Constructing Hom-Poisson Color Algebras

Ibrahima Bakayoko

University of N’Zérékoré, BP: 50, N’Zérékoré, Guinea

Binko Mamady Touré

University of N’Zérékoré, BP: 50, N’Zérékoré, Guinea

Abstract

We give some constructions of Hom-Poisson color algebras first from a Hom-associative color algebra which twisting map is an averaging operator, then from a given Hom-Poisson color algebra and an averaging operator and finally from a Hom-post-Poisson color algebra. Then we show that any Hom-pre-Poisson color algebra leads to a Hom-Poisson color algebra. Conversely, we prove that any Hom-Poisson color algebra turn to a Hom-pre-Poisson color algebra via Rota-Baxter operator.

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1 Introduction

Hom-algebraic structures were appeared for the first time in the works of Aizawa N. and Sato H. [14] as a generalization of Lie algebras. Other interesting Hom-type algebraic structures of many classical structures were studied as G-Hom-associative algebras [3], Hom-alternative algebras, Hom-Malcev algebras and Hom-Jordan algebras [7]. However, Post-Lie algebras first arise from the work of Bruno Vallette [4] in 2007 through the purely operadic technique of
Koszul dualization. Some examples and related structures are given. Among other results, the authors proved in [5] that tridendriform and Rota-Baxter Lie algebras lead to the structures of post-Lie algebras. Post-Poisson algebras [6] are algebraic structures containing post-Lie algebras [4], [9] and pre-Poisson algebras [13] which contain pre-Lie algebras (or left-symmetric algebras) and Zinbiel algebras, both structures being connected by some compatibility conditions.

Classical (or ordinary) algebras were extended to generalized (or color or graded) algebras, among which one can cite: Simple Jordan color algebras arising from associative graded algebras [11], Representations and cocycle twists of color Lie algebras [16], On the classification of 3-dimensional coloured Lie algebras [15]. These structures are well-known to physicists and to mathematicians studying differential geometry and homotopy theory. They were extended to the Hom-setting by studying Hom-Lie superalgebras, Rota-Baxter operator on pre-Lie superalgebras and beyond [8] and Hom-Lie color algebras [12]. Color Hom-Poisson algebras were introduced in [10] as generalization of Hom-Poisson algebras [1].

The aim of this paper is to introduce and study the relationship among commutative Hom-tridendriform color algebras and Hom-post-Poisson color algebras. The paper is organized as follows. In section 2, we recall definition of Hom-Poisson color algebras and define averaging operator on Hom-associative and Hom-Lie color algebras. We introduce Hom-post-Lie color algebras, and give some procedures of construction of Hom-post-Lie color algebras form either a given one or from Hom-Lie Rota-Baxter color algebras. The section 3 is devoted to the main results of this paper i.e. some constructions of Hom-Poisson color algebras from other algebraic structures.

Throughout this paper, all graded vector spaces are assumed to be over a field $\mathbf{K}$ of characteristic different from 2.

## 2 Preliminaries

We give basic notions and give some of their properties.

Let $G$ be an abelian group. A vector space $V$ is said to be a $G$-graded if, there exists a family $(V_a)_{a \in G}$ of vector subspaces of $V$ such that

$$V = \bigoplus_{a \in G} V_a.$$  

An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$. We denote $\mathcal{H}(V)$ the set of all homogeneous elements in $V$.

Let $V = \oplus_{a \in G} V_a$ and $V' = \oplus_{a \in G} V'_a$ be two $G$-graded vector spaces. A linear mapping $f : V \to V'$ is said to be homogeneous of degree $b$ if

$$f(V_a) \subseteq V'_{a+b}, \forall a \in G.$$
If, $f$ is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$, then $f$ is said to be even.

An algebra $(A, \mu)$ is said to be $G$-graded if its underlying vector space is $G$-graded i.e. $A = \bigoplus_{a \in G} A_a$, and if furthermore $\mu(A_a, A_b) \subseteq A_{a+b}$, for all $a, b \in G$. Let $A'$ be another $G$-graded algebra. A morphism $f : A \to A'$ of $G$-graded algebras is by definition an algebra morphism from $A$ to $A'$ which is, in addition an even mapping.

**Definition 2.1** Let $G$ be an abelian group. A map $\varepsilon : G \times G \to K^*$ is called a skew-symmetric bicharacter on $G$ if the following identities hold,

(i) $\varepsilon(a, b)\varepsilon(b, a) = 1$,

(ii) $\varepsilon(a, b+c) = \varepsilon(a, b)\varepsilon(a, c)$,

(iii) $\varepsilon(a+b, c) = \varepsilon(a, c)\varepsilon(b, c)$,

$a, b, c \in G$.

Observe that

$$\varepsilon(a, e) = \varepsilon(e, a) = 1, \quad \varepsilon(a, a) = \pm 1 \quad \text{for all } a \in G,$$

where $e$ is the identity of $G$.

If $x$ and $y$ are two homogeneous elements of degree $a$ and $b$ respectively and $\varepsilon$ is a skew-symmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$.

**Definition 2.2** By a color Hom-algebra we mean a quadruple $(A, \mu, \varepsilon, \alpha)$ in which $A$ is a $G$-graded vector space, $\mu : A \times A \to A$ is an even bilinear map, $\varepsilon : G \times G \to K^*$ is a bicharacter and $\alpha : A \to A$ is an even linear map.

**Definition 2.3** A Hom-associative color algebra is a color Hom-algebra $(A, \mu, \varepsilon, \alpha)$ such that

$$as_\mu(x, y, z) = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = 0,$$

for any $x, y, z \in A$.

If in addition $\mu = \varepsilon(\cdot, \cdot)\mu^{op}$ i.e. $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$, for any $x, y \in \mathcal{H}(A)$, the Hom-associative color algebra $(A, \mu, \varepsilon, \alpha)$ is said to be a commutative Hom-associative color algebra.

**Remark** : When $\alpha = Id$, we recover the classical associative color algebra.
Definition 2.4 A Hom-Lie color algebra is a color Hom-algebra \((A, [\cdot, \cdot], \varepsilon, \alpha)\) such that
\[
[x, y] = -\varepsilon(x, y)[y, x],
\]
\[
\varepsilon(z, x)[\alpha(x), [y, z]] + \varepsilon(x, y)[\alpha(y), [z, x]] + \varepsilon(y, z)[\alpha(z), [x, y]] = 0,
\]
for any \(x, y, z \in \mathcal{H}(A)\).

By the \(\varepsilon\)-skew-symmetry (2), the \(\varepsilon\)-Hom-Jacobi identity (3) is equivalent to
\[
[\alpha(z), [x, y]] + \varepsilon(x + z, y)[\alpha(y), [z, x]] + \varepsilon(z + y, x)[\alpha(x), [y, z]] = 0.
\]

Example 2.5 It is clear that Lie color algebras are examples of Hom-Lie color algebras by setting \(\alpha = \text{id} \). If, in addition, \(\varepsilon(x, y) = 1\) or \(\varepsilon(x, y) = (-1)^{|x||y|}\), then the Hom-Lie color algebra is nothing but a classical Lie algebra or Lie superalgebra. Hom-Lie algebras and Hom-Lie superalgebras are also obtained when \(\varepsilon(x, y) = 1\) and \(\varepsilon(x, y) = (-1)^{|x||y|}\) respectively. See [12] for other examples.

Proposition 2.6 Let \((A, [\cdot, \cdot], \varepsilon, \alpha_1)\) and \((L, \{-, -\}, \varepsilon, \alpha_2)\) be a commutative Hom-associative color algebra and a Hom-Lie color algebra respectively. Then the tensor product \(A \otimes L\) endowed with the even linear map \(\alpha = \alpha_1 \otimes \alpha_2 : A \otimes L \to A \otimes L\) and the even bilinear map \(\{-, -\} : (A \otimes L) \times (A \otimes L) \to A \otimes L\) defined, for any \(a, b \in \mathcal{H}(A), x, y \in \mathcal{H}(L)\), by
\[
\alpha(a \otimes x) := \alpha_1(a) \otimes \alpha_2(x),
\]
\[
\{a \otimes x, b \otimes y\} := \varepsilon(x, b)(a \cdot b) \otimes [x, y],
\]
is a Hom-Lie color algebra.

Proof : The proof follows from a direct computation; it is essentially based on the fact that \(A\) is a commutative Hom-associative color algebra.

Definition 2.7 A Hom-post-Lie color algebra \((L, [-, -], \cdot, \varepsilon, \alpha)\) is a Hom-Lie color algebra \((L, [-, -], \varepsilon, \alpha)\) together with an even bilinear map \(\cdot : L \otimes L \to L\) such that
\[
\alpha(z) \cdot [x, y] - [z \cdot x, \alpha(y)] - \varepsilon(z, x)[\alpha(x), z \cdot y] = 0,
\]
\[
as(x, y, z) - \varepsilon(x, y)as(y, x, z) = -[x, y] \cdot \alpha(z),
\]
for any \(x, y, z \in \mathcal{H}(L)\).

If in addition, \(\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)\) and \(\alpha([x, y]) = [\alpha(x), \alpha(y)]\), then \((L, [-, -], \cdot, \varepsilon, \alpha)\) is said to be a multiplicative Hom-post-Lie color algebra.
Example 2.8 A color post-Lie algebra is a Hom-post-Lie color algebra with \( \alpha = \text{Id} \).

We need the following definition in the next proposition.

Definition 2.9 A Rota-Baxter Hom-Lie color algebra of weight \( \lambda \in K \) is a Hom-Lie color algebra \((L, [-,-], \varepsilon, \alpha)\) together with an even linear map \( R : L \to L \) that satisfies the identities

\[
R \circ \alpha = \alpha \circ R,
\]

\[
[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y]),
\]

for all \( x, y \in \mathcal{H}(L) \).

Proposition 2.10 Let \((L, [-,-], \varepsilon, \alpha, R)\) be a Rota-Baxter Hom-Lie color algebra of weight 1. Then \((L, [-,-], \cdot, \varepsilon, \alpha)\) is a Hom-post-Lie color algebra, where \( x \cdot y = [R(x), y] \), for any \( x, y \in \mathcal{H}(L) \).

Proof: For any \( x, y, z \in \mathcal{H}(L) \), we have

\[
\alpha(x) \cdot (y \cdot z) - \varepsilon(x, y)\alpha(y) \cdot (x \cdot z) + \varepsilon(x, y)(y \cdot x) \cdot \alpha(z) - (x \cdot y) \cdot \alpha(z) + \varepsilon(x, y)[y, x] \cdot \alpha(z) =
\]

\[
= [R(\alpha(x)), [R(y), z]] - \varepsilon(x, y)[R(\alpha(y)), [R(x), z]] + \varepsilon(x, y)[R([R(y), x]), \alpha(z)]
\]

\[
- [R([R(x), y]), \alpha(z)] + \varepsilon(x, y)[R([y, x]), \alpha(z)]
\]

\[
= [\alpha(R(x)), [R(y), z]] - \varepsilon(x, y)[\alpha(R(y)), [R(x), z]] - [R([x, R(y)]), \alpha(z)]
\]

\[
- [R([R(x), y]), \alpha(z)] - [R([x, y]), \alpha(z)]
\]

The last line vanishes by Rota-Baxter identity (8). Next,

\[
\alpha(z) \cdot [x, y] - [z \cdot x, \alpha(y)] - \varepsilon(z, x)[\alpha(x), z \cdot y] =
\]

\[
= [R(\alpha(z)), [x, y]] - [[R(z), x], \alpha(y)] - \varepsilon(z, x)[\alpha(x), [R(z), y]]
\]

\[
= [\alpha(R(z)), [x, y]] + \varepsilon(x + z, y)[\alpha(y), [R(z), x]] + \varepsilon(z, x + y)[\alpha(x), [y, R(z)]]
\]

The last line vanishes by using the \( \varepsilon \)-Hom-Jacobi identity (4).

Proposition 2.11 Let \((L, [-,-], \cdot, \varepsilon, \alpha)\) be a Hom-post-Lie color algebra. Let us define the products \( \cdot : L \times L \to L \) and \( [-,-]' : L \times L \to L \) as

\[
x \cdot y := x \cdot y + [x, y], \quad [x, y]' := -[x, y].
\]

Then, \((L, [x, y]', \cdot, \varepsilon, \alpha)\) is also a Hom-post-Lie color algebra.
Proof: The skew-symmetry of the bracket $[-,-]'$ is obvious. Prove (6) for $'$ and $[-,-]'$. For any $x,y,z \in \mathcal{H}(L)$,

$$
\alpha(z) \cdot \ [x,y]' - [z \cdot x, \alpha(y)]' - \varepsilon(z,x)[\alpha(x), z \cdot y]' = \\
= -\alpha(z) \cdot [x,y] + [z \cdot x, \alpha(y)] + \varepsilon(z,x)[\alpha(x), z \cdot y] \\
= -\alpha(z) \cdot [x,y] - [\alpha(z), [x,y]] + [z \cdot x, \alpha(y)] \\
+[[z,x], \alpha(y)] + \varepsilon(z,x)[\alpha(x), z \cdot y] + \varepsilon(z,x)[\alpha(x), [z,y]].
$$

Using axiom (6) and $\varepsilon$-Hom-Jacobi identity, it is immediate to see that (6) holds.

Now, we prove (5) for $'$ and $[-,-]'$. For any $x,y,z \in \mathcal{H}(L)$,

$$
\alpha(z) \cdot \ (y \cdot x) - \varepsilon(z,y)\alpha(y) \cdot (z \cdot x) + \varepsilon(z,y)(y \cdot x) \cdot \alpha(x) \\
-(z \cdot y) \cdot \alpha(x) + \varepsilon(z,y)[y,z]' \cdot \alpha(x) = \\
= \alpha(z) \cdot (y \cdot x + [y,x]) + [\alpha(z), y \cdot x + [y,x]] - \varepsilon(z,y)\alpha(y) \cdot (z \cdot x + [z,x]) \\
-\varepsilon(z,y)[\alpha(y), z \cdot x + [z,x]] + \varepsilon(z,y)(y \cdot z + [y,z]) \cdot \alpha(x) \\
+\varepsilon(z,y)[y \cdot z + [y,z], \alpha(x)] - (z \cdot y + [z,y]) \cdot \alpha(x) - [z \cdot y + [z,y], \alpha(x)] \\
-\varepsilon(z,y)[y,z] \cdot \alpha(x) - \varepsilon(z,y)[[y,z], \alpha(x)].
$$

The left hand side of the last equality vanishes by axioms in Definition 2.7.

**Definition 2.12** A non-commutative Hom-Poisson color algebra consists of a $G$-graded vector space $A$, a multiplication $\mu : A \times A \rightarrow A$, an even bilinear bracket $\{\cdot,\cdot\} : A \times A \rightarrow A$ and an even linear map $\alpha : A \rightarrow A$ such that:

1) $(A, \mu, \varepsilon, \alpha)$ is a Hom-associative color algebra,

2) $(A, \{\cdot,\cdot\}, \varepsilon, \alpha)$ is a Hom-Lie color algebra,

3) the Hom-Leibniz color identity

$$
\{\alpha(x), \mu(y,z)\} = \mu(\{x,y\}, \alpha(z)) + \varepsilon(x,y)\mu(\alpha(y), \{x,z\}),
$$

is satisfied for any $x,y,z \in \mathcal{H}(A)$.

A non-commutative Hom-Poisson color algebra $(A, \mu, \{\cdot,\cdot\}, \varepsilon, \alpha)$ in which $\mu$ is $\varepsilon$-commutative is said to be a commutative Hom-Poisson color algebra.

A Rota-Baxter operator of weight $\lambda$ on a (non-)commutative Hom-Poisson color algebra $(A, \cdot, [-,-], \varepsilon, \alpha)$ is an even linear map that commutes with $\alpha$ and satisfies (8) for both products $\cdot$ and $[-,-]$.

Any Hom-associative color algebra carries a structure of a non-commutative Hom-Poisson color algebra with the $\varepsilon$-commutator bracket. More precisely:
Lemma 2.13 [10] Let \((A, \mu, \varepsilon, \alpha)\) be a Hom-associative color algebra. Then \((A, \mu, \{\cdot, \cdot\} = \mu - \varepsilon(\cdot, \cdot)\mu^{op}, \varepsilon, \alpha)\) is a non-commutative Hom-Poisson color algebra.

Now we have the following definitions.

Definition 2.14 A commutative Hom-tridendriform color algebra is a quintuple \((T, \star, \ast, \varepsilon, \alpha)\) in which \((T, \ast, \varepsilon, \alpha)\) is a commutative Hom-associative color algebra and \(\star : T \otimes T \rightarrow T\) is an even bilinear operation such that:

\[
(x \ast y) \ast \alpha(z) = \alpha(x) \ast (y \ast z),
\]

\[
(x \ast y + \varepsilon(x, y) y \ast x + x \ast y) \ast \alpha(z) = \alpha(x) \ast (y \ast z),
\]

for any \(x, y, z \in \mathcal{H}(T)\).

Definition 2.15 A Hom-post-Poisson color algebra is a \(G\)-graded vector space \(P\) equipped with four even bilinear maps \([-,-], \cdot, \ast, \varepsilon, \alpha\) such that

1) \((P, [-,-], \cdot, \ast, \varepsilon, \alpha)\) is a Hom-post-Lie color algebra,

2) \((P, \ast, \ast, \varepsilon, \alpha)\) is a commutative Hom-tridendriform color algebra, and they are compatible in the sense that

3)

\[
[\alpha(x), y \ast z] = [x, y] \ast \alpha(z) + \varepsilon(x, y) \alpha(y) \ast [x, z],
\]

\[
\alpha(x) \ast [y, z] = (x \cdot y) \ast \alpha(z) + \varepsilon(x, y) \alpha(y, x \ast z],
\]

\[
\alpha(x) \cdot (y \ast z) = (x \cdot y) \ast \alpha(z) + \varepsilon(x, y) \alpha(y) \ast (x \cdot z),
\]

\[
(x \ast y + \varepsilon(x, y)y \ast x + x \ast y) \alpha(z) = \alpha(x) \ast (y \ast z) + \varepsilon(x, y) \alpha(y) \ast (xz),
\]

\[
(xy - \varepsilon(x, y)yx + [x, y]) \ast \alpha(z) = \alpha(x)(y \ast z) - \varepsilon(x, y) \alpha(y) \ast (xz),
\]

for any \(x, y, z \in \mathcal{H}(P)\).

Remark that \((P, \ast, [-,-], \varepsilon, \alpha)\) is a commutative Hom-Poisson color algebra.

Definition 2.16 A Hom-pre-Poisson color algebra is a quintuple \((A, \ast, \cdot, \varepsilon, \alpha)\) in which

1) \((A, \ast, \cdot, \varepsilon, \alpha)\) is a Hom-Zinbiel color algebra i.e.

\[
(x \ast y) \ast \alpha(z) + \varepsilon(x, y)(y \ast x) \ast \alpha(z) = \alpha(x) \ast (y \ast z),
\]

for any \(x, y, z \in \mathcal{H}(P)\).
2) \((A, \cdot, \varepsilon, \alpha)\) is a Hom-left-symmetric color algebra i.e.
\[
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = \varepsilon(x, y) \left( (y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z) \right) \quad (17)
\]

3) and the following two conditions :
\[
(x \cdot y - \varepsilon(x, y)y \cdot x) \star \alpha(z) = \alpha(x) \cdot (y \star z) - \varepsilon(x, y)\alpha(y) \star (x \cdot z), \quad (18)
\]
\[
(x \star y + \varepsilon(x, y)y \star x) \cdot \alpha(x) = \alpha(x) \star (y \cdot z) + \varepsilon(x, y)\alpha(y) \star (x \cdot z), \quad (19)
\]

are satisfied for any \(x, y, z \in \mathcal{H}(A)\).

**Definition 2.17**
1) An averaging operator over a Hom-associative color algebra \((A, \mu, \varepsilon, \alpha)\) is an even linear map \(\beta : A \to A\) such that \(\alpha \circ \beta = \beta \circ \alpha\) and
\[
\beta(\mu(x, y)) = \mu(\beta(x), \beta(y)) = \beta(\mu(x, \beta(y))), \quad (20)
\]
for all \(x, y \in \mathcal{H}(A)\).

2) An averaging operator over a Hom-Lie color algebra \((A, [\cdot, \cdot], \varepsilon, \alpha)\) is an even linear map \(\beta : A \to A\) such that \(\alpha \circ \beta = \beta \circ \alpha\) and
\[
\beta([\beta(x), y]) = [\beta(x), \beta(y)] = \beta([x, \beta(y)]), \quad (21)
\]
for all \(x, y \in \mathcal{H}(A)\).

For example, any differential operator \(d : A \to A\) (i.e a derivation \(d\) such that \(d^2 = 0\)) is an averaging operator.

**Proposition 2.18** Let \((A, [\cdot, \cdot], \varepsilon)\) be a Lie color algebra and \(\alpha : A \to A\) an averaging operator. Define a new operation \(\{\cdot, \cdot\} : A \times A \to A\) by
\[
\{x, y\} := [\alpha(x), y].
\]
Then \((A, \{\cdot, \cdot\}, \varepsilon, \alpha)\) is a Hom-Lie color algebra.

**Proof** : For any \(x, y, z \in \mathcal{H}(A)\),
\[
\{\alpha(x), \{y, z\}\} &= [[\alpha^2(x), [\alpha(y), z]] + [\alpha(x), \alpha(y)]\alpha(z)] + \varepsilon(x, y)[\alpha^2(y), [\alpha(x), z]] \\
&= [\alpha([\alpha(x), y]), \alpha(z)] + \varepsilon(x, y)[\alpha^2(y), [\alpha(x), z]] \\
&= \{[\alpha(x), y], \alpha(z)\} + \varepsilon(x, y)\{\alpha(y), [\alpha(x), z]\} \\
&= \\{\{x, y\}, \alpha(z)\} + \varepsilon(x, y)\{\alpha(y), \{x, z\}\}.
\]
3 Main results

**Theorem 3.1** Let \((A, \cdot, \alpha, \varepsilon)\) be a Hom-associative color algebra such that \(\alpha : A \to A\) be an averaging operator on \(A\). Then \((A, \mu, \{\cdot, \cdot\} = \mu - \varepsilon(\cdot, \cdot) \mu^{op}, \varepsilon, \alpha)\) is a Hom-Poisson color algebra, where \(\mu(x, y) = \alpha(x) \cdot y\).

**Proof :** We only have to prove the color Hom-associativity. Then for any \(x, y, z \in \mathcal{H}(A)\), we have

\[
\begin{align*}
\text{as}_\alpha(x, y, z) &= (x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) \\
&= \alpha(\alpha(x) \cdot y) \cdot \alpha(z) - \alpha^2(x) \cdot (\alpha(y) \cdot z)
\end{align*}
\]

Then, for the Lie structure, we have for any \(x, y, z \in \mathcal{H}(A)\),

\[
\{\alpha(x), \{y, z\}\} = [\beta(\alpha(x)), [\beta(y), z]] = [\alpha(\beta(x)), [\beta(y), z]]
\]

The final conclusion follows from Lemma 2.13.

**Theorem 3.2** Let \((A, \cdot, [-, -], \varepsilon, \alpha)\) be a non-commutative Hom-Poisson color algebra and \(\beta : A \to A\) an averaging operator. Let us define two new operations \(* : A \times A \to A\) and \(\{,\} : A \times A \to A\) by

\[
x \ast y := \beta(x) \cdot y, \quad \{x, y\} := [\beta(x), y].
\]

Then \((A, *, \{\cdot, \cdot\}, \varepsilon, \alpha)\) is also a non-commutative Hom-Poisson color algebra.

**Proof :** First, let us prove the color Hom-associativity; for any \(x, y, z \in \mathcal{H}(A)\), we have

\[
\begin{align*}
\text{as}_*(x, y, z) &= (x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) \\
&= \beta(\beta(x) \cdot y) \cdot \alpha(z) - \beta(\alpha(x)) \cdot (\beta(y) \cdot z)
\end{align*}
\]

Then, for the Lie structure, we have for any \(x, y, z \in \mathcal{H}(A)\),

\[
\{\alpha(x), \{y, z\}\} = [\beta(\alpha(x)), [\beta(y), z]] = [\alpha(\beta(x)), [\beta(y), z]]
\]
Finally, let us prove the compatibility condition. For any \( x, y, z \in \mathcal{H}(A) \),
\[
\begin{align*}
\{\alpha(x), y \ast z\} &= [\beta(\alpha(x)), \beta(y) \cdot z] = [\alpha(\beta(x)), \beta(y) \cdot z] \\
&= [\beta(x), \beta(y)] \cdot \alpha(z) + \varepsilon(x, y)\alpha(\beta(y)) \cdot [\beta(x), z] \\
&= \beta([\beta(x), y]) \cdot \alpha(z) + \varepsilon(x, y)\beta(\alpha(y)) \cdot [\beta(x), z] \\
&= \beta([\{x, y\}] \cdot \alpha(z) + \varepsilon(x, y)\beta(\alpha(y)) \cdot \{x, z\} \\
&= \{x, y\} \ast \alpha(z) + \varepsilon(x, y)\alpha(y) \ast \{x, y\}.
\end{align*}
\]

This completes the proof.

**Lemma 3.3** Let \((A, \ast, \ast, \varepsilon, \alpha)\) be a commutative Hom-tridendriform color algebra. Define the even bilinear map \(\odot : A \times A \to A\) by
\[
x \odot y := x \ast y + \varepsilon(x, y)y \ast x + x \ast y.
\]

Then \((A, \odot, \varepsilon, \alpha)\) is a commutative Hom-associative color algebra.

**Proof:** It is clear that the product \(\odot\) is \(\varepsilon\)-commutative. Now, we prove the color Hom-associativity. One has for any \(x, y, z \in \mathcal{H}(A)\),
\[
as_{\odot}(x, y, z) = (x \odot y) \odot \alpha(z) - \alpha(x) \odot (y \odot z) \\
= (x \ast y + \varepsilon(x, y)y \ast x + x \ast y) \ast \alpha(z) \\
+ \varepsilon(x + y, z)\alpha(z) \ast (x \ast y + \varepsilon(x, y)y \ast x + x \ast y) \\
+ (x \ast y + \varepsilon(x, y)y \ast x + x \ast y) \ast \alpha(z) \\
- \alpha(x) \ast (y \ast z + \varepsilon(y, z)z \ast y + y \ast z) \\
- \varepsilon(x, y + z)(y \ast z + \varepsilon(y, z)z \ast y + y \ast z) \ast \alpha(x) \\
- \alpha(x) \ast (y \ast z + \varepsilon(y, z)z \ast y + y \ast z).
\]

By expanding the right hand side, we get
\[
as_{\odot}(x, y, z) = (x \ast y) \ast \alpha(z) + \varepsilon(x, y)(y \ast x) \ast \alpha(z) + (x \ast y) \ast \alpha(z)
\]
\[
+ \varepsilon(x + y, z)\alpha(z) \ast (x \ast y) + \varepsilon(x, y)\varepsilon(x + y, z)\alpha(z) \ast (y \ast x)
\]
\[
+ \varepsilon(x + y, z)\alpha(z) \ast (x \ast y) + (x \ast y) \ast \alpha(z) + \varepsilon(x, y)(y \ast x) \ast \alpha(z)
\]
\[
+ (x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) - \varepsilon(y, z)\alpha(x) \ast (z \ast y) - \alpha(x) \ast (y \ast z)
\]
\[
- \varepsilon(x, y + z)(y \ast z) \ast \alpha(x) - \varepsilon(x, y + z)\varepsilon(y, z)(z \ast y) \ast \alpha(x)
\]
\[
- \varepsilon(x, y + z)(y \ast z) \ast \alpha(x) - \alpha(x) \ast (y \ast z) - \varepsilon(y, z)\alpha(x) \ast (z \ast y)
\]
\[
- \alpha(x) \ast (y \ast z).
\]
By (10) and the $\varepsilon$-commutativity, $a_1 + a_2 = 0$ and $c_1 + c_2 + c_3 = 0$. By (9) and the $\varepsilon$-commutativity, $d_1 + d_2 = 0$ and $e_1 + e_2 = 0$. And by the color Hom-associativity, $g_1 + g_2 = 0$. It follows that

$$as_{\circ}(x, y, z) = \varepsilon(x + y, z)\alpha(z) * (x * y) + \varepsilon(x, y)(y * x) * \alpha(z)$$

$$- \varepsilon(y, z)\alpha(x) * (z * y) - \alpha(x) * (y * z).$$

Now, by adding and subtracting pairwise below last eight terms, we obtain

$$as_{\circ}(x, y, z) = \varepsilon(x + y, z)\alpha(z) * (x * y) + \varepsilon(x, y)(y * x) * \alpha(z) - \varepsilon(y, z)\alpha(x) * (z * y)$$

$$- \alpha(x) * (y * z) + \varepsilon(y, z)(x * z) * \alpha(y) - \varepsilon(y, z)(x * z) * \alpha(y)$$

$$+ \varepsilon(x + y, z)(z * x) * \alpha(y) - \varepsilon(x + y, z)(z * x) * \alpha(y)$$

$$+ \varepsilon(x + y, z)(z * x) * \alpha(y) - \varepsilon(x + y, z)(z * x) * \alpha(y)$$

$$+ \varepsilon(x, y)\alpha(y) * (x * z) - \varepsilon(x, y)\alpha(y) * (x * z).$$

By (10) and the $\varepsilon$-commutativity, $b_1 + b_2 + b_3 + b_4 = 0$ and $h_1 + h_2 + h_3 + h_4 = 0$. By (9) and the $\varepsilon$-commutativity, $f_1 + f_2 = 0$ and $i_1 + i_2 = 0$.

This completes the proof.

**Lemma 3.4** Let $(A, [-,-], \cdot, \varepsilon, \alpha)$ be a Hom-post-Lie color algebra. Define the even bilinear map $\{x, y\} : A \times A \to A$ by

$$\{x, y\} := x \cdot y - \varepsilon(x, y)y \cdot x + [x, y].$$

Then $(A, \{-,-\}, \varepsilon, \alpha)$ is a Hom-Lie color algebra.

**Proof:** It is clear that the bracket $\{-,-\}$ is $\varepsilon$-skew-symmetric. Then, for any $x, y, z \in \mathcal{H}(A)$,

$$\{\alpha(x), \{y, z\}\} = \{\alpha(x), y \cdot z - \varepsilon(y, z)z \cdot y + [y, z]\}$$

$$= \alpha(x) \cdot (y \cdot z - \varepsilon(y, z)z \cdot y + [y, z])$$

$$- \varepsilon(x, y + z)(y \cdot z - \varepsilon(y, z)z \cdot y + [y, z]) \cdot \alpha(x)$$

$$+ [\alpha(x), y \cdot z - \varepsilon(y, z)z \cdot y + [y, z]]$$

$$= \alpha(x) \cdot (y \cdot z) - \varepsilon(x, y + z)(y \cdot z) \cdot \alpha(x) + [\alpha(x), y \cdot z]$$

$$- \varepsilon(y, z)\alpha(x) \cdot (y \cdot z) + \varepsilon(y, z)\varepsilon(x, y + z)(y \cdot z) \cdot \alpha(x)$$

$$- \varepsilon(y, z)\alpha(x) \cdot (z \cdot y) + \alpha(x) \cdot [y, z] - \varepsilon(x, y + z)[y, z] \cdot \alpha(x)$$

$$+ [\alpha(x), [y, z]].$$
By expanding the cyclic sum, we get

$$
\sum_{x,y,z} \varepsilon(z,x)\{\alpha(x), \{y,z\}\} =
$$

$$
= \varepsilon(z,x)\alpha(x) \cdot (y \cdot z) - \varepsilon(x,y)(y \cdot z) \cdot \alpha(x) + \varepsilon(z,x)[\alpha(x), y \cdot z]
$$

$$
- \varepsilon(z,x)\varepsilon(y,z)\alpha(x) \cdot (z \cdot y) + \varepsilon(y,z)\varepsilon(x,y)(z \cdot y) \cdot \alpha(x) - \varepsilon(z,x)\varepsilon(y,z)[\alpha(x), z \cdot y]
$$

$$
+ \varepsilon(z,x)\alpha(x) \cdot [y,z] - \varepsilon(x,y)[y,z] \cdot \alpha(x) + \varepsilon(z,x)[\alpha(x), [y,z]]
$$

$$
+ \varepsilon(x,y)\alpha(y) \cdot (z \cdot x) - \varepsilon(y,z)(z \cdot x) \cdot \alpha(y) + \varepsilon(x,y)[\alpha(y), z \cdot x]
$$

$$
- \varepsilon(x,y)\varepsilon(z,x)\alpha(y) \cdot (z \cdot x) + \varepsilon(z,x)\varepsilon(y,z)(z \cdot x) \cdot \alpha(y)
$$

$$
- \varepsilon(z,x)\varepsilon(y,z)(x \cdot z) \cdot \alpha(y) - \varepsilon(z,x)\varepsilon(y,z)[\alpha(y), x \cdot z]
$$

$$
+ \varepsilon(x,y)\alpha(y) \cdot [z,x] - \varepsilon(y,z)[z,x] \cdot \alpha(y) + \varepsilon(x,y)[\alpha(y), [z,x]]
$$

$$
+ \varepsilon(y,z)\alpha(z) \cdot (x \cdot y) - \varepsilon(z,x)(x \cdot y) \cdot \alpha(z) + \varepsilon(y,z)[\alpha(z), x \cdot y]
$$

$$
- \varepsilon(y,z)\varepsilon(x,y)\alpha(z) \cdot (y \cdot x) + \varepsilon(x,y)\varepsilon(z,x)(y \cdot x) \cdot \alpha(z)
$$

$$
- \varepsilon(y,z)\varepsilon(x,y)[\alpha(z), y \cdot x] + \varepsilon(y,z)[\alpha(z), [x,y]]
$$

$$
- \varepsilon(z,x)[x,y] \cdot \alpha(z) + \varepsilon(y,z)[\alpha(z), [x,y]]
$$

The sums $a_1 + a_2 + a_3 + a_4 + a_5, b_1 + b_2 + b_3 + b_4 + b_5$ and $d_1 + d_2 + d_3 + d_4 + d_5$ vanish by (6) and the $\varepsilon$-skew-symmetry, $c_1 + c_2 + c_3, f_1 + f_2 + f_3$ and $e_1 + e_2 + e_3$ vanish by (5) and the $\varepsilon$-skew-symmetry, $g_1 + g_2 + g_3$ vanishes by (3).

It follows that $\sum_{x,y,z} \varepsilon(z,x)\{\alpha(x), \{y,z\}\} = 0$.

Now, we can state the following result.

**Theorem 3.5** Let $(P, [-,-], \cdot, \ast, \varepsilon, \alpha)$ be a Hom-post-Poisson color algebra. Define two new even bilinear operations

$$
x \odot y = x \ast y + \varepsilon(x,y)y \ast x + x \ast y \quad \text{and} \quad \{x,y\} = x \cdot y - \varepsilon(x,y)y \cdot x + [x,y]
$$

for any $x, y \in P$. Then $(P, \odot, \{-,-\}, \varepsilon, \alpha)$ is a commutative Hom-Poisson color algebra.

**Proof:** The commutative color Hom-associativity and the Hom-Lie color algebra structure follows from Lemma 3.3 and Lemma 3.4 respectively. Let us
prove the compatibility condition. One has for any \( x, y \in P \),

\[
\begin{align*}
\{\alpha(x), y \circ z\} - \{x, y\} \circ \alpha(z) - \varepsilon(x, y)\alpha(y) \circ \{x, z\} &= \\
\alpha(x) &\cdot (y \ast z + \varepsilon(y, z)z \ast y + y \ast z) \\
- \varepsilon(x, y + z)(y \ast z + \varepsilon(y, z)z \ast y + y \ast z) \cdot \alpha(x) \\
+[\alpha(x), y \ast z + \varepsilon(y, z)z \ast y + y \ast z] - (x \cdot y - \varepsilon(x, y)y \cdot x + [x, y]) \ast \alpha(z) \\
- \varepsilon(x + y, z)\alpha(z) \ast (x \cdot y - \varepsilon(x, y)y \cdot x + [x, y]) \\
-(x \cdot y - \varepsilon(x, y)y \cdot x + [x, y]) \ast \alpha(z) \\
- \varepsilon(x, y)\alpha(y) \ast (x \cdot z - \varepsilon(x, z)z \cdot x + [x, z]) \\
+\varepsilon(y, x + z)(x \cdot z - \varepsilon(x, z)z \cdot x + [x, z]) \ast \alpha(y) \\
+\alpha(y) \ast (x \cdot z - \varepsilon(x, z)z \cdot x + [x, z])
\end{align*}
\]

\[
= \alpha(x) \cdot (y \ast z) + \varepsilon(y, z)\alpha(x) \cdot (z \ast y) + \alpha(x) \cdot (y \ast z) - \varepsilon(x, y + z)(y \ast z) \cdot \alpha(x)
\]

\[
- \varepsilon(x, y + z)\varepsilon(y, z)(z \ast y) \cdot \alpha(x) - \varepsilon(x, y + z)(y \ast z) \cdot \alpha(x)
\]

\[
+ [\alpha(x), y \ast z] + \varepsilon(y, z)\alpha(x) \cdot z \ast y + [\alpha(x), y \ast z]
\]

\[
- (x \cdot y) \ast \alpha(z) + \varepsilon(x, y)(y \cdot x) \ast \alpha(z) - [x, y] \ast \alpha(z) - \varepsilon(x + y, z)\alpha(z) \ast (x \cdot y)
\]

\[
+ \varepsilon(x + y, z)\varepsilon(x, y)\alpha(z) \ast (y \cdot x) - \varepsilon(x + y, z)\alpha(z) \ast [x, y] - (x \cdot y) \ast \alpha(z)
\]

\[
+ \varepsilon(x, y)(y \cdot x) \ast \alpha(z) - [x, y] \ast \alpha(z) - \varepsilon(x, y)\alpha(y) \ast (x \cdot z)
\]

\[
+ \varepsilon(x, y + z)\alpha(y) \ast (z \cdot x) - \varepsilon(x, y)\alpha(y) \ast [x, z]
\]

\[
- \varepsilon(y, z)(x \cdot z) \ast \alpha(y) + \varepsilon(x + y, z)(z \cdot x) \ast \alpha(y) - \varepsilon(y, z)[x, z] \ast \alpha(y)
\]

\[
- \varepsilon(x, y)\alpha(y) \ast (x \cdot z) + \varepsilon(x, y + z)\alpha(y) \ast (z \cdot x) - \varepsilon(x, y)\alpha(y) \ast [x, z].
\]

Observe that \( a_1 + a_2 + a_3 = 0 \) and \( b_1 + b_2 + b_3 = 0 \) by (15), \( c_1 + c_2 + c_3 = 0 \) by (13), \( d_1 + d_2 + d_3 + d_4 = 0 \) by (14), \( e_1 + e_2 + e_3 = 0 \) and \( f_1 + f_2 + f_3 = 0 \) by (12) and \( \varepsilon \)-commutativity, \( g_1 + g_2 + g_3 = 0 \) by (11).

Therefore \( (P, \ast, \{\cdot, \cdot\}, \varepsilon, \alpha) \) is a commutative Hom-Poisson color algebra.

From the trivial associative and Lie products in the theorem 3.5 we obtain the following corollary.

**Corollary 3.6** Let \((A, \ast, \cdot, \varepsilon, \alpha)\) be a Hom-pre-Poisson color algebra. Then \((A, \ast, \{\cdot, \cdot\}, \varepsilon, \alpha)\) is a commutative Hom-Poisson color algebra, with \( x \ast y = x \ast y + \varepsilon(x, y)y \ast x \) and \( \{x, y\} = x \cdot y - \varepsilon(x, y)y \cdot x \), for any \( x, y \in \mathcal{H}(A) \).
The below lemma is the graded version of a result in [2].

**Lemma 3.7** Let \((L, [-, -], \varepsilon, \alpha, R)\) be a Rota-Baxter Hom-Lie color algebra of weight 0. Then \(L\) is a Hom-left-symmetric color algebra with

\[
x \cdot y = [R(x), y],
\]

for \(x, y \in \mathcal{H}(L)\).

**Proof:** For any \(x, y \in \mathcal{H}(L)\), we have

\[
as_{\alpha}(x, y, z) - \varepsilon(x, y) as_{\alpha}(y, x, z) = \]
\[
= (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) - \varepsilon(x, y)((y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z)) \]
\[
= [R([R(x), y]), \alpha(z)] - [R(\alpha(x)), [R(y), z]] \]
\[
- \varepsilon(x, y)([R([R(y), x]), \alpha(z)] - [R(\alpha(y)), [R(x), z]]).
\]

By the Rota-Baxter identity (8) and the \(\varepsilon\)-skew-symmetry of the bracket \([-,-]\), we obtain

\[
as_{\alpha}(x, y, z) - \varepsilon(x, y) as_{\alpha}(y, x, z) = \]
\[
= [R(x), R(y)], \alpha(z)] - [\alpha(R(x)), [R(y), z]] + \varepsilon(x, y)[\alpha(R(y)), [R(x), z]] \]
\[
- \varepsilon(x, z)(\varepsilon(y, z)[\alpha(z), [R(x), R(y)]] + \varepsilon(z, x)[\alpha(R(x)), [R(y), z]] \]
\[
+ \varepsilon(x, y)[\alpha(R(y)), [z, R(x)]]).
\]

The \(\varepsilon\)-Hom-Jacobi identity (3) finishes the proof.

**Proposition 3.8** Let \((A, \ast, [-, -], \varepsilon, \alpha)\) be a (non-)commutative Hom-Poisson color algebra and \(R : A \to A\) be a Rota-Baxter operator of weight 0 on \(A\). Define the two products \(\ast : A \times A \to A\) and \(\cdot : A \times A \to A\) by

\[
x \ast y := R(x) \ast y \quad \text{and} \quad x \cdot y := [R(x), y].
\]

Then \((A, \ast, \cdot, \varepsilon, \alpha)\) is a Hom-pre-Poisson color algebra.

**Proof:** The proof follows from a direct computation and Lemma 3.7.

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