, m)–semigroups is a vector variety of (n, m)–semigroups iff $\mathcal{V}$ is closed under the operations of taking (n, m)–subsemigroups, homomorphic images and direct products, and the homomorphic extension $\bar{i} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{F}$ of the identity map $i : \mathbb{N} \to \mathbb{N}$, where $\boldsymbol{F}$ is the free (n, m)–semigroup in $\mathcal{V}$ with basis $\mathbb{N} \subseteq F$, satisfies the condition: $(\exists M \neq \emptyset)$ such that $M \subseteq \ker\bar{i} \subseteq \widehat{M}$ and $M = \bigcup\limits_{(\mathbf{w},\mathbf{z})} \{((i, \mathbf{w}), (i, \mathbf{z}))|i = \overline{1, m}\}$ for some pairs of elements $(\mathbf{w}, \mathbf{z})$ such that $\mathbf{w}, \mathbf{z} \in \mathbb{N}^m \cup \mathbb{N}^{m,k}$.

Proof. $(\Longleftarrow)$. Let $\mathcal{V}$ be a non-trivial class of (n, m)–semigroups that is closed under the operations of taking (n, m)–subsemigroups, homomorphic images and direct products. By Theorem 1.6, Proposition 1.4 and Proposition 1.5 we have that $\mathcal{V}$ contains a free (n, m)–semigroup $\boldsymbol{F}$ with basis $\mathbb{N}$ such that $\mathbb{N} \subseteq F$, and, the identity map $i : \mathbb{N} \to \mathbb{N}$ has a unique homomorphic extension $\bar{i} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{F}$. Assume that $\ker\bar{i}$ satisfies the condition above. By Theorem 3.1 and its proof we have that $\mathcal{V} = Var\ker\bar{i}$. Moreover, $\ker\bar{i}$ is a complete system of (n, m)–identities for $\mathcal{V}$ (by Remark 3.2) and $\widehat{\ker\bar{i}} = \ker\bar{i}$ (by Theorem 2.5). Therefore, $Var\ker\bar{i}$ satisfies the conditions given in Theorem 2.11, and thus we conclude that $\mathcal{V}$ is a vector variety of (n, m)–semigroups.
$(\Longrightarrow)$. If $\mathcal{V}$ is a non-trivial class of (n, m)–semigroups which is a vector variety of (n, m)–semigroups, there exists a set of vector (n, m)–identities $\Theta'_{\#}$ such that $\mathcal{V} = Var\Theta'_{\#}$. Theorem 3.1 implies that $\mathcal{V}$ is closed under the operations of taking (n, m)–subsemigroups, homomorphic images and direct products. Proposition 2.6 implies that $\mathcal{V}$ contains a free (n, m)–semigroup with basis $\mathbb{N}$ which is the factor (n, m)–semigroup $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta'_{\#}}$. Moreover, the class $\mathcal{V}$ is non-trivial and thus $j^{\widehat{\Theta'_{\#}}} = \{j\}, j \in \mathbb{N}$, so we can take

$\mathbb{N} \subseteq \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta'_{\#}}$. Now, for the the corresponding homomorphic extension $\bar{i} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta'_{\#}}$ of the identity map $i : \mathbb{N} \to \mathbb{N}$, it follows that $\bar{i} = nat\widehat{\Theta'_{\#}}$ (from $(nat\widehat{\Theta'_{\#}})_{|\mathbb{N}} = i$ and the uniqueness of $\bar{i}$.) Thus, we obtain that $\ker \bar{i} = \ker(nat\widehat{\Theta'_{\#}})$ and therefore $\ker \bar{i} = \widehat{\Theta'_{\#}}$. Applying Theorem 2.11 it follows that $\widehat{\Theta'_{\#}}$ satisfies the condition above, so the same is true for $\ker \bar{i}$. □

Theorem 3.4. A non-trivial class $\mathcal{V}$ of (n, m)–semigroups is a vector variety of (n, m)–semigroups $Var\Theta'_{\#}$ such that all nontrivial $(i_1^p, j_1^q) \in \Theta'$ have lengths $p, q > m$ iff $\mathcal{V}$ is closed under the operations of taking (n, m)–subsemigroups, homomorphic images and direct products, and the homomorphic extension $\bar{i} : \psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)}) \to \boldsymbol{F}$ of the identity map $i : \mathbb{N} \to \mathbb{N}$ where $\boldsymbol{F}$ is the free (n, m)–semigroup in $\mathcal{V}$ with basis $\mathbb{N} \subseteq F$, satisfies the conditions:

1) $\ker \bar{i} \cap \mathbb{N}^2_{(1)} = \bigcup\limits_{(\mathbf{w},\mathbf{z})} \{((i, \mathbf{w}), (i, \mathbf{z}))|i = \overline{1, m}\}$ for some pairs of elements $(\mathbf{w}, \mathbf{z})$ such that $\mathbf{w}, \mathbf{z} \in \mathbb{N}^m \cup \mathbb{N}^{m,k}$

2) $\ker \bar{i} \subseteq (\widehat{\ker \bar{i} \cap \mathbb{N}^2_{(1)}})$.

Proof. Completely analogical as Theorem 3.3, applying Theorem 2.12. □

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