Abstract

In this paper, we study a Poisson algebra $A$ constructed from the $\mathbb{C}$-algebra $T$ with generators $x, y, z$ and $t^{\pm 1}$ and subject to the relations as follows:

\[
xy - t^{-2}yx = (1 - t^{-2})z, \\
yz - t^{-2}zy = (1 - t^{-2})x, \\
zx - t^{-2}xz = (1 - t^{-2})y,
\]

and

\[
x t = t x, \quad y t = t y, \quad z t = t z, \quad t t^{-1} = 1 = t^{-1} t,
\]

where $a, b, c \in \mathbb{C} \setminus \{0\}$ and $t$ is a central element of $T$. The main result shows that there are five $d$-dimensional simple modules over Poisson algebra $A$ for any $d \geq 1$.

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1 Introduction

Poisson geometry is originated in classical mechanics and Poisson bracket has lately been playing an important role in algebraic geometry, mathematical
physics and other subjects. Poisson modules can be considered as certain representations of Poisson algebras, which forms an abelian category. The related Poisson cohomology theory is essential in Kontsevich’s deformation quantization of Poisson manifolds [6].

Let $A$ be a commutative algebra over the complex numbers $\mathbb{C}$. A Poisson algebra $A$, that is, there is a bilinear map $\{-,-\} : A \times A \rightarrow A$ such that $(A, \{-,-\})$ is a Lie algebra and satisfies the Leibniz identity $\{ab,c\} = a\{b,c\} + \{a,c\}b$, for all $a, b, c \in A$. Then we say that $\{-,-\}$ is a Poisson bracket on $A$.

Consider the $\mathbb{C}$-algebra $T$ with generators $x, y, z$ and $t^{\pm 1}$ and subject to the relations as follows:

\[
xy - t^{-2}yx = (1 - t^{-2})z, \\
yz - t^{-2}zy = (1 - t^{-2})x, \\
zx - t^{-2}xz = (1 - t^{-2})y,
\]

and

\[
xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t,
\]

where $a, b, c \in \mathbb{C}\{0\}$ and $t$ is a central element of $T$.

In this algebra $T$, $t$ is a central non-unit non-zero-divisor element such that $A := T/(t - 1)T$ is commutative and isomorphic to $\mathbb{C}[x, y, z]$, then there is a Poisson bracket $\{-,-\}$ on $A$ such that $\{\overline{x}, \overline{y}\} = (t - 1)^{-1}[x, y]$ for all $\overline{x}, \overline{y} \in A$. We can view $T$ as a quantization of the Poisson algebra $A$. Therefore there is a Poisson bracket $\{-,-\}$ on $A$ such that

\[
\{x, y\} = 2(z - yx), \quad \{y, z\} = 2(x - zy), \quad \{z, x\} = 2(y - xz). \tag{1}
\]

The purpose of this paper is to classify all finite-dimensional simple Poisson modules over $A$. We find that the annihilator of a finite-dimensional simple Poisson module is a Poisson maximal ideal. Therefore, in section 2 and section 3, we briefly review some basic concepts related to Poisson modules and present five Poisson maximal ideals of $A$, which is denoted by $J_1, J_2, J_3, J_4$ and $J_5$. Section 4 and section 5 are devoted to classifying the finite-dimensional simple Poisson modules annihilated by $J_1$ and $J_2$. In the next section, we twist the finite-dimensional simple Poisson modules annihilated by $J_2$ to $J_3, J_4$ and $J_5$ by a certain Poisson automorphism.

2 Preliminaries

In this section we will recall some definitions and basic results.

**Definition 2.1.** A Poisson algebra is a commutative algebra $A$ equipped with a Lie bracket $\{-,-\}$ such that

\[
\{ab, c\} = a\{b, c\} + \{a, c\}b,
\]
for any $a, b, c \in A$. We call $\{-, -\}$ a Poisson bracket on $A$.

**Definition 2.2.** Let $A$ be a Poisson algebra and $I$ is an ideal of $A$. If $\{i, a\} \in I$ for any $i \in I$ and $a \in A$, we call $I$ a *Poisson ideal* of $A$. Moreover, if a Poisson ideal $I$ of $A$ is also a maximal ideal of $A$, we say $I$ a *Poisson maximal ideal*.

**Definition 2.3.** Let $A$ be a commutative Poisson algebra with Poisson bracket $\{-, -\}$. An $A$-module $M$ becomes a *Poisson module* if there exists a bilinear map $\{-, -\}_M : A \times M \to M$ such that

1. $\{a, bm\}_M = \{a, b\}M + b\{a, m\}_M$,
2. $\{ab, m\}_M = a\{b, m\}_M + b\{a, m\}_M$,
3. $\{\{a, b\}, m\}_M = \{a, \{b, m\}_M\}_M - \{b, \{a, m\}_M\}_M$,

for any $a, b \in A$ and $m \in M$.

Similarly as in the module category, we can define Poisson submodules and simple Poisson modules in the category of Poisson modules over $A$.

**Lemma 2.4.** Let $A$ be a Poisson $\mathbb{C}$-algebra, and $M$ be a Poisson $A$-module.

1. The annihilator of $M$ is a Poisson ideal of $A$.
2. If $M$ is a simple Poisson module, then the annihilator of $M$ is a prime Poisson ideal of $A$.
3. If $M$ is a finite-dimensional simple Poisson module, then the annihilator of $M$ is a Poisson maximal ideal of $A$.

**Proof.** See [7, Lemma 4.1.1].

**Lemma 2.5.** Let $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$ with a Poisson bracket $\{-, -\}$. Let $V = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_n$ and let $M$ be an $A$-module. Suppose that there is a bilinear form $\{-, -\}_M : V \times M \to M$. Extend this to a bilinear form $\{-, -\}_M : A \times M \to M$ using Definition 2.3 (2) and $\{1, m\}_M = 0$. If Definition 2.3 (1)&(3) hold, for any $m \in M$, whenever $a = x_i$ and $b = x_j$ for $1 \leq i < j \leq n$, then Definition 2.3 (1)&(3) hold for all $a, b \in A$.

**Proof.** See [7, Lemma 4.1.2].
3 Poisson maximal ideals

In this section, we find the Poisson maximal ideals of a Poisson algebra \( A \).

Lemma 3.1. Let \( A = \mathbb{C}[x_1, x_2, \ldots, x_n] \) be the polynomial ring over \( \mathbb{C} \) in the \( n \) indeterminates \( x_1, x_2, \ldots, x_n \). The ideal \( J \) is a maximal ideal if and only if there exist \( a_1, a_2, \ldots, a_n \) such that \( J = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \).

Proof. See [8, Theorem 1.4.6].

Lemma 3.2. Let \( A \) be the Poisson algebra with Poisson bracket as (1). There are only two Poisson maximal ideals of \( A \). They are:

\[
\begin{align*}
J_1 &= xA + yA + zA; \\
J_2 &= (x + 1)A + (y + 1)A + (z - 1)A; \\
J_3 &= (x + 1)A + (y - 1)A + (z + 1)A; \\
J_4 &= (x - 1)A + (y + 1)A + (z + 1)A; \\
J_5 &= (x - 1)A + (y - 1)A + (z - 1)A.
\end{align*}
\]

Proof. Let \( J \) be a Poisson maximal ideal of \( A \). Since \( A \) is a commutative polynomial ring over \( \mathbb{C} \), by Lemma 3.1, \( J = (x-m, y-n, z-l) \) for \( m, n, l \in \mathbb{C} \). Since \( J \) is a Poisson maximal ideal, \( \{x, J\} \subseteq J, \{y, J\} \subseteq J, \{z, J\} \subseteq J \). Consider

\[
\begin{align*}
2(y - x) &= \{x, y - n\} \in J, \\
2(x - zy) &= \{y, z - l\} \in J, \\
-2(y - zx) &= \{x, z - l\} \in J.
\end{align*}
\]

This implies \( l - mn = m - nl = n - ml = 0 \). Therefore, we can get \( m = n = l = 0; m = -1, n = -1, l = 1; m = -1, n = 1, l = -1; m = 1, n = -1, l = -1; \) Or \( m = 1, n = 1, l = 1 \). There are five solutions and so the Poisson maximal ideals are obtained.

4 Poisson modules annihilated by \( J_1 \)

Throughout the remaining of the paper, we set \( A = \mathbb{C}[x, y, z] \) as the polynomial algebra on 3 variable equipped with the Poisson bracket given in (1). Our goal is to classify all finite-dimensional simple Poisson modules over \( A \).

Lemma 4.1. Let \( M \) be a Poisson module annihilated by \( J_1 \) and \( m \in M \), then we have:
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(1) $xm = ym = zm = 0$;

(2) $\{xy, m\} = \{yz, m\} = \{zx, m\} = 0$;

(3) $\{x, \{y, m\}\} - \{y, \{x, m\}\} = 2\{z, m\}$;
   $\{y, \{z, m\}\} - \{z, \{y, m\}\} = 2\{x, m\}$;
   $\{z, \{x, m\}\} - \{x, \{z, m\}\} = 2\{y, m\}$.

Proof. It is a routine calculation by Definition 2.3.

Lemma 4.2. Let $M$ be a Poisson module annihilated by $J_1$, and $\lambda \in \mathbb{C}$ be such that $\{x, m\} = \lambda m$ for some $0 \neq m \in M$. Then

(1) $\{x, \{y, m\}\} = 2\{z, m\} + \lambda \{y, m\}$;

(2) $\{x, \{z, m\}\} = \lambda \{z, m\} - 2\{y, m\}$;

(3) $\{y, \{z, m\}\} - \{z, \{y, m\}\} = 2\lambda m$.

Proof. By Definition 2.3 (3), we check (1),

\[
\{x, \{y, m\}\} = \{(x, y), m\} + \{y, \{x, m\}\} \\
= 2(z - yx, m) + \lambda \{y, m\} \\
= 2\{z, m\} + \lambda \{y, m\}.
\]

The proof of (2), (3) is similar to (1).

In order to simplify our classification of finite-dimensional simple Poisson modules over $A$, we shall replace the generators $x, y$ and $z$ of $A$ by $x, u = \frac{1}{2}(y - iz)$ and $v = \frac{1}{2}(z - iy)$ ($y = u + vi$ and $z = v + ui$). By a straightforward calculation, we get the Poisson bracket on $A = \mathbb{C}[x, u, v]$ is

\[
\{x, u\} = 2(-vix + ui); \\
\{x, v\} = 2(uix - vi);\tag{2} \\
\{u, v\} = x - (v^2 + u^2)i.
\]

Lemma 4.3. Let $M$ be a Poisson $A$-module annihilated by

$J_1 = xA + uA + vA$.

For any $m \in M$, we have:

(1) $xm = um = vm = 0$;
\[ \{xu, m\}_M = \{xv, m\}_M = \{uv, m\}_M = 0; \]

\[ \{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = 2i\{u, m\}_M; \]

\[ \{x, \{v, m\}_M\}_M - \{v, \{x, m\}_M\}_M = -2i\{v, m\}_M; \]

\[ \{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \{x, m\}_M. \]

**Proof.** (1),(2) is a routine calculation.

(3) By using Definition 2.3 (3),

\[ \{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = \{\{x, u\}, m\}_M = 2i\{u, m\}_M. \]

The rest two formulas can be proved similarly.  

**Lemma 4.4.** Let \( M \) be a Poisson \( A \)-module annihilated by \( J_1 \), and \( 0 \neq m \in M \) such that \( \{x, m\}_M = \lambda m \) for some \( \lambda \in \mathbb{C} \). Then

(1) \( \{x, \{u, m\}_M\}_M = (\lambda + 2i)\{u, m\}_M; \)

(2) \( \{x, \{v, m\}_M\}_M = (\lambda - 2i)\{v, m\}_M; \)

(3) \( \{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \lambda m. \)

**Proof.** It is a routine calculation by using Lemma 4.3.  

**Lemma 4.5.** For every \( d \geq 1 \), there is a \( d \)-dimensional Poisson \( A \)-module \( M \), with the basis \( \{m_1, m_2, \ldots, m_d\} \), such that \( M \) is annihilated by \( J_1 \) and

(1) \( \{x, m_j\}_M = (\lambda + 2(j - 1)i)m_j \), for \( 1 \leq j \leq d; \)

(2) \( \{v, m_1\}_M = 0 \) and \( \{v, m_j\}_M = -(j - 1)(\lambda + (j - 2)i)m_{j-1} \) for \( 1 < j \leq d; \)

(3) \( \{u, m_d\}_M = 0 \) and \( \{u, m_j\}_M = m_{j+1} \) for \( 1 \leq j < d; \)

with \( \lambda = (1 - d)i. \)

**Proof.** By Lemma 2.5, it suffices to show that Definition 2.3 (1)&(3) hold for \( m = m_j \) and \( \{a, b\} = \{x, u\} \) or \( \{x, v\} \) or \( \{u, v\} \) for the bracket \( \{-, -\}_M : A \times M \to M \) defined above. We then can extend the bracket from \( \mathbb{C}x \oplus \mathbb{C}u \oplus \mathbb{C}v \) to \( A = \mathbb{C}[x, u, v] \) on \( M \) using Definition 2.3 (2). Note that the conclusion of Lemma 4.3 (1)&(2) hold in our case.

To show that Definition 2.3 (3) holds, we first show that

\[ \{\{x, u\}, m_j\}_M = \{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M, \text{ for } 1 \leq j \leq d. \]
In the case $1 \leq j < d$, by Lemma 4.3 (3), it follows from
\[
\{x, u, m_j\}_M = \{2(-vix + ui), m_j\}_M = 2i\{u, m_j\}_M = 2im_{j+1},
\]
and
\[
\{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M = \{x, m_{j+1}\}_M - \{u, (\lambda + 2(j-1)i)m_j\}_M = (\lambda + 2ji)m_{j+1} - (\lambda + 2(j-1)i)m_{j+1} = 2im_{j+1}.
\]
In the case $j = d$, consider that \(\{x, u, m_d\}_M = 2i\{u, m_d\}_M = 0\), and
\[
\{x, \{u, m_d\}_M\}_M - \{u, \{x, m_d\}_M\}_M = 0 - (\lambda + 2(d-1)i)\{u, m_d\}_M = 0.
\]
Then we can check that
\[
\{x, v, m_j\}_M = \{x, v, m_j\}_M - \{v, \{x, m_j\}_M\}_M, \text{ for } 1 \leq j \leq d;
\]
\[
\{u, v, m_j\}_M = \{u, v, m_j\}_M - \{v, \{u, m_j\}_M\}_M, \text{ for } 1 \leq j \leq d.
\]
It remains to show that Definition 2.3 (1) holds for $m = m_j$ and $\{a, b\} = \{x, u\}$ or $\{x, v\}$ or $\{u, v\}$. First of all,
\[
\{x, um_j\}_M = 0,
\]
and by (2)
\[
\{x, u\}_m_j + u\{x, m_j\}_M = 2(-vix + ui)m_j = 0.
\]
Secondly,
\[
\{x, vm_j\}_M = 0,
\]
and by (2)
\[
\{x, v\}_m_j + v\{x, m_j\}_M = 2(uix - vi)m_j = 0.
\]
Finally,
\[
\{u, vm_j\}_M = 0,
\]
and by (2)
\[
\{u, v\}_m_j + v\{u, m_j\}_M = (x - (v^2 + u^2)i)m_j = 0.
\]
This completes the proof.
Lemma 4.6. For every \( d \geq 1 \), the \( d \)-dimensional Poisson module \( M \) over \( A \) constructed in Lemma 4.5 is simple as a Poisson module.

Proof. Suppose that the \( d \)-dimensional Poisson module \( M \) isn’t simple. Let \( \lambda_j = \lambda + 2(j - 1)i \), \( 1 \leq j \leq d \) where \( \lambda = (1 - d)i \) and \( \lambda_j \neq \lambda_k \) when \( j \neq k \). Suppose \( N \) is a non-zero submodule of \( M \). Let \( 0 \neq n = \sum_{j=1}^{d} \alpha_j m_j \in N \) be such that minimally many of the coefficients \( \alpha_j \in \mathbb{C} \) are non-zero. We choose \( k \) such that \( \alpha_k \neq 0 \). Consider

\[
\{ x, n \}_M - \lambda_k n = \sum_{j=1}^{d} \alpha_j (\lambda_j - \lambda_k)m_j,
\]

which has one fewer non-zero coefficient than \( n \). By the minimality of \( n \), we get \( \{ x, n \}_M - \lambda_k n = 0 \) and \( \alpha_j = 0 \) when \( j \neq k \), that is \( n = \alpha_k m_k \). So \( m_k \in N \). By the Poisson action of \( u \) and \( v \), \( m_j \in N \) for all \( j \). So \( N = M \), that is \( M \) is a simple Poisson module. \( \square \)

Lemma 4.7. Let \( M \) be a finite-dimensional simple Poisson \( A \)-module annihilated by \( J_1 \) and let \( n \leq \dim M \). Then there exist \( \lambda \in \mathbb{C} \) and \( n \) linearly independent elements \( m_1, m_2, \ldots, m_n \in M \) such that

1. \( \{ x, m_j \}_M = (\lambda + 2(j - 1)i)m_j \) for \( 1 \leq j \leq n \);
2. \( \{ v, m_1 \}_M = 0 \) and \( \{ v, m_j \}_M = -(j - 1)(\lambda + (j - 2)i)m_{j-1} \) for \( 1 < j \leq n \);
3. \( \{ u, m_j \}_M = m_{j+1} \) for \( 1 \leq j < n \).

Proof. We prove the result by induction on \( n \). When \( n = 1 \). Let \( \theta = \{ \lambda \in \mathbb{C} \mid \{ x, m \}_M = \lambda m \} \) for some \( 0 \neq m \in M \). Since \( M \) is finite-dimensional, the linear endomorphism of \( M \) given by \( m \mapsto \{ x, m \}_M \) for any \( m \in M \) has an eigenvalue. Therefore \( \theta \neq \emptyset \). Choose \( \lambda \in \theta \). Then there is some \( 0 \neq m_1 \in M \) such that \( \{ x, m_1 \} = \lambda m_1 \). It is clear that \( m_1 \) satisfies all the conditions (1)-(3) above.

Suppose the result holds for any \( \dim M \geq n \). Now let \( \dim M \geq n + 1 \). By induction hypothesis, there are linearly independent elements \( m_1, m_2, \ldots, m_n \) of \( M \) satisfying conditions (1)-(3) above for some \( \lambda \in \mathbb{C} \). We set \( m_{n+1} = \{ u, m_n \}_M \). It is clear that \( m_{n+1} = \{ u, m_n \}_M \neq 0 \). Otherwise, it would imply that the \( n \)-dimensional subspace spanned by \( m_1, m_2, \ldots, m_n \) is a Poisson sub-module of \( M \). This contradicts the fact that \( M \) is simple with \( \dim M \geq n + 1 \).

It remains to show that \( m_{n+1} \) satisfies conditions (1)-(3). Since \( \{ x, m_n \}_M = (\lambda + 2(n - 1)i)m_n \), we get

\[
\{ x, m_{n+1} \}_M = \{ x, \{ u, m_n \}_M \}_M = (\lambda + 2ni)m_{n+1},
\]

by Lemma 4.4 (1). Hence (1) holds. For (2), we have

\[
\{ u, \{ v, m \}_M \}_M - \{ v, \{ u, m \}_M \}_M = k_1 k_2 b \{ x, m \}_M,
\]

where
by Lemma 4.3 (3). Let \( m = m_n \). Then

\[
-(n-1)(\lambda + (n-2)i)\{u, m_{n-1}\}_M - \{v, m_{n+1}\}_M = \{x, m_n\}_M.
\]

So

\[
\{v, m_{n+1}\}_M = -n(\lambda + (n-1)i)m_n.
\]

Then (2) holds and (3) is trivial. Finally, \( m_1, m_2, \ldots, m_{n+1} \) are eigenvectors for \( \{x, -\}_M \) with different eigenvalues. So they are linearly independent. This includes the induction step.

**Theorem 4.8.** Let \( M \) be a \( d \)-dimensional simple Poisson \( A \)-module annihilated by \( J_1 \). Then \( M \) has a basis \( m_1, m_2, \ldots, m_d \) such that

\[
\begin{align*}
(1) \quad & \{x, m_j\}_M = (\lambda + 2(j - 1)i)m_j \text{ for } 1 \leq j \leq d; \\
(2) \quad & \{v, m_1\}_M = 0 \text{ and } \{v, m_j\}_M = -(j-1)(\lambda + (j-2)i)m_{j-1} \text{ for } 1 < j \leq d; \\
(3) \quad & \{u, m_d\}_M = 0 \text{ and } \{u, m_j\}_M = m_{j+1} \text{ for } 1 \leq j < d;
\end{align*}
\]

where \( \lambda = (1 - d)i \).

**Proof.** According to Lemma 4.7, \( M \) has a basis \( m_1, m_2, \ldots, m_d \) satisfying conditions (1)-(3) above for some there exist \( \lambda \in \mathbb{C} \) except for \( \{u, m_d\}_M = 0 \). By Lemma 4.4 (1), we get

\[
\{x, \{u, m_d\}_M\}_M = (\lambda + 2di)\{u, m_d\}_M.
\]

But \( \lambda + 2di \) is not an eigenvalue of \( \{x, -\}_M \) since \( M \) is spanned by the eigenvectors \( m_j \) with distinguished eigenvalues \( \lambda + 2i(j - 1) \) for \( 1 \leq j \leq d \). So \( \{u, m_d\}_M = 0 \). By Lemma 4.3 (3), we get

\[
\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \{x, m\}_M,
\]

and

\[
-(d-1)(\lambda + (d-2)i)m_d = (\lambda + 2(d-1)i)m_d.
\]

So \( d(\lambda + (d-1)i) = 0 \), which implies that \( \lambda = (1 - d)i \).

**Example 4.9.** Let \( A = \mathbb{C}[x, u, v] \) with the Poisson bracket as in (2). The matrices representing the action \( \{x, -\}_M, \{u, -\}_M, \{v, -\}_M \) on the 3-dimensional and 4-dimension simple Poisson modules, with respect to the basis in Theorem 4.8, are shown below:

3-dimensional simple Poisson module:
\{x, -\}_M = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}, \{u, -\}_M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \{v, -\}_M = \begin{pmatrix} 0 & 2i & 0 \\ 0 & 0 & 2i \\ 0 & 0 & 0 \end{pmatrix}

4-dimension simple Poisson module:
\{x, -\}_M = \begin{pmatrix} -3i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}, \{u, -\}_M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \{v, -\}_M = \begin{pmatrix} 0 & 3i & 0 \\ 0 & 0 & 4i \\ 0 & 0 & 3i \end{pmatrix}

Theorem 4.10. For every \(d \geq 1\), there is a unique \(d\)-dimensional simple Poisson module over \(A\) annihilated by \(J_1\). Moreover, it has a basis \(m_1, m_2, \ldots, m_d\) such that

1. \(\{x, m_j\}_M = (\lambda + 2(j - 1)i)m_j\), for \(1 \leq j \leq d\);
2. \(\{y, m_1\}_M = \frac{1}{i}m_2\);
   \(\{y, m_j\}_M = m_{j+1} - i(j - 1)(\lambda + (j - 2)i)m_{j-1}, 1 < j < d\);
   \(\{y, m_d\}_M = -i(d - 1)(\lambda + (d - 2)i)m_{d-1}\);
3. \(\{z, m_1\}_M = im_2\);
   \(\{z, m_j\}_M = -(j - 1)(\lambda + (j - 2)i)m_{j-1} + im_{j+1}\) for \(1 < j < d\);
   \(\{z, m_d\}_M = -(d - 1)(\lambda + (d - 2)i)m_{d-1}\);

where \(\lambda = (1 - d)i\).

Proof. It directly follows from Theorem 4.8 by a change of variables according to (2).

5 Piosson modules annihilated by \(J_2\)

In this section, we classify finite-dimensional simple Poisson module over \(A\) by Poisson maximal ideal \(J_2 = (x + 1)A + (y + 1)A + (z - 1)A\).

Lemma 5.1. Let \(M\) be a Poisson module annihilated by \(J_2\) and \(m \in M\), then we have:

1. \(xm = -m, ym = -m, zm = m\);
\[
\{xy, m\}_M = -\{y, m\}_M - \{x, m\}_M;
\]
\[
\{yz, m\}_M = -\{z, m\}_M + \{y, m\}_M;
\]
\[
\{zx, m\}_M = -\{z, m\}_M + \{x, m\}_M;
\]
\[
\{z, \{y, m\}_M\}_M - \{x, \{z, m\}_M\}_M = 2\{z, m\}_M - 2\{x, m\}_M + 2\{y, m\}_M.
\]

Proof. It is a routine calculation by Definition 2.3.

In order to simplify our classification of finite-dimensional simple Poisson modules over \(A\), we shall replace the generators \(x, y\) and \(z\) of \(A\) by \(x, y\) and \(s = x + y + z\). By a straightforward calculation, we get the Poisson bracket on \(A = \mathbb{C}[x, y, s]\) is
\[
\{x, y\} = -2((x + 1)y - s + x);
\]
\[
\{x, s\} = -2(x + 1)(x + 2y - s);
\]
\[
\{y, s\} = 2(y + 1)(2x + y - s).
\]
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\{x, m_j\}_M = 2(j - 1)(\lambda + 2(j - 2))m_{j-1}, \text{ for } 1 \leq j \leq d;

(2) \{y, m_j\}_M = m_{j+1}, \text{ for } 1 \leq j < d \text{ and } \{y, m_d\}_M = 0;

(3) \{z, m_j\}_M = (\lambda + 4(j - 1))m_j - 2(j - 1)(\lambda + 2(j - 2))m_{j-1} - m_{j+1} \text{ for } 1 \leq j < d;

\{z, m_d\}_M = (\lambda + 4(d - 1))m_d - 2(d - 1)(\lambda + 2(d - 2))m_{d-1};

where \( \lambda = 2(1 - d) \).

6 Poisson automorphism

In this section, by Poisson automorphism, we can classify the finite-dimensional simple Poisson modules annihilated by \( J_2 \).

**Definition 6.1.** Let \((A, \{-, -\})\) be a Poisson algebra. We say that a \( \mathbb{C} \)-algebra automorphism \( \theta: A \rightarrow A \) is a Poisson automorphism if \( \theta(\{x, y\}) = \{\theta(x), \theta(y)\} \) for any \( x, y \in A \).

**Lemma 6.2.** Let \( A \) be a Poisson algebra, \( \alpha \) be a Poisson automorphism of \( A \) and \( M \) be a Poisson \( A \)-module. Then \( M^\alpha \) is a Poisson module where \( M^\alpha = M \) as vector spaces and \( a^\alpha m = \alpha(a)m \) and \( \{a, m\}^\alpha_M = \{\alpha(a), m\}_M \) for any \( a \in A \) and \( m \in M \).

**Proof.** See [7, Theorem 4.4.2]. \( \square \)

**Remark 6.3.** Let \( A \) be a Poisson algebra, \( \alpha \) be a Poisson automorphism of \( A \) and \( M \) be a simple Poisson \( A \)-module. Let \( M^\alpha \) be defined as in Lemma 6.2. Then \( M^\alpha \) is a simple Poisson \( A \)-module and \( \text{ann}_A(M^\alpha) = \alpha^{-1}(\text{ann}_A(M)) \).

Consider the Poisson algebra \( A \) with Poisson bracket as (1)

\[ \{x, y\} = 2(z - yx), \{y, z\} = 2(x - zy), \{z, x\} = 2(y - xz), \]

where \( a, b, c \in \mathbb{C} \setminus \{0\} \). Let \( \alpha \) be the Poisson automorphism of \( A \) such that

\[ \alpha(x) = x, \alpha(y) = -y, \alpha(z) = -z. \]

Observe that \( \alpha(J_2) = J_3 \) and \( \alpha^2 \) is the identity. So the simple Poisson modules annihilated by \( J_3 \) are precisely the Poisson module \( M^\alpha \), where \( M \) is a simple Poisson module annihilated by \( J_2 \). Besides, we can conclude that for each \( d \geq 1 \) there is one \( d \)-dimensional simple Poisson module annihilated by \( J_3 \).

For the Poisson module annihilated by \( J_4 \) and \( J_5 \), we use the same method in \( J_3 \) which defined \( \mathbb{C} \)-automorphisms \( \beta \) and \( \gamma \) by

\[ \beta(x) = -x, \beta(y) = y, \beta(z) = -z. \]
\[ \gamma(x) = -x, \gamma(y) = -y, \gamma(z) = z. \]

Therefore the $d$-dimensional simple Poisson modules $M^\beta$ and $M^\gamma$ can be obtained annihilated by $J_4$ and $J_5$.

**Theorem 6.4.** For $d \geq 1$, the Poisson algebra $A$ in (1) has five $d$-dimensional simple Poisson modules.

**Proof.** By Lemma 2.4, every simple Poisson $A$-modules is annihilated by a maximal Poisson ideal of $A$. There are five maximal Poisson ideals by Lemma 3.2. The result is a corollary of Theorem 4.10 and Theorem 5.3. \qed

**References**


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