On Vector Varieties of \((n,m)\)-Semigroups

Irena Stojmenovska

SCSIT
University American College Skopje
Republic of Macedonia

Dončo Dimovski

Macedonian Acad. of Sci. and Arts
Ss. Cyril and Methodius University Skopje
Republic of Macedonia

Copyright © 2018 Irena Stojmenovska and Dončo Dimovski. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract
We explore a special class of varieties of \((n,m)\)-semigroups called vector varieties of \((n,m)\)-semigroups. In [8] we gave a direct description of the complete system of \((n,m)\)-identities for a variety of \((n,m)\)-semigroups. Now we obtain some properties of the complete system of \((n,m)\)-identities for vector varieties of \((n,m)\)-semigroups satisfying certain conditions. As a consequence, we give a characterization of such varieties. We also show that the class of vector varieties of \((n,m)\)-semigroups is a proper subset of the class of varieties of \((n,m)\)-semigroups (when \(m \geq 2\)). We give necessary and sufficient conditions for a variety of \((n,m)\)-semigroups to be a vector variety.

Mathematics Subject Classification: 20M07, 20M10, 20N99

Keywords: \((n,m)\)-semigroup, [vector] \((n,m)\)-identity, [vector] variety of \((n,m)\)-semigroups

1 Introduction

Vector valued algebraic structures are a generalization of the usual binary, i.e. \((2,1)\) algebraic structures. The theory of \((n,m)\)-structures starts with
defining the notion of an \((n, m)\)-semigroup, that is a set having a multivariable vector-valued associative operation (see for example the expository paper [2]). This notion was introduced in [1]. A canonical form of a free \((n, m)\)-semigroup generated by a nonempty set \(B\) is given in [4]. This led to the introduction of a combinatorial theory of \((n, m)\)-semigroups (see [3]), which was followed by new interesting investigations and results (see for example [5],[6],[7]). The results in this paper contribute to the combinatorial theory of \((n, m)\)-semigroups and hopefully to its future development, especially in varieties of \((n, m)\)-semigroups.

Throughout the sequel we assume that \(Q \neq \emptyset, n, m \in \mathbb{N}\) and \(n-m = k \geq 1\). We denote by \(\mathbb{N}_0\) the set \(\mathbb{N} \cup \{0\}\), and by \(\mathbb{N}_t\) and \(\mathbb{N}_{t,0}\) the sets \(\{1, 2, \ldots, t\}\) and \(\{0, 1, 2, \ldots, t\}\), \(t \in \mathbb{N}\). If \(x = (a_1, a_2, \ldots, a_t) \in Q^t\) (where \(Q^t\) is the cartesian product of \(t\) copies of \(Q\)), we write \(x = a_t^1\), and we identify \(x\) with the word \(a_1a_2 \ldots a_t\). For such an \(x\) we say that its length \(|x|\) is \(t\). The notation \(a_r^t\), where \(r > t\) will be identified with the empty word (denoted by \(1\)). Let \(Q^+\) be the union of all the cartesian products \(Q^t, t \in \mathbb{N}\), which is the free semigroup generated by \(Q\).

We denote the set \(\{x|x \in Q^+, |x| = m + sk, s \in \mathbb{N}\}\) by \(Q^{m,k}\).

A map \(f: Q^n \to Q^m\) is called an \((n, m)\)-operation on \(Q\), and \((Q, f)\) is called an \((n, m)\)-groupoid. An \((n, m)\)-groupoid \((Q, f)\) is called an \((n, m)\)-semigroup, if the \((n, m)\)-operation is associative, i.e. if \(f(xf(y)z) = f(uf(v)w)\), for any \(xyz = uvw \in Q^{n+k}, y, v \in Q^n\) and \(x, z, u, w \in Q^* = Q^+ \cup \{1\}\).

For \(m = 1 = k\), the above notions are the usual notions of binary groupoids and semigroups, and for \(m = 1, k > 1\) they are the notions of \(n\)-groupoids and \(n\)-semigroups.

From now on we will assume that \(m \geq 2\).

An \((n, m)\)-groupoid \((Q, f)\) can be considered as an algebra with \(m\) \(n\)-ary operations \(f_1, f_2, \ldots, f_m: Q^n \to Q\), such that \(f(x) = f_1(x)f_2(x) \ldots f_m(x)\). These operations \(f_1, f_2, \ldots, f_m\) are called component operations for the \((n, m)\)-operation \(f\). In general, for an associative \((n, m)\)-operation \(f\), the corresponding component operations do not have to be associative. This implies that there is a big difference between studying \((n, m)\)-semigroups and \(n\)-semigroups.

Given an \((n, m)\)-groupoid \((Q, f)\) the component operations \(f_j: Q^n \to Q\), \(j \in \mathbb{N}_m\) can be extended to an infinite family of operations \(f_{j,s}: Q^{n+sk} \to Q\) for \(s \in \mathbb{N}\), where for a given \(s\), there are more than one operation \(f_{j,s}\). When \((Q, f)\) is an \((n, m)\)-semigroup, for each \(s \in \mathbb{N}\), there is only one operation \(f_{j,s}: Q^{n+sk} \to Q\) whose union is a map \(f_j: Q^{m,k} \to Q\). This leads to the following notions.

A map \(g: Q^{m,k} \to Q^n\) is called a poly-(\(n, m)\)-operation and the structure \(Q = (Q, g)\) is called a poly-(\(n, m)\)-groupoid. A poly-(\(n, m)\)-groupoid \(Q = (Q, g)\) is called a poly-(\(n, m)\)-semigroup if \(g(xg(y)z) = g(xyz)\), for each \(xyz \in Q^{m,k}\), \(y \in Q^{m,k}\) and \(x, z \in Q^*\), \(xz \neq 1\).
Remark 1.1. There is no essential difference between the notions of \((n,m)\)-semigroups and poly-\((n,m)\)-semigroups, due to the General Associative Law which holds in \((n,m)\)-semigroups. (For more details see [4], [2]).

Similarly as above, a poly-\((n,m)\)-groupoid \((Q,g)\) can be considered as an algebra with \(m\) poly-\(n\)-ary operations, \(g_1, g_2, \ldots, g_m : Q^{m,k} \to Q\) where \(g(x) = g_1(x)g_2(x)\ldots g_m(x)\). It is easy to see that the usual notions of universal algebra can be extended to \(\text{poly-}(n,m)\)-structures. Thus, they will be considered well known.

We recall the construction of a canonical form of a free poly-\((n,m)\)-groupoid \(F(B) = (F(B), f)\) with a basis \(B \neq \emptyset\) (see [4], [5]).

\[B_0 = B, \quad B_{p+1} = B_p \cup (N_m \times B^{m,k}_p), \quad F(B) = \bigcup_{p \geq 0} B_p.\]

The poly-\((n,m)\)-operation \(f\) on \(F(B)\) is defined by \(f(x) = (1, x)(2, x)\ldots (m, x)\). Hierarchy of the elements of \(F(B)\) is a map \(\chi : F(B) \to \mathbb{N}_0\) defined by \(\chi(u) = \min\{p \mid u \in B_p\}\), \(u \in F(B)\). The norm on \(F(B)\) is a map \(\|\cdot\| : F(B) \to \mathbb{N}\) defined by induction on the hierarchy:

\[\|u\| = 1\text{ for } u \in B_0 \text{ and }\]
\[\|(i, u_1^{m+sk})\| = \|u_1\| + \ldots + \|u_{m+sk}\| \text{ for } (i, u_1^{m+sk}) \in B_{p+1} \setminus B_p \quad (s \geq 1).\]

We recall the construction of a canonical form of a free \((n,m)\)-semigroup generated by a nonempty set \(B\) ([4]). Define a map \(\psi_0 : F(B) \to F(B)\) as follows: \(\psi_0(b) = b, b \in B\); Let \(u = (i, u_1^{m+sk}) \in F(B)\setminus B\) \((s \geq 1)\), assume that \(\psi_0(v) \in F(B)\) is defined for all \(v \in F(B)\) with \(\|v\| < \|u\|\) and \(\psi_0(v) \neq v\) implies \(\|\psi_0(v)\| < \|v\|\). Then \(v_\lambda = \psi_0(u_\lambda)\) is well defined for all \(\lambda \in \mathbb{N}_m^{m+sk}\) and thus \(v = (i, v_\lambda^{m+sk}) \in F(B)\). If there exists a \(\lambda' \in \mathbb{N}_m^{m+sk}\) such that \(v_{\lambda'} \neq u_{\lambda'}\) then \(\|v\| < \|u\|\) and consequently define \(\psi_0(u) = \psi_0(v)\); If \(v_\lambda = u_\lambda\) for all \(\lambda \in \mathbb{N}_m^{m+sk}\) and if \(u = (i, u_1^j(1, x)\ldots (m, x)u_{m+sk}^{j+m+1})\) where \(x \in F(B)^{m,k}\) and \(j \in \mathbb{N}_{sk,0}\) is the smallest such index, then \(\psi_0(u) = \psi_0(i, u_1\lambda x u_{m+sk}^m)\). If \(u\) does not satisfy any of the conditions above, then \(\psi_0(u) = u\).

Consider the set \(\psi_0(F(B)) = \{u \in F(B) \mid \psi_0(u) = u\}\) and define a poly-\((n,m)\)-operation \(\mid \mid : (\psi_0(F(B)))^{m,k} \to (\psi_0(F(B)))^m\) by

\[\mid \mid u_1^{m+sk} = \psi_0(i, u_1^{m+sk}) \iff (\forall i \in \mathbb{N}_m) v_i = \psi_0(i, u_1^{m+sk}).\]

Theorem 1.2 ([4]). \((\psi_0(F(B)), \|\\|)\) is a free \((n,m)\)-semigroup generated by \(B\).

Such \((n,m)\)-semigroup is denoted by \(\psi_0(F(B))\) i.e. \(\psi_0(F(B))) = (\psi_0(F(B)), \|\|)\).

This work concerns varieties of \((n,m)\)-semigroups i.e. we investigate a special class of varieties of \((n,m)\)-semigroups called vector varieties of \((n,m)\)-semigroups. For that purpose we make the following remarks.

Remark 1.3. From now on, for \(u \in F(B)\) we will also use the notation \((i, u_i^{-1}u_w^{m+1})\) where \(i \in \mathbb{N}_m\) and \(u_w \in F(B)\) for \(v \in \mathbb{N}_m \setminus \{i\}\). Hence,

\[u \in F(B) \iff u = (i, x)\text{ where } i \in \mathbb{N}_m, x = u_1^{m+sk}, u_\eta \in F(B)\text{ and } s \in \mathbb{N}_0.\]
Remark 1.4. Each element from $F(B)\setminus B$ has a unique representation as $(i,u_1^{m+sk})$, with $i \in \mathbb{N}_m$ and $s \geq 1$.

Given an $(n,m)$-semigroup $(Q;g)$, for the corresponding (poly-)$(n,m)$-operation $g : Q^{m,k} \to Q^m$ we will allow the case when $s = 0$, by setting $g(a_1^m) = a_1^m$.

2 Varieties of $(n,m)$-semigroups

A class of $(n,m)$-semigroups is said to be a variety of $(n,m)$-semigroups if it has an axiom system which is a set of identities. We recall the definition of an identity in the class of poly-$(n,m)$-groupoids ([3]).

Consider the free poly-$(n,m)$-groupoid $F(\mathbb{N})$ with a basis $\mathbb{N}$.

Let $Q = (Q,h)$ be a poly-$(n,m)$-groupoid. For each $\tau \in F(\mathbb{N})$ there exists a smallest $t \in \mathbb{N}$ such that $\tau \in F(\mathbb{N}_t)$ and the element $\tau$ defines a $t$-ary operation on $Q$ (denoted by $\tau$) as follows:

i) If $\tau = j \in \mathbb{N}_t$ and $a = a_1^j \in Q^t$ then $\tau(a) = a_j$;

ii) If $\tau = (i,\tau_1^{m+sk})$, $s \geq 1$, $a = a_1^i \in Q^t$ and $\tau_\nu(a) = b_\nu$, $v \in \mathbb{N}_{m+sk}$, then $\tau(a) = h_i(b_1^{m+sk})$, i.e. $\tau(a) = h_i(\tau_1(a) \ldots \tau_{m+sk}(a))$ (induction on $\chi$).

For $q \in \mathbb{N}$ and $q > t$ it is clear that $\tau \in F(\mathbb{N}_q)$ and $\tau$ defines a $q$-ary operation on $Q$ as well, which we also denote by $\tau$. (It is easy to verify that $\tau(a_1^q) = \tau(a_1^1)$ for all $a_q \in Q$).

Let $\tau,\omega \in F(\mathbb{N})$. Then $\tau,\omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$.

We say that the poly-$(n,m)$-groupoid $Q$ satisfies the $(n,m)$-identity $(\tau,\omega)$, and write $Q \models (\tau,\omega)$, if $\tau(a) = \omega(a)$ for any $a = a_1^i \in Q^t$.

An $(n,m)$-identity $(\tau,\omega)$ is said to be reduced, if $\psi_0(\tau) = \tau$ and $\psi_0(\omega) = \omega$.

The only reduced $(n,m)$-identity satisfied by all $(n,m)$-semigroups is the trivial $(n,m)$-identity $(\tau,\tau)$. Also, every $(n,m)$-semigroup $G$ satisfies the identity $(\tau,\psi_0(\tau))$, $\tau \in F(\mathbb{N})$. (For more details see [3]). As a consequence, we get

Corollary 2.1. If $G$ is an $(n,m)$-semigroup and $(\tau,\omega)$ is an $(n,m)$-identity, then

$$G \models (\tau,\omega) \iff G \models (\psi_0(\tau),\psi_0(\omega)).$$

Let $\Theta \subseteq F(\mathbb{N}) \times F(\mathbb{N})$. Then we say that $\Theta$ is a set of $(n,m)$-identities.

By $Var\Theta$ we denote the class of all $(n,m)$-semigroups $G$ such that $G \models \Theta$, where $G \models \Theta$ if and only if $(\tau,\omega) \in \Theta$ for every $(\tau,\omega) \in \Theta$.

We say that $Var\Theta$ is a variety of $(n,m)$-semigroups generated by $\Theta$.

A class of $(n,m)$-semigroups $\mathcal{V}$ is a variety if and only if (iff) there exists a set of $(n,m)$-identities $\Theta$ such that $G \in \mathcal{V} \iff G \models \Theta$. (In this case, $\mathcal{V} = Var\Theta$).

Since we explore varieties of $(n,m)$-semigroups, we are interested only in $(n,m)$-identities satisfied by poly-$(n,m)$-groupoids that are $(n,m)$-semigroups.
By Corollary 2.1 and the results in [3], for a given \((n,m)\)-identity \((\tau, \omega)\) we can always assume that \((\tau, \omega) \in \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\) i.e. that \((\tau, \omega)\) is reduced.

From now on, an \((n,m)\)-identity will always be a reduced \((n,m)\)-identity.

In [8] we gave a description of the complete system of \((n,m)\)-identities for a variety \(\text{Var}\Theta\), that is the set of all \((n,m)\)-identities satisfied by all \((n,m)\)-semigroups in \(\text{Var}\Theta\). We recall its construction.

Consider the free \((n,m)\)-semigroup \(\psi_0(F(\mathbb{N})) = (\psi_0(F(\mathbb{N})), [ \ ] )\) generated by \(\mathbb{N}\). Every element \((\tau, \omega) \in \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\) \((t \in \mathbb{N})\), induces a set \((\tau, \omega))(\psi_0(F(\mathbb{N}))) \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\) defined by

\[
(\tau, \omega)(\psi_0(F(\mathbb{N}))) = \{ (\tau(u), \omega(u)) | u \in (\psi_0(F(\mathbb{N})))^t \}. 
\]

Consequently, given a set of \((n,m)\)-identities \(\Theta \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\), it induces a set of \((n,m)\)-identities \(\Theta(\psi_0(F(\mathbb{N}))) \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\) defined by

\[
\Theta(\psi_0(F(\mathbb{N}))) = \bigcup_{(\tau, \omega) \in \Theta} (\tau, \omega)(\psi_0(F(\mathbb{N}))) = \\
\{ (\tau(u), \omega(u)) | (\tau, \omega) \in \Theta, \tau, \omega \in \psi_0(F(\mathbb{N})), u = u_1^t \in (\psi_0(F(\mathbb{N})))^t, t \in \mathbb{N} \}. 
\]

We denote by \(\hat{\Theta}\) the smallest congruence on \(\psi_0(F(\mathbb{N}))\) which contains the set \(\Theta(\psi_0(F(\mathbb{N})))\). Below we give its description. First define relations \(\vdash_\Theta^\alpha\) \((\alpha \in \mathbb{N}_0)\) on \(\psi_0(F(\mathbb{N}))\) as follows: \(u \vdash_\Theta^\alpha v \iff (u, v) \in \Theta(\psi_0(F(\mathbb{N})))\); Assume that \(\vdash_\Theta^\alpha\) is defined on \(\psi_0(F(\mathbb{N}))\) and define \(\vdash_\Theta^{\alpha+1} \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))\) by \(u \vdash_\Theta^{\alpha+1} v \iff u = \psi_0(i, xu'y), v = \psi_0(i, xu'y)\) and \(u' \vdash_\Theta^\alpha v', x, y \in F(\mathbb{N})^+ \cup \{1\}, |xy| \geq m, i \in \mathbb{N}_m\). Now, define relations \(\vdash_\Theta\) and \(\sim_\Theta\) on \(\psi_0(F(\mathbb{N}))\) by: \(u \vdash_\Theta v \iff (\exists \alpha \in \mathbb{N}_0) (u \vdash_\Theta^\alpha v)\) and \(u \sim_\Theta v \iff u \vdash_\Theta v \vee v \vdash_\Theta u\). Then \(\hat{\Theta}\) is the reflexive and transitive closure of \(\sim_\Theta\) i.e. \(u \hat{\Theta} v\) iff \(u = v\) or there exists a sequence \(u_0, u_1, \ldots, u_{r-1}, u_r \in \psi_0(F(\mathbb{N}))\) such that \(u = u_0, v = u_r\) and \(u_j \sim_\Theta u_j\) \(\forall j \in \mathbb{N}, r \geq 1\).

Sometimes instead of \(\vdash_\Theta, \vdash_\Theta\) and \(\sim_\Theta\) we will use the notations \(\vdash^\alpha, \vdash\) and \(\sim\) (for the cases when it is clear what is the set of the \((n,m)\)-identities).

**Theorem 2.2** ([8]). The set \(\hat{\Theta}\) is a complete system of \((n,m)\)-identities for \(\text{Var}\Theta\) and it consists of all \((n,m)\)-identities satisfied by all \((n,m)\)-semigroups in \(\text{Var}\Theta\).

**Corollary 2.3** ([8]). \(\hat{\Theta} = \hat{\Theta}\).

**Corollary 2.4** ([8]). Let \(\Theta\) and \(\Sigma\) be sets of \((n,m)\)-identities. Then \(\text{Var}\Theta = \text{Var}\Sigma \iff \Theta = \Sigma\).

### 3 Vector varieties of \((n,m)\)-semigroups

The idea of defining \((n,m)\)-identities arises from the notion of \((n,m)\)-relations on a nonempty set \(B\) (for more details see [3]).
Definition 3.1. Let \( p = m + sk, q = m + rk \), where \( s, r \geq 0 \) and let \((i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+\). We say that an \((n, m)\)-semigroup \( G = (G; g) \) satisfies the vector \((n, m)\)-identity \((i_1^p, j_1^q)\) and write \( G \models (i_1^p, j_1^q) \), if \( g(a_{i_1} \ldots a_{i_p}) = g(a_{j_1} \ldots a_{j_q}) \) for every \( a_i^t \in G^t \) where \( t = \max\{i_\mu, j_\nu\} \) \((\mu \in \mathbb{N}_p, \nu \in \mathbb{N}_q)\).

If \( \Theta' = \{(i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+ | p = m + sk, q = m + rk, s, r \geq 0\} \) then we say that \( \Theta' \) is a set of vector \((n, m)\)-identities.

Every vector \((n, m)\)-identity \((i_1^p, j_1^q)\) induces a set of \((n, m)\)-identities
\[
(i_1^p, j_1^q)_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))
\]
defined by
\[
(i_1^p, j_1^q)_\# = \{((i, i_1^p), (i, j_1^q)) | i \in \mathbb{N}_m\}.
\]

For an \((n, m)\)-semigroup \( G = (G; g) \) and \( a_i^t \in G^t \) where \( t = \max\{i_\mu, j_\nu\} \),
\[
g(a_{i_1} \ldots a_{i_p}) = g(a_{j_1} \ldots a_{j_q}) \iff g_i(a_{i_1} \ldots a_{i_p}) = g_i(a_{j_1} \ldots a_{j_q}) \quad \text{for all} \quad i \in \mathbb{N}_m,
\]
which implies that \( G \models (i_1^p, j_1^q) \iff G \models (i_1^p, j_1^q)_\# \).

Consequently, if \( \Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+ \) is a set of vector \((n, m)\)-identities then it induces a set of \((n, m)\)-identities \( \Theta'_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \) where
\[
\Theta'_\# = \bigcup_{(i_1^p, j_1^q) \in \Theta'} (i_1^p, j_1^q)_\# = \{((i, i_1^p), (i, j_1^q)) | (i_1^p, j_1^q) \in \Theta', i \in \mathbb{N}_m\}
\]
and
\[
G \models \Theta' \iff G \models \Theta'_\#.
\]

We say that \( \Theta'_\# \) is a set of vector \((n, m)\)-identities induced by \( \Theta' \).

From now on, by a set of vector \((n, m)\)-identities we will usually mean an induced set \( \Theta'_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N})) \) where \( \Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+ \) is a corresponding set of vector \((n, m)\)-identities in terms of Definition 3.1.

Definition 3.2. A variety of \((n, m)\)-semigroups \( \mathcal{V} \) is said to be a vector variety of \((n, m)\)-semigroups, if there exists a set of vector \((n, m)\)-identities \( \Theta'_\# \) such that \( \mathcal{V} = \text{Var}\Theta'_\# \).

Example 3.3. \( \text{Var}\{(i_1^m, j_1^m)\}_\# \) is the variety of all \((n, m)\)-semigroups for \( i_1^m = j_1^m \), or the variety of trivial \((n, m)\)-semigroups for \( i_1^m \neq j_1^m \).

Proposition 3.4. A variety of \((n, m)\)-semigroups \( \text{Var}\Theta \) is a vector variety of \((n, m)\)-semigroups iff there exists a set of vector \((n, m)\)-identities \( \Theta'_\# \) such that \( \hat{\Theta} = \hat{\Theta}'_\# \).

Proof. Applying Definition 3.2 and Corollary 2.4. \( \text{Var}\Theta \) is a vector variety of \((n, m)\)-semigroups iff there exists a set of vector \((n, m)\)-identities \( \Theta'_\# \) such that \( \text{Var}\Theta = \text{Var}\Theta'_\# \) iff there exists a set of vector \((n, m)\)-identities \( \Theta'_\# \) such that \( \hat{\Theta} = \hat{\Theta}'_\# \).
Theorem 3.5. A variety of \((n, m)\)-semigroups \(\text{Var}\Theta\) is a vector variety of \((n, m)\)-semigroups iff there exists a nonempty set \(M \subseteq \hat{\Theta}\) which satisfies the conditions:

1) If \((\tau, \omega) \in M\) then there exist some \(j \in \mathbb{N}_m\) and \(w, z \in \mathbb{N}^m \cup \mathbb{N}^{m,k}\) such that 
\[
\tau = (j, w), \ \omega = (j, z) \text{ and } ((i, w), (i, z)) \in M \ \text{for all} \ i \in \mathbb{N}_m;
\]
2) \(\hat{\Theta} \subseteq \hat{M}\).

Proof. (⇒). Let \(\text{Var}\Theta\) be a vector variety of \((n, m)\)-semigroups. Then \(\text{Var}\Theta = \text{Var}\Theta'\) for some set of vector \((n, m)\)-identities \(\Theta'\) and \(\hat{\Theta} = \hat{\Theta'}\). Since \(\Theta'_\# = \{(i, i^p, j^q), (i, j^q)\} \in \Theta', i \in \mathbb{N}_m\} \) where \(\Theta' \subseteq (\mathbb{N}^m \cup \mathbb{N}^{m,k}) \times (\mathbb{N}^m \cup \mathbb{N}^{m,k})\), by taking \(M = \Theta'_\#\) we get a nonempty set \(M \subseteq \Theta'_\# = \hat{\Theta}\), that satisfies the condition 1). Since \(\hat{M} = \Theta'_\#,\) the condition 2) is satisfied as well.

(⇐). Let \(\text{Var}\Theta\) be a variety of \((n, m)\)-semigroups and let \(M \subseteq \hat{\Theta}\) be a nonempty set that satisfies the conditions 1) and 2). The condition 1) implies that \(M\) is a union of sets \(((i, w), (i, z))| i \in \mathbb{N}_m\} \) over some elements \((w, z) \in (\mathbb{N}^m \cup \mathbb{N}^{m,k}) \times (\mathbb{N}^m \cup \mathbb{N}^{m,k})\). Let \(\Theta'\) be the union of such elements. Then \(M = \Theta'_\#\), i.e. \(M\) is a set of vector \((n, m)\)-identities. Since \(M \subseteq \hat{\Theta}\), the description of the congruence \(\hat{\sim}\) and Corollary 2.3 imply that \(\hat{M} \subseteq \hat{\Theta} = \hat{\Theta}\), and applying the condition 2) we get that \(\hat{\Theta} = \hat{M}\). Hence, \(\text{Var}\Theta\) is a vector variety of \((n, m)\)-semigroups by Proposition 3.4.

Bellow we explore the properties of the complete system of \((n, m)\)-identities for vector varieties of \((n, m)\)-semigroups such that all (nontrivial) identities \((i^p_1, j^q_1)\) \(\in \Theta'\) satisfy the condition \(p, q > m\). Then we give a characterization of such varieties.

Lemma 3.6. Let \(\text{Var}\Theta'_\#\) be a vector variety of \((n, m)\)-semigroups such that all \((i^p_1, j^q_1)\) \(\in \Theta'\) satisfy the condition \(p, q > m\). If \((\tau, \omega) \in \Theta'_\#\) then \(\tau = \omega \in \mathbb{N}\) or \(\tau = (j, x), \ \omega = (j, y)\) where \(j \in \mathbb{N}_m, |x|, |y| > m\), and \(((i, x), (i, y)) \in \Theta'_\#\) for all \(i \in \mathbb{N}_m\).

Proof. Recall the description of \(\hat{\Theta'}_\#\). The definition of \(\hat{\sim}^0\) implies that, if \(\hat{\tau}, \hat{\omega} \in \hat{\sim}^0\) then \((\tau, \omega) = (\psi_0(j, u_1, \ldots, u_p), \psi_0(j, u_1, \ldots, u_q))\) for some \(((i, i^p_1), (j^q_1)) \in \Theta'_\#\) and \(u_1, \ldots, u_t \in \psi_0(F(\mathbb{N}))\), where \(t = \max \{i_{\mu}, j_{\nu}\}\), \((i^p_1, j^q_1) \in \Theta'\) and \(j \in \mathbb{N}_m\). Since \(p, q > m\) and the mapping \(\psi_0\) does not change the first coordinate nor will reduce (to \(m\)) the length of the second one, and since we have that \((\psi_0(i, u_1, \ldots, u_p), \psi_0(i, u_1, \ldots, u_q)) \in \Theta'_\#(\psi_0(F(\mathbb{N})))\) for all \(i \in \mathbb{N}_m\) (due to the fact that \(((i, i^p_1), (i, j^q_1)) \in \Theta'_\#\) for all \(i \in \mathbb{N}_m\), it follows that the lemma is valid on the set \(\hat{\sim}^0\). (Also, note that \((\tau, \omega) \in \hat{\sim}^0\) implies \(\tau, \omega \notin \mathbb{N}\)). Next, if \((\tau, \omega) \in \hat{\sim}^0\) for some \(\alpha \geq 1\), then \(\tau = \psi_0(j, wu'z)\) and \(\omega = \psi_0(j, wu'z)\).
where \( j \in \mathbb{N}_m, \ u'^{\alpha-1} v', \ w, z \in F(N)^+ \cup \{1\} \) and \(|w^u'z| = |w^v'z| > m\). Just like above, we conclude that \( \tau \) and \( \omega \) will have equal first coordinates while the lengths of their second coordinates will be greater than \( m \) (and thus \( \tau, \omega \notin \mathbb{N} \)). Applying the definition of the relations \( \vdash^\alpha \), we also get that \((\psi_0(v, w^u'z), \psi_0(v, w^v'z)) \in \vdash^\alpha\) for all \( i \in \mathbb{N}_m \). Hence, the lemma is valid on the sets \( \vdash^\alpha \), \( \alpha \in \mathbb{N}_0 \), and so, it is valid on the set \( \vdash \) and on the set \( \sim \) as well. Moreover, \( (\tau, \omega) \in \sim \) implies \( \tau \notin \mathbb{N} \) and \( \omega \notin \mathbb{N} \). Let \( (\tau, \omega) \in \widetilde{\Theta}_\#^\sim \). If \( \tau = \omega \notin \mathbb{N} \) then \( \tau = (j, x) = \omega \) for some \( j \in \mathbb{N}_m \) where \(|x| > m \) (Remark 1.4), and clearly \((\tilde{(i, x)}, (i, x)) \in \widetilde{\Theta}_\#^\sim\) for all \( i \in \mathbb{N}_m \). Let \( \tau \neq \omega \). The case when \( (\tau, \omega) \in \sim \) is proven, so let \( \tau \sim z_1 \sim z_2 \sim \ldots \sim z_{c-1} \sim \omega \) where \( c \geq 2 \). Since the lemma is valid for the elements \((\tau, z_1), (z_1, z_2), \ldots, (z_{c-1}, \omega)\) respectively, there exists some \( j \in \mathbb{N}_m \) such that \( \tau = (j, x), \omega = (j, y) \) and \(|x|, |y| > m\). (Hence, \( \tau, \omega \notin \mathbb{N} \)). Moreover, we have that the first coordinate of the elements \( z_1, z_2, \ldots, z_{c-1} \) is \( j \), and the second one has a length greater than \( m \). Now, for each \( i \in \mathbb{N}_m \) we construct sequences which correspond to the sequence \((j, x), z_1, z_2, \ldots, z_{c-1}, (j, y)\) as follows: We put the first coordinates of the elements above to be \( i \) (instead of \( j \)), while the second coordinates remain the same. Then, having \( \tau \sim z_1 \sim z_2 \sim \ldots \sim z_{c-1} \sim \omega \) and since the lemma is valid for \( \sim \), we conclude that for each \( i \in \mathbb{N}_m \) the obtained sequences will link the elements \((i, x)\) and \((i, y)\) by \( \widetilde{\Theta}_\#^\sim \).

\[\Box\]

**Remark 3.7.** Lemma 3.6 is also valid in the case when \( \Theta' \) contains pairs \((i^m_1, i^m_1')\). (The presence of trivial identities in \( \Theta' \) i.e. in \( \Theta'_\# \) makes no difference for \( \widetilde{\Theta}_\#^\sim \).) If \( \Theta' = \{(i^m_1, i^m_1')\} \) then \( \widetilde{\Theta}_\#^\sim = \{(u, u)|u \in \psi_0(F(N))\} \) and the lemma is valid as well.

Lemma 3.6 is not valid in the general case. For example, if \( \text{Var}(\Theta'_\#) \) is the variety of trivial \((n, m)\)-semigroups then \( \widetilde{\Theta}_\#^\sim = \psi_0(F(B)) \times \psi_0(F(B)) \) and Lemma 3.6 is not valid.

**Example 3.8.** Consider a vector variety of \((n, m)\)-semigroups \( \text{Var}(\Theta'_\#) \) such that \( \Theta' \) contains pairs \((i^p_1, j^m_1), p' > m \) and \((l_1 j^m_1 j_3^m, k^q_1)\), \( q' > m \). Then \((1, i^p_1) \sim (1, j^m_1) \sim j_1 = (2, l_1 j^m_1 j_3^m) \sim (2, k^q_1) \) i.e. \((1, i^p_1), (2, k^q_1) \in \widetilde{\Theta}_\#^\sim\). Since \( 1 \neq 2 \), Lemma 3.6 is not valid.

The example bellow shows that the class of vector varieties of \((n, m)\)-semigroups is a proper subset of the class of varieties of \((n, m)\)-semigroups (when \( m \geq 2 \)).

**Example 3.9.** Let \( \Theta = \{((1, x), (2, y))\} \subseteq \psi_0(F(N)) \times \psi_0(F(N)) \) where \(|x|, |y| > m\). Suppose that there exists a set of vector \((n, m)\)-identities \( \Theta'_\# \) such that \( \text{Var}(\Theta) = \text{Var}(\Theta'_\#) \). If \( \Theta'_\# \) is such that all \((i^p, j^q) \in \Theta' \) satisfy \( p, q > m \) or \((i^p_1, j^q_1) = (i^m_1, i^m_1)\), by Corollary 2.4 we have \(((1, x), (2, y)) \in \widetilde{\Theta}_\#^\sim\), which
is in contradiction with Lemma 3.6 together with Remark 3.7. Assume that there exists a nontrivial element \((i_1^m, j_1^0) \in \Theta\), where \(q \geq m\), \(i_1^m \neq j_1^0\). Then, \((i_i, (i, j_1^0)) \in \Theta^\#\) for all \(i \in \mathbb{N}_m\). Choose an element \((i_j, j_1^0) \in \Theta^\#_\), \((j \in \mathbb{N}_m)\) such that \(i_j \neq (j, j_1^0)\). Such \(j\) always exists since \(i_1^m \neq j_1^0\). (Note that for \(q > m\) this is true for all \(j \in \mathbb{N}_m\)). By Corollary 2.4 we have that \((i_j, (j, j_1^0)) \in \widehat{\Theta}\). Thus, \(i_j \sim_{\Theta} (j, j_1^0)\) or there exists a sequence \(u_1, \ldots, u_c \in \psi_0(F(N))\) \((c \geq 1)\) such that \(i_j \sim_{\Theta} u_1 \sim_{\Theta} \ldots \sim_{\Theta} u_c \sim_{\Theta} (j, j_1^0)\). On the other hand, since \(\Theta = \{(1, x), (2, y)\}\) \(\{x, y\} > m\) and since \(\Theta = \Theta(\psi_0(F(N))\), the set \(\Theta\) does not contain pairs \((l, z)\) or \((z, l)\) where \(l \in \mathbb{N}\) and \(z \in \psi_0(F(N))\). This follows from the definition of \(\Theta(\psi_0(F(N))\) and the fact that the mapping \(\psi_0\) does not reduce \((m)\) the length of second coordinate. The same is true for all \(\Theta\), \(\alpha \geq 1\), which follows by their definitions and the above property of \(\psi_0\). Consequently, there is no \(z \in \psi_0(F(N))\) such that \(i_j \sim_{\Theta} z\) or \(z \sim_{\Theta} i_j\), and therefore there is no \(z \in \psi_0(F(N))\) such that \(i_j \sim_{\Theta} z\). All this is a contradiction. If we assume that \(\Theta^\prime\) contains a nontrivial identity \((i_1^m, j_1^0) \in \Theta^\prime\) where \(p \geq m\), the conclusion is the same. Since there are no other possibilities for \(\Theta^\prime_\) (i.e. for the corresponding \(\Theta^\prime\)), we have shown that a set of vector \((n, m)\)-identities \(\Theta^\prime_\) such that \(\text{Var} \Theta = \text{Var} \Theta^\prime_\) does not exist. Thus \(\text{Var} \Theta\) is not a vector variety of \((n, m)\)-semigroups.

Now consider \(F(N)\) - the free poly \((n, m)\)-groupoid with a basis \(\mathbb{N}\). (Recall its construction). To avoid confusion with the notations of the subsets of \(\mathbb{N}\), the components of the union \(F(N)\) will be denoted by \(\mathbb{N}(p)\) i.e. \(F(N) = \bigcup_{p \geq 0} \mathbb{N}(p)\).

Thus, \(\mathbb{N}(0) = \mathbb{N}\) and \(\mathbb{N}(1) = \mathbb{N} \cup (\mathbb{N}_m \times \mathbb{N}^{m,k}) = \mathbb{N}_m \times (\mathbb{N}^{m} \cup \mathbb{N}^{m,k})\) (by Remark 1.3). Note that \(\mathbb{N}(1) \subseteq \psi_0(F(N))\). The direct product \(\mathbb{N}(1) \times \mathbb{N}(1)\) will be denoted by \(\mathbb{N}^{2}(1)\).

**Proposition 3.10.** Let \(\text{Var} \Theta^\prime_\) be a vector variety of \((n, m)\)-semigroups. The following conditions are equivalent

(I) \(\text{All } (i_1^p, j_1^q) \in \Theta' \text{ are such that } p, q > m \text{ or } (i_1^p, j_1^q) = (i_1^m, i_1^m)\).

(II) \(\text{If } (\tau, \omega) \in \Theta^\prime_\cap \mathbb{N}^{2}(1) \text{ then } \tau = (j, w) \text{ and } \omega = (j, z), \text{ for some } j \in \mathbb{N}_m \text{ and } w, z \in \mathbb{N}^{m} \cup \mathbb{N}^{m,k}\).

**Proof.** (I) \(\Rightarrow\) (II). Let \((\tau, \omega) \in \Theta^\prime_\cap \mathbb{N}^{2}(1), \text{ If } \tau = \omega, \text{ there exists some } j \in \mathbb{N}_m \text{ and a sequence } w \in \mathbb{N}^{m} \cup \mathbb{N}^{m,k} \text{ such that } \tau = (j, w) = \omega. \text{ This follows by Remark 1.3 and the fact that } (\tau, \omega) \in \mathbb{N}^{2}(1). \text{ If } \tau \neq \omega, \text{ Lemma 3.6 (together with Remark 3.7) implies that there exists some } j \in \mathbb{N}_m \text{ such that } \tau = (j, w), \omega = (j, z) \text{ and } |w|, |z| > m. \text{ Since } (\tau, \omega) \in \mathbb{N}^{2}(1), \text{ it follows that } w, z \in \mathbb{N}^{m,k}.

(II) \(\Rightarrow\) (I). Assume that there exists a pair \((i_1^p, j_1^q) \in \Theta'\) where \(p \geq m\) and \(i_1^p \neq j_1^q\). If \(p = m\), \(i_1^p \neq j_1^m\) implies that \(i_1 \neq j_l\) for some \(l \in \mathbb{N}_m\). Since \(((l, i_1^m), (l, j_1^m)) = (i_1, j_l) \in \Theta^\prime_\) we have that \(\Theta^\prime_\) = \(\psi_0(F(N)) \times \psi_0(F(N))\), and
thus $\hat{\Theta}' \cap N^2_{(1)} = N^2_{(1)}$. So, the condition (II) will not be satisfied. Suppose that $p > m$. Let $t = \max_{\mu \in N^p, \nu \in N^m} \{i_\mu, j_\nu\}$ and choose a sequence $u_\mu = a \in \mathbb{N}$ for all $r \in \mathbb{N}_t$. Then we obtain the following: $((1, i_1^p)(u_1^1), j_1(u_1^1)) = (\psi_0(1, u_1^1), u_{j_1}) = (\psi_0(1, a), a) = ((1, a), a)$ and $((2, i_1^p)(u_2^1), j_2(u_2^1)) = (\psi_0(2, u_2^1), u_{j_2}) = (\psi_0(2, a), a) = ((2, a), a)$. Since $((1, i_1^p), j_1), ((2, i_1^p), j_2) \in \Theta'$, it follows that $((1, a), a), (2, a), a) \in \hat{\Theta}'_#$ and thus $((1, a), (2, a)) \in \hat{\Theta}'_#$. Moreover, $((1, a), (2, a)) \in \hat{\Theta}'_# \cap N^2_{(1)}$, since $a \in \mathbb{N}^{m,k}$. This is in contradiction with the condition (II).

Symmetrically, the existence of an identity $(i_1^m, j_1^m) \in \Theta'$ where $q \geq m$ and $i_1^m \neq j_1^m$ leads to a contradiction as well. (The proof is completely analogous).

Hence, the set $\Theta'$ does not contain nontrivial pairs $(i_1^p, j_1^q)$ such that $p = m$ and/or $q = m$.

Finally, we get the following criteria for a variety of $(n, m)$-semigroups to be a variety vector, such that all nontrivial (generating) $(n, m)$-identities have lengths greater than $m$.

**Theorem 3.11.** A variety of $(n, m)$-semigroups $\text{Var} \Theta$ is a vector variety $\text{Var} \Theta'$ such that all nontrivial $(i_1^p, j_1^q) \in \Theta'$ have lengths $p, q > m$, iff the following two conditions are satisfied

1) If $(\tau, \omega) \in \hat{\Theta} \cap N^2_{(1)}$ then $\tau = (j, w)$ and $\omega = (j, z)$ for some $j \in \mathbb{N}_m$, w, z $\in \mathbb{N}^m \cup \mathbb{N}^{m,k}$, and $((i, w), (i, z)) \in \hat{\Theta} \cap N^2_{(1)}$ for all $i \in \mathbb{N}_m$.

2) $\hat{\Theta} \subseteq \hat{\Theta} \cap N^2_{(1)}$.

**Proof.** ($\Rightarrow$). Let $\text{Var} \Theta = \text{Var} \Theta'$ for some set of vector $(n, m)$-identities $\Theta'$ where all $(i_1^p, j_1^q) \in \Theta'$ are such that $p, q > m$ or $(i_1^p, j_1^q) = (i_1^m, i_1^m)$. Then $\hat{\Theta} = \hat{\Theta}'_#$ (by Corollary 2.4), which implies that $\hat{\Theta} \cap N^2_{(1)} = \hat{\Theta}'_# \cap N^2_{(1)}$. Hence, condition 1) follows by Lemma 3.6, Remark 3.7 and Proposition 3.10. (Note that all $w, z \in \mathbb{N}^m \cup \mathbb{N}^{m,k}$ are such that $|w|, |z| > m$ or $w = z \in \mathbb{N}^m$). Moreover, since $\Theta' \subseteq \hat{\Theta}'_# \cap N^2_{(1)}$, we obtain that $\Theta' \subseteq \hat{\Theta}'_# \cap N^2_{(1)}$. Thus, $\hat{\Theta} \subseteq \hat{\Theta} \cap N^2_{(1)}$ and condition 2) is satisfied as well.

($\Leftarrow$). Let $\text{Var} \Theta$ be a variety of $(n, m)$-semigroups such that its complete system of $(n, m)$-identities $\hat{\Theta}$ satisfies the conditions 1) and 2). The condition 1) implies that $\hat{\Theta} \cap N^2_{(1)}$ is a set of vector $(n, m)$-identities. Namely, it follows that $\hat{\Theta} \cap N^2_{(1)}$ is a union of sets $\{(i, w), (i, z) | i \in \mathbb{N}_m\}$ over some elements $(w, z) \in (\mathbb{N}^m \cup \mathbb{N}^{m,k}) \times (\mathbb{N}^m \cup \mathbb{N}^{m,k})$. If we take $\Theta'$ to be the union of such elements, then $\hat{\Theta} \cap N^2_{(1)} = \Theta'_#$. The fact that $\hat{\Theta} \cap N^2_{(1)} \subseteq \hat{\Theta}$ together with Corollary 2.3 imply that $\hat{\Theta} \cap N^2_{(1)} \subseteq \hat{\Theta} = \hat{\Theta}'. Now applying condition 2) we get that $\hat{\Theta} = \hat{\Theta} \cap N^2_{(1)}$. Proposition 3.4 implies that $\text{Var} \Theta$ is a vector
variety of \((n,m)\)-semigroups i.e. \(\text{Var}\Theta = \text{Var}\left(\hat{\Theta} \cap \mathbb{N}_2(1)\right) = \text{Var}\Theta'\). Since \(\hat{\Theta}' \cap \mathbb{N}_2(1) = \hat{\Theta} \cap \mathbb{N}_2(1)\) by Proposition 3.10 (together with condition 1)) we conclude that all \((w,z) \in \Theta'\) are such that \(|w|, |z| > m\) or \(w = z \in \mathbb{N}_m\). \(\square\)

References


Received: September 24, 2018; Published: October 23, 2018