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Singularities of Quasi Mappings from $\mathbb{R}^2$ to $\mathbb{R}^3$

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Abstract

We obtain a list of all simple classes of singularities of map germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ with respect to the quasi equivalence relation. A comparison between quasi and Mond's classes and their relations with boundary (Arnold's) and corner classes are discussed.

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1 Introduction

In [7], Vladimir Zakalyukin classified projections of hypersurfaces with respect to a new and special equivalence relation which he named quasi equivalence. The relation is more rough than the standard group of diffeomorphisms preserving a given projection [5]. The main difference with standard one is that the quasi equivalence does not preserve the sets of points in the same fibre (multiple singular points on one fibre can go to different fibres). In particular, it is aimed to control positions of the fibers containing the critical points of the projections and allow absolute freedom outside the critical locus.

In [1], the first steps in the study of the quasi equivalence of maps were taken within the approach similar to the one introduced by Zakalyukin [7]. Two situations were investigated there: curves (irreducible and reducible ones) in real spaces of any dimension and maps from plane to plane (see [4, 5] for the corresponding results for the $\mathcal{A}$-equivalence).

In the current paper we investigate the quasi equivalence relation of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. The corresponding results with respect to the $\mathcal{A}$-equivalence relation were obtained by David Mond [6]: A map germ from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^3, 0)$ is $\mathcal{A}$-simple if and only if it is $\mathcal{A}$-equivalent to one of the following germs

1. (x_1, x_2),
2. $\mathbf{S}_0 : (x_1, x_2^2, x_1x_2)$,
3. $\mathbf{S}_k^{\pm} : (x_1, x_2^2, x_2^3 \pm x_1^{k+1}x_2), k \geq 1$,
4. $\mathbf{B}_k^{\pm} : (x_1, x_2^2, x_1^2x_2 \pm x_2^{2k+1}), k \geq 2$,
5. $\mathbf{C}_k^{\pm} : (x_1, x_2^2, x_1x_2^3 \pm x_1^kx_2), k \geq 3$,
6. $\mathbf{F}_4 : (x_1, x_2^2, x_1^3x_2 + x_2^5)$,
7. $\mathbf{H}_k : (x_1, x_1x_2 + x_2^{3k-1}, x_2^3), k \geq 2$.

This classification was used later by many authors in various applications in differential and algebraic geometries. The further research which goes beyond the present work might be related to the interesting question to the applications of our classification in analogue way to that of standard classes.

The paper is organized as follows. In Section 2 we recall from [1] the main notions and basic results of the quasi equivalence relation of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. In Section 3, we study the case when $m = 2$ and $n = 3$ and classify simple classes. Finally, in Section 4 we compare between the quasi singularities and the corresponding results obtained by Mond.

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2 Quasi equivalence relation of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$

Consider a $\mathbf{C}^\infty$ map germ $F : (\mathbb{R}^m, 0) \to \mathbb{R}^n$, $x = (x_1, \ldots, x_m) \mapsto y = (y_1, \ldots, y_n)$, $y_i = f_i(x)$, where $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ is a smooth function-germ. Denote by $\mathbb{C}_n^m$ the space of all such maps. Let $\mathbb{R}^p = \{(x, y) : x \in \mathbb{R}^m, y \in \mathbb{R}^n\}$ and set $w = (x, y)$ for the whole set coordinates on $\mathbb{R}^p$ when the distinction between x and y is not crucial. Denote by $\mathbb{C}_w$ the ring of all $\mathbf{C}^\infty$ function germs $h_i : (\mathbb{R}^p, 0) \to \mathbb{R}$ and by $\mathbb{M}_w$ the maximal ideal in $\mathbb{C}_w$.

Since $\mathbb{C}_n^m$ is a vector space, sometimes its elements will be written as column vectors:

$$F = (f_1, f_2, \ldots, f_n)^t = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Consider the graph Λ_F of F, that is $\Lambda_F = \{(x, y) : y_i = f_i(x), i = 1, 2, \ldots, n\} \subset \mathbb{R}^p$, with trivial fibration structure $\pi : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(x, y) \mapsto y$.

Definition 2.1 [1] Two map germs F_0 and F_1 are called *quasi equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$, such that $\Phi(\Lambda_{F_1}) = \Lambda_{F_0}$ and the derivative of Φ preserves the direction of the projection at the points which lie on Λ_{F_1}.

Remarks 2.2

1. The quasi equivalence is an equivalence relation.
2. Clearly, if two map germs F_0 and F_1 are $\mathcal{A}$-equivalent then they are quasi equivalent.

Denote by Q_F the quasi equivalence class of a map germ F and call it a *quasi orbit*. Then, the tangent space TQ_F to Q_F has the following description.

Lemma 2.3 [1] *TQ_F is the set of all expressions of the form*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{pmatrix} + \begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \vdots \\ \dot{Y}_n \end{pmatrix},$$

where

$$\frac{\partial \dot{Y}_i}{\partial x_j} = \sum_{r=1}^{n} A_{ir} \frac{\partial f_r}{\partial x_j}, \quad \text{and} \quad \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_m \in \mathbb{C}_x,$$

with $A_{ir} \in \mathbb{C}_x$ for all i and j.

Let $\theta(F)$ be the set of all vector fields along F. Set $\theta_m = \theta(Id_m)$ and $\theta_n = \theta(Id_n)$, where Id_m and Id_n denote the germs at 0 of the identity maps on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Denote by $T\mathcal{A}_F$ the tangent space of the standard $\mathcal{A} -$ *orbit* at the map germ F. Recall that $T\mathcal{A}_F$ is equal to $tf(\theta_m) + \omega F(\theta_n)$, where $tf : \theta_n \to \theta(F), tF(\zeta) = dF \circ \zeta$ and $\omega F : \theta_n \to \theta(F)$, $\omega f(\eta) = \eta \circ F$.

Lemma 2.4 $T\mathcal{A}_F \subseteq TQ_F$.

Proof. Suppose that $v = v_1 + v_2 \in T\mathcal{A}_F$, where $v_1 \in tF(\theta_m)$ and $v_2 \in \omega F(\theta_n)$. Therefore, v_2 is the column matrix $(\dot{Y}_i)$, $i = 1, 2, \ldots, n$, where $\dot{Y}_i = U_i \circ F$ and $U_i \in \mathbb{M}_y$. Notice that $\frac{d\dot{Y}_i}{dx_j} = \sum_{r=1}^{m} \frac{df_r}{dx_j}\frac{\partial U_i}{\partial f_r}$. Setting $A_{ir} = \frac{\partial U_i}{\partial f_r}$ implies that $v \in TQ_F$, and the result follows. □

Due to the inclusion in Lemma (2.4), we have

Proposition 2.5 *If map germ F has a finite codimension with respect to the $\mathcal{A}$-equivalence, then F has a finite codimension with respect to the quasi equivalence.*

Following [2], we call a map germ $F : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ *simple* if its sufficiently small neighbourhood in the space of all map germs from $(\mathbb{R}^m, 0)$ to $(\mathbb{R}^n, 0)$ contains only a finite number of quasi equivalence classes.

3 Classification of simple maps from $\mathbb{R}^2$ to $\mathbb{R}^3$

We start this section with the techniques of the classification. Then, we recall the $\mathcal{A}$-classification of the 2-jets of map germs from $\mathbb{R}^2$ to $\mathbb{R}^3$, from [6]. After that, we introduce auxiliary results on prenormal forms with respect to the quasi equivalence relation. Finally, we classify simple quasi singularities, giving details of the proof.

3.1 Basic techniques

The finiteness of $\mathbb{C}_2^3 \big/ TQ_F$ implies finite determinacy of map germs under the quasi equivalence. Thus, a relevant version of Arnold's spectral sequence of semiquasihomogeneous maps [2] works for the normal form reduction. We recall the following related concepts.

- The components $f_1, f_2, \ldots$ and f_n of a map germ F generate an ideal in $\mathbb{C}_x$ formed from all linear combinations $u_1 f_1 + u_2 f_2 + \cdots + u_n f_n$ with coefficients u_i from $\mathbb{C}_x$. We shall denote this ideal by $I_F =< f_1, \ldots, f_n >$.

  Definition 3.1 [2] The quotient-algebra

  $$\mathbb{Q}(F) = \mathbb{C}_x / I_F$$

  is said to be *the local algebra* of the map F.

Definition 3.2 [2] The map F is said to be *non-degenerate* if its multiplicity at 0 is finite, that is if the local algebra $\mathbb{Q}(F)$ has finite dimension over $\mathbb{R}$; this dimension $\mu = dim_{\mathbb{R}}\mathbb{Q}(F)$ is said to be the multiplicity of F at 0.

- Let us fix the type of quasihomogeneity $\boldsymbol{\alpha}= (\alpha_1, \alpha_2)$ in the space $\mathbb{R}^2$ with local coordinates x_1 and x_2. Let $\boldsymbol{d}= (d_1, d_2, d_3)$ be a vector with nonnegative components.

Definition 3.3 [2] The map $F = \big(f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2)\big)$ is said to be *quasihomogeneous* of degree $\boldsymbol{d}$ (of type $\boldsymbol{\alpha}$) if every component f_i is a quasihomogeneous function of degree d_i of the same type $\boldsymbol{\alpha}$.

The map F is said to be *semiquasihomogeneous* map if $F = F_0 + \widetilde{F}$, where F_0 is a nondegenerate quasihomogeneous map and every component $\widetilde{f_i}$ of $\widetilde{F}$ has order greater than the degree of the corresponding component $F_{i(0)}$ of F_0. The map F_0 will be referred as the principal part of F.

3.2 The 2-jets

Recall that the k-jet $j^kF(0)$ of F at $0 \in \mathbb{R}^m$ is the Taylor expansion of F about the point 0 truncated at degree k. The k-jet space $J^k(m, n)$ is the vector space of all polynomial maps of degree k from $(\mathbb{R}^m, 0)$ to $(\mathbb{R}^n, 0)$. The group $\mathcal{A}$ acts on $J^k(m, n)$ and this action factors via the projection $\mathcal{A} \to \mathcal{A}^k$(the set of $k-$jets of elements of $\mathcal{A}$). The Lie group $\mathcal{A}^k$ acts smoothly on $J^k(m, n)$, and two $k-$jets are $\mathcal{A}^k-$*equivalent* if they lie in the same $\mathcal{A}^k$-orbit. The set of k-jets in $J^k(m, n)$ of corank i will be denoted by $\sum^i J^k(m, n)$.

Note that map germs $F : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ of corank one can be reduced by the $\mathcal{A}$-equivalence relation to the form:

$$(x_1, x_2) \mapsto \big(x_1, \varphi(x), \psi(x)\big), \quad \varphi, \psi \in \mathcal{M}_x^2.$$

Henceforth all germs of corank one and jets that we consider will be of this form.

The classification of the 2-jets of map germs $F : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ of corank one with respect to the $\mathcal{A}$-equivalence relation is given as follows.

Proposition 3.4 [6] *The table of adjacencies of the $\mathcal{A}^2$-orbits in $\sum^1 J^2(2, 3)$ is*

$$I\colon (x_1, x_2^2, x_1x_2) \leftarrow II\colon (x_1, x_2^2, 0) \leftarrow III\colon (x_1, x_1x_2, 0) \leftarrow V\colon (x_1, 0, 0).$$

Moreover,

Proposition 3.5 [6] *The 3-jets of map germs whose 2-jet is $\mathcal{A}$-equivalent to $(x_1, 0, 0)$ lie in the following $\mathcal{A}^3$-orbits:*

Orbit	*Codimension*	*Orbit*	*Codimension*
$(x_1, x_2^3, x_1^2x_2 + x_1x_2^2)$	*6*	$(x_1, x_1x_2^2, x_1^2x_2)$	*8*
$(x_1, x_2^3 \pm x_1^2x_2, x_1x_2^2)$	*6*	$(x_1, x_1x_2^2, 0)$	*9*
$(x_1, x_2^3, x_1^2x_2)$	*7*	$(x_1, x_2^3, 0)$	*9*
$(x_1, x_2^3, x_1x_2^2)$	*7*	$(x_1, x_1^2x_2, 0)$	*10*
$(x_1, x_2^3 + x_1^2x_2, 0)$	*8*	$(x_1, 0, 0)$	*12*

The classification of the 2-jets of these germs with respect to the standard $\mathcal{A}$-equivalence is described in the following.

Proposition 3.6 [6] The table of adjacencies of the $\mathcal{A}^2$-orbits in $\sum^2 J^2(2,3)$ is

$\mathbb{I} : (x_1^2, x_1x_2, x_2^2) \leftarrow (\mathbb{III})^{\pm} : (x_1x_2, x_1^2 \pm x_2^2, 0) \leftarrow \mathbb{IIII} : (x_1^2, x_1x_2, 0) \leftarrow \mathbb{V} : (x_1^2 \pm x_2^2, 0, 0) \leftarrow \mathbb{IV} : (x_1^2, 0, 0)$.

Remark 3.7 The normal forms of the k-jets in Proposition 3.4, Proposition 3.5 and Proposition 3.6 remain the same for the quasi equivalence due to the coincidence of the k-jets of the standard and quasi tangent spaces of their respective orbits .

3.3 Prenormal forms

Suppose that $F_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $(x_1, x_2) \mapsto \big(x_1, \varphi_t(x), \psi_t(x)\big)$, be a family of quasi equivalent map germs at the origin with respect to a smooth family Φ_t of germs of diffeomorphisms, $F_t(\Phi_t) = F_0$, preserving the first component of F_t, where $t \in [0, 1]$ and $\varphi_t, \psi_t \in \mathcal{M}_x^2$. So, in order to classify map germs of corank one, it is convenient to calculate the *stationary* algebra of the admissible vector fields $W_t = (\dot{X}_1, \dot{X}_2; \dot{Y}_1, \dot{Y}_2, \dot{Y}_3)$, the flow of which provides a quasi equivalence of the family F_t.

Lemma 3.8 *The stationary vector field W_t satisfies*

$$\begin{pmatrix} \dot{\varphi}_t \\ \dot{\psi}_t \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_t}{\partial x_1}\dot{X}_1 + \frac{\partial \varphi_t}{\partial x_2}\dot{X}_2 \\ \frac{\partial \psi_t}{\partial x_1}\dot{X}_1 + \frac{\partial \psi_t}{\partial x_2}\dot{X}_2 \end{pmatrix} + \begin{pmatrix} \dot{Y}_2 \\ \dot{Y}_3 \end{pmatrix},$$

where $\dot{\varphi}_t = \frac{\partial \varphi_t}{\partial t}$, $\dot{\psi}_t = \frac{\partial \psi_t}{\partial t}$, and its components have the forms

$$\dot{X}_1 = -\dot{Y}_1, \quad \dot{X}_2 \in \mathbb{C}_x, \quad \dot{Y}_i = h_i + \int_0^{x_2} \big(a_i \tfrac{\partial \varphi_t}{\partial x_2} + b_i \tfrac{\partial \psi_t}{\partial x_2}\big)\, dx_2,$$

with $h_i \in \mathbb{C}_{x_1}$ and $a_i, b_i \in \mathbb{C}_x$, $\forall i \in \{1,2,3\}$.

Proof. Differentiating $F_t(\Phi_t) = F_0$ with respect to t provides the following homological equation

$$\begin{pmatrix} 0 \\ \dot{\varphi}_t \\ \dot{\psi}_t \end{pmatrix} = \begin{pmatrix} \dot{X}_1 \\ \frac{\partial \varphi_t}{\partial x_1}\dot{X}_1 + \frac{\partial \varphi_t}{\partial x_2}\dot{X}_2 \\ \frac{\partial \psi_t}{\partial x_1}\dot{X}_1 + \frac{\partial \psi_t}{\partial x_2}\dot{X}_2 \end{pmatrix} + \begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \dot{Y}_3 \end{pmatrix}. \tag{1}$$

The first row of (1) imply that $\dot{X}_1 = -\dot{Y}_1$ and hence the first part of the Lemma follows. Clearly, we have $\dot{X}_2 \in \mathbb{C}_x$. On the other hand, the $\dot{Y}_i$ components satisfy the following quasi constraints

$$\frac{\partial \dot{Y}_i}{\partial x_1} = a_i + b_i \frac{\partial \varphi_t}{\partial x_1} + c_i \frac{\partial \psi_t}{\partial x_1} \tag{2}$$

and

$$\frac{\partial \dot{Y}_i}{\partial x_2} = b_i \frac{\partial \varphi_t}{\partial x_2} + c_i \frac{\partial \psi_t}{\partial x_2}, \tag{3}$$

where $a_i, b_i, c_i \in \mathbb{C}_x$, depending on t. Since a_i is an arbitrary germ, (2) and (3) imply that

$$\dot{Y}_i = h_i + \int_0^{x_2} \big(b_i \frac{\partial \varphi_t}{\partial x_2} + c_i \frac{\partial \psi_t}{\partial x_2}\big)\, dx_2,$$

where $h_i \in \mathbb{C}_{x_1}$, as required. □

Remark 3.9 The quasi classification of corank one germs $F : (x_1, x_2) \mapsto \big(x_1, \varphi(x), \psi(x)\big)$, at the origin, reduces to the classification of the orbits of the action of the stationary vector fields W on maps $F^* : (x_1, x_2) \mapsto \big(y_2 = \varphi(x), y_3 = \psi(x)\big)$. In this direction, the quasi tangent space to the orbit at F^* will be denoted by TQS_{F^*}.

Map germs can be reduced to simpler forms by quasi equivalence. We first treat the case germs with corank one.

Lemma 3.10 *Let F be a map germ whose 2-jet is $\mathcal{A}$-equivalent to II or III. Then, F is quasi equivalent to a map germ described in the following table:*

2-jet type	*Reduced form*	*Restrictions*
II	$\big(x_1, x_2^2, x_2\varphi(x_1)\big)$	$\varphi \in \mathbb{M}_{x_1}^2$.
III	$\big(x_1, x_1x_2 + \vartheta(x_2), \psi(x_2)\big)$,	$\vartheta, \psi \in \mathbb{M}_{x_2}^3$.

Proof. Consider the 2-jet of type II and let $F = \big(x_1, x_2^2 + h(x), g(x)\big)$, where $h, g \in \mathbb{M}_x^3$. Consider the principal part $F_0^* = \big(x_2^2, 0\big)$ of the semiquasihomogeneous map germ F^*. Using Remark 3.9, we have

$$\mathbb{C}_2^2 \Big/ TQS_{F_0^*} \equiv \left\{ \begin{pmatrix} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + x_2 q(x_1) \end{pmatrix} : \alpha_i, \beta_i \in \mathbb{R}, q \in \mathbb{M}_{x_1} \right\}.$$

Thus, Arnold's spectral method and considering the constraints in the lemma on h and g yield that F is quasi equivalent to a germ of the form $\big(x_1, x_2^2, x_2\varphi(x_1)\big)$, where $\varphi \in \mathbb{M}_{x_1}^2$, as claimed.

Similarly, we can prove the second claim of the Lemma. □

Lemma 3.11 *Let f be a map germ whose 3-jet is $\mathcal{A}$-equivalent to $\big(x_1, x_2^3, x_1x_2^2\big)$. Then, F is quasi equivalent to a map germ of the form $\big(x_1, x_2^3 + x_2\varphi(x_1), x_1x_2^2\big)$, where $\varphi \in \mathbb{M}_{x_1}^3$.*

Proof. Similar to proof of Lemma 3.10. □

Now, assume that f has corank two.

Lemma 3.12 *If the 2-jet of F has type $\mathbb{I}$, then F is quasi equivalent to (x_1^2, x_1x_2, x_2^2).*

Proof. Let $F_0 = (x_1^2, x_1x_2, x_2^2)$ be the principal part of F. Then, TQ_{F_0} is the set of all vectors of the form

$$\begin{pmatrix} 2x_1\dot{X}_1 + \dot{Y}_1 \\ x_2\dot{X}_1 + x_1\dot{X}_2 + \dot{Y}_2 \\ 2x_2\dot{X}_2 + \dot{Y}_3 \end{pmatrix},$$

where $\dot{X}_1, \dot{X}_2 \in \mathbb{C}_x$ and $\dot{Y}_i$ satisfies the following constraints

$$\frac{\partial \dot{Y}_i}{\partial x_1} = 2x_1 a_i + x_2 b_i + 0.c_i \tag{4}$$

and

$$\frac{\partial \dot{Y}_i}{\partial x_2} = 0.a_i + x_1 b_i + 2x_2 c_i, \tag{5}$$

for some $a_i, b_i, c_i \in \mathbb{C}_x$. By the Hadamrd Lemma, we can always write

$$b_i = \alpha_i + x_1 d_i + x_2 e_i,$$

where $d_i, e_i \in \mathbb{C}_x$ and $\alpha_i \in \mathbb{R}$. Thus, (4) and (5) can be written equivalently as

$$\frac{\partial \dot{Y}_i}{\partial x_1} = x_1 \widetilde{a}_i + x_2 \widetilde{e}_i \tag{6}$$

and

$$\frac{\partial \dot{Y}_i}{\partial x_2} = x_1 \widetilde{d}_i + x_2 \widetilde{c}_i, \tag{7}$$

respectively, where $\widetilde{a}_i = 2a_i + x_2 d_i, \widetilde{e}_i = \alpha_i + x_2 e_i, \widetilde{d}_i = \alpha_i + x_1 d_i$, and $\widetilde{c}_i = 2c_i + x_1 e_i$. Clearly, (6) and (7) imply that $\dot{Y}_i \in \mathcal{M}_x^2$. Thus, we have

$$\mathbb{C}_2^3 \Big/ TQ_{F_0} \equiv \left\{ \begin{pmatrix} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \\ \beta_0 + b_1 x_1 + \gamma_2 x_2 \\ \gamma_0 + \beta_1 x_1 + \beta_2 x_2 \end{pmatrix} : \alpha_i, \beta_i, \gamma_i \in \mathbb{R} \right\}.$$

Therefore, Arnold's spectral method imply that F is quasi equivalent to F_0 as claimed. □

3.4 Simple quasi maps

Classifications of simple quasi singularities of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is as follows.

Theorem 3.13 *Assume a map germ* $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is simple with respect to the quasi equivalence relation. Then,* F *is quasi equivalent to one of the following maps.*

Notation	*Normal form*	*Restrictions*	*Codimension*
$\mathbb{E}_1$	$(x_1, x_2, 0)$	–	*0*
$\mathbb{S}_0$	$(x_1, x_2^2, x_1 x_2)$	–	2
$\mathbb{S}_k$	$(x_1, x_2^2, x_1^{k+1} x_2)$	$k \geq 1$	$k+2$
$\mathbb{H}_k$	$(x_1, x_1 x_2, x_2^k)$	$k \geq 3$	$2(k-1)$
$\mathbb{H}_{l,k}$	$(x_1, x_1 x_2 + x_2^l, x_2^k)$	$k > l \geq 3$	$l+k-2$
$\mathbb{D}_k$	$(x_1, x_2^3 + x_1^k x_2, x_1 x_2^2)$	$k \geq 2$	$k+4$
$\mathbb{E}_2$	$(x_1, x_2^3, x_1^2 x_2 + x_1 x_2^2)$	–	*6*
$\mathbb{E}_3$	$(x_1, x_2^3, x_1^2 x_2)$	–	7
$\mathbb{E}_4$	$(x_1^2, x_1 x_2, x_2^2)$	–	*8*

Remarks 3.14 1. The classes $\mathbb{H}_k$ can be written equivalently as: $(x_1, x_1x_2 + x_2^k, x_2^k)$. So, we may include them in the series $\mathbb{H}_{l,k}$ as $\mathbb{H}_{k,k}$.

2. The codimension is nicely calculated. Each class can be put in the form $(x_1, x_2^i + x_1x_2^j, x_2^l + x_1^k x_2)$, for some values i, j, l and k. Then, the codimension is $i + j + l + k - 4$.

To prove Theorem 3.13, we need the following auxiliary results.

Lemma 3.15 *Map germs F whose 2-jet is equivalent to $(x_1x_2, x_1^2 \pm x_2^2, 0)$ are non-simple with respect to the quasi equivalence relation.*

Proof. Consider the principal part $F_0 = (x_1x_2, x_1^2 \pm x_2^2, h)$ of F, where h is a cubic polynomial in x_1 and x_2. Therefore, TQ_{F_0} is the set of all expressions of the form

$$\begin{pmatrix} x_2\dot{X}_1 + x_1\dot{X}_2 + \dot{Y}_1 \\ 2x_1\dot{X}_1 \pm 2x_2\dot{X}_2 + \dot{Y}_2 \\ \frac{\partial h}{\partial x_1}\dot{X}_1 + \frac{\partial h}{\partial x_2}\dot{X}_2 + \dot{Y}_3 \end{pmatrix}, \tag{8}$$

where $\dot{X}_1, \dot{X}_2 \in \mathbb{C}_x$ and $\dot{Y}_i$ satisfies the following constraints

$$\frac{\partial \dot{Y}_i}{\partial x_1} = x_2a_i + 2x_1b_i + \tfrac{\partial h}{\partial x_1}c_i \quad \text{and} \quad \frac{\partial \dot{Y}_i}{\partial x_2} = x_1a_i \pm 2x_2b_i + \tfrac{\partial h}{\partial x_2}c_i,$$

where $a_i, b_i, c_i \in \mathbb{C}_x$. The quadratic and cubic terms in the first, second and third row of (8) are obtained from

$$x_2\dot{X}_{1j}, \quad x_1\dot{X}_{2j}, \quad \gamma_{11}x_1x_2, \quad \gamma_{12}(x_1^2 \pm x_2^2), \quad \gamma_{13}h,$$

$$2x_1\dot{X}_{1j}, \quad \pm 2x_2\dot{X}_{2j}, \quad \gamma_{21}x_1x_2, \quad \gamma_{22}(x_1^2 \pm x_2^2), \quad \gamma_{23}h \quad \text{and}$$

$$\dot{X}_{1j}\tfrac{\partial h}{\partial x_1}, \quad \dot{X}_{2j}\tfrac{\partial h}{\partial x_2}, \quad \gamma_{31}x_1x_2, \quad \gamma_{32}(x_1^2 \pm x_2^2), \quad \gamma_{33}h,$$

respectively, where $\gamma_{11}, \dots, \gamma_{33} \in \mathbb{R}$ and $\dot{X}_{ij} = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \beta_{i1}x_1^2 + \beta_{i2}x_1x_2 + \beta_{i3}x_2^2$ with $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i = 1, 2$ and $j = 1, 2, \dots, 6$. These terms form a subspace of dimension at most 19. The dimension of the space of maps (x_1, J_1^3, J_2^3), where J_1^3 and J_2^3 are spaces of 3-jets with no linear part, is 21 which is greater than the subspace dimension. This means that the germ F_0 is non-simple with respect to the quasi equivalence relation. □

Lemma 3.16 *Map germs F whose 3-jet is equivalent to either $(x_1, x_2^3 + x_1^2x_2, 0)$ or $(x_1, x_1x_2^2, x_1^2x_2)$ are non-simple with respect to the quasi equivalence relation.*

Proof. The results are proved by using Remark 3.9 and following similar approach to proof of Lemma 3.17. □

3.4.1 Proof of the main Theorem 3.13

If F is an immersion, then F is $\mathcal{A}$-equivalent (and hence F is quasi equivalent) to $\mathbb{E}_1 : (x_1, x_2, 0)$. Next, assume that F has corank one and consider the $\mathcal{A}^2$-orbits in $\sum^1 J^2(2,3)$, described in Proposition 3.4 and ordered by increasing their codimensions. Then, using Arnold's spectral method, one can easily prove the following results.

- If $j^2F(0)$ is $\mathcal{A}^2-$equivalent to I, then F is $\mathcal{A}$-equivalent (and hence F is quasi equivalent) to $\mathbb{S}_0 : (x_1, x_2^2, x_1x_2)$ (see Theorem 4.3 in [6]).
- If $j^2F(0)$ is $\mathcal{A}^2-$equivalent to II or III, then Lemma 3.10 yields that F can be reduced to either $\big(x_1, x_2^2, x_2\varphi_1\big)$ or $(x_1, x_1x_2 + \vartheta_2, \varphi_2)$, respectively, where $\varphi_1 \in \mathcal{M}^2_{x_1}$ and $\vartheta_2, \varphi_2 \in \mathcal{M}^3_{x_2}$. Let $\alpha_i x_1^i$, $\beta_j x_2^j$ and $\gamma_l x_2^l$ be the lowest non zero terms in φ_1, ϑ_2 and φ_2, respectively. Then, F is quasi equivalent to one of the germs:
    1. $\mathbb{S}_{i-1} : (x_1, x_2^2, x_2x_1^i)$, $i \geq 2$.
    2. $\mathbb{H}_l : (x_1, x_1x_2, x_2^l)$, if $\quad j \geq l \geq 3$.
    3. $\mathbb{H}_{j,l} : (x_1, x_1x_2 + x_2^j, x_2^l)$, if $\quad l > j \geq 3$.
- If $j^3F(0)$ is $\mathcal{A}^3-$equivalent to one of the $\mathcal{A}^3$-orbits in $\sum^1 J^2(2,3)$ over $(x_1, 0, 0)$ described in Proposition 3.5, then Lemma 3.16 implies that simple singularities are among those whose codimension is either 6 or 7 in $J^3(2,3)$. In particular, the simple classes are the germs:
    1. $\mathbb{D}_k : (x_1, x_2^3 + x_1^k x_2, x_1x_2^2)$.
    2. $\mathbb{E}_2 : (x_1, x_2^3, x_1^2x_2 + x_1x_2^2)$.
    3. $\mathbb{E}_3 : (x_1, x_2^3, x_1^2x_2)$.

Finally, consider the remaining case of map germs of corank two. Lemma 3.12 and Lemma 3.17 imply that any simple germ is quasi equivalent to $\mathbb{E}_4 : (x_1^2, x_1x_2, x_2^2)$. This finishes the proof.

3.5 A comparison between quasi and Mond's classes and their relations with boundary (Arnold's) and corner classes

A part from the immersion case, All Mond's singularities are among germs whose 2-jets are equivalent to either $(x_1, x_2^2, 0)$ or $(x_1, x_1x_2, 0)$. In the first case, Lemma 4.1:4 in [6] implies that every germ having the 2-jet $(x_1, x_2^2, 0)$ is reduced to

$$\big(x_1, x_2^2, x_2\Lambda(x_1, x_2^2)\big), \tag{9}$$

where $\Lambda \in \mathbb{M}^2_{x_1,x_2^2}$. The method which was used to classify (9) is done via diffeomorphisms preserving the Whiteny fold $(x_1, x_2) \mapsto (x_1, x_2^2)$. Hence, the classification reduces to consider germs of functions $\Lambda(x_1, x_2^2)$ defined on the half-plane $\mathbf{H}^1$ [8]. This is equivalent to consider Arnold's classes of germs of functions defined on a manifold with the boundary $\{x_2 = 0\}$ [3]. In particular, a normal form of a simple germ on $\mathbf{H}^1$ corresponds to a simple boundary class, obtained by dividing every exponent of x_2 by 2. For this reason, all of the names given to Mond's simple singularities, except for $\mathbf{S}_k^{\pm}$ and $\mathbf{H}_k^{\pm}$, are those Arnold gave to the corresponding germs in his lists. In the case of $\mathbf{S}_k^{\pm}$, the corresponding germs in $\big(\mathbf{H}^1, 0\big)$ are non-singular. However, Lemma 3.10 implies that(9) can be reduced further by the quasi equivalence to the form:

$$\big(x_1, x_2^2, x_2^3 + x_2\Psi(x_1)\big), \tag{10}$$

where $\Psi \in \mathbb{M}^2_{x_1}$. Therefore, we may assume that the quasi classification of (10) reduces to classifying the non-singular functions $f = x_2 + \psi(x_1)$ on a manifold with the boundary $\{x_2 = 0\}$. So, we obtain the quasi classes $\mathbb{S}_k$: $(x_1, x_2^2, x_2^3 + x_1^{k+1}x_2), k \geq 1$. This means that $\mathbf{C}_k^{\pm}, \mathbf{B}_k^{\pm}, \mathbf{F}_4$ and $\mathbf{S}_k$ merge into a single series of quasi classes.

On the other hand, functions defined on a manifold with a corner $\{x_1x_2 = 0\}$ can be lifted to symmetric functions on two copies of the half-plane $\mathbf{H}^2$ [8]. Therefore, a normal form of a germ $\Lambda(x_1^2, x_2^2)$ on $\mathbf{H}^2$ corresponds to a germ on $\{x_1x_2 = 0\}$, obtained by dividing every exponent of x_2 and x_2 by 2. Analogue to the boundary case, the standard $\mathcal{A}$-classes of map germs of the form

$$\big(x_1, x_1x_2, x_2(\Lambda(x_1^2, x_2^2))\big)$$

can be achieved via diffeomorphisms preserving the Saddle surface $(x_1, x_2) \mapsto (x_1, x_1x_2)$. Hence, it is enough to consider the classification of $\Lambda(x_1^2, x_2^2)$ on $\mathbf{H}^2$. Notice that all Mond's series of classes $\mathbf{H}_k$ are contained in the quasi class $\mathbb{H}_2$ which can be written equivalenty as $(x_1, x_1x_2, x_2(x_1^2 + x_2^2))$. It follows that $\mathbf{H}_k$ corresponds also to the non-singular (and the unique simple class) function $f = x_1 + x_2$ on a manifold with the corner $\{x_1x_2 = 0\}$.

The above discussion implies the following.

Proposition 3.17 *Every Mond's class corresponds to a non singular function defined on a manifold with either a boundary or a corner.*

The explicit comparison between quasi singularities and the standard ones is given in the following table.

Mond's singularities	The quasi normal form
$\mathbf{S}_0$	$\mathbb{S}_0 : (x_1, x_2^2, x_1x_2)$
$\mathbf{S}_k^{\pm}$	$\mathbb{S}_k : (x_1, x_2^2, x_1^{k+1}x_2)$, $k \geq 1$
$\mathbf{B}_k^{\pm}$	$\mathbb{S}_1 : (x_1, x_2^2, x_1^2x_2)$
$\mathbf{C}_k^{\pm}$	$\mathbb{S}_{k-1} : (x_1, x_2^2, x_1^kx_2)$, $k \geq 3$
$\mathbf{F}_4$	$\mathbb{S}_2 : (x_1, x_2^2, x_1^3x_2)$
$\mathbf{H}_k$	$\mathbb{H}_2 : (x_1, x_1x_2, x_2^3)$

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