Embedding Problems for Real Division Algebras

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Abstract

By using the well-known processes of isotopy and Cayley-Dickson duplication, we plunge every two-dimensional real division algebra into an algebra isotopic to \( \mathbb{H} \) and every real division algebra isotopic to \( \mathbb{H} \) into an algebra isotopic to \( \mathbb{O} \). Also these processes permit us to show that every four-dimensional real power commutative division algebra, not necessarily isotopic to \( \mathbb{H} \), can be embedded into an eight-dimensional real division algebra which is power commutative. This partially answers a question asked in J. Algebra 282 (2004), 758-796. Such an embedding is refined in a way to provide examples where the triviality of the Lie algebra of derivations is preserved.

Keywords: Quaternions, Octonions, Real division algebra, Duplication processes, Quadratic (Flexible, Power-commutative, Third-power associative) algebra, Derivations, Automorphisms, Isotopy

1. Introduction

One of the fundamental and powerful results in the theory of real division algebras (RDA) asserts that every finite dimensional RDA has dimension equal to 1, 2, 4, or 8 [9, 26, 24]. The classification of RDA of dimension \( d \) is known for \( d \in \{1, 2\} \) and is partial for \( d \in \{4, 8\} \). Concretely, it is established, by mean of algebraic tools, that an odd dimensional RDA is isomorphic to \( \mathbb{R} \) [36, Proposition 1.2]. The case \( d = 2 \) was elucidated only a decade ago by establishing that a RDA of dimension 2 is isotopic to \( \mathbb{C} \) [5, 25, 21]. However, the classification of RDA of dimension \( d = 4 \) or 8 is still an open problem and we have only some partial results. Indeed,
(1) for $d = 4$, we have a classification of absolute-valued algebras [33, 36, 12], and power-commutative RDA [15, 18], including both quadratic RDA [30, 20, 22] and flexible RDA [29, 8, 13, 14],

(2) the case $d = 8$ is more difficult and there is only a classification of absolute-valued algebras [35, 11] and flexible RDA [29, 8, 13, 14].

In [6], another approach has been proposed for the study of a RDA using its Lie algebra of derivations. This approach leads to some partial results [7, 34, 23, 31, 16, 2]. One can also find some studies in dimension 4 in [10, (15.7), p. 179] and [3, 4, 38].

Here, we study the embedding problem of a RDA as subalgebras of another RDA. The existence of non-zero idempotents in every RDA (see [37]) shows that any 1-dimensional RDA can be embedded into every RDA of dimension $n \geq 2$. It has been proved in [23, Theorem 4.2] that every RDA of dimension 2 can be embedded into a RDA of dimension 4 and 8. So it is natural to ask if every RDA of dimension 4 can be embedded into some RDA of dimension 8. This problem appeared in [23, Problem 2, p. 793].

Our starting point is a refinement of [23, Theorem 4.2]. By using isotopy and Cayley-Dickson duplication process, we prove that every 2-dimensional RDA can be embedded into a 4-dimensional RDA which is isotopic to $\mathbb{H}$ (Proposition 2 (1)). However, unlike 2-dimensional algebras there are 4-dimensional quadratic flexible RDA that are not isotopic to $\mathbb{H}$ (Remark 1). With the same method we show that every 4-dimensional RDA isotopic to $\mathbb{H}$ can be embedded into an 8-dimensional RDA which is isotopic to $\mathbb{O}$ (Proposition 2 (2)).

Also, by using the fact that every quadratic RDA of dimension 4 is embedded via the Cayley-Dickson process into a quadratic RDA of dimension 8 (see [28]), we show that any power commutative RDA of dimension 4, not necessarily isotopic to $\mathbb{H}$, can be embedded into a power commutative RDA of dimension 8 (Theorem 1).

These results gives a partial answer to [23, Problem 2, p. 793]. Moreover, by using a well known generalized Cayley-Dickson process (see [13, Definition 3.1]) the embedding is refined in a way to provide examples where the finiteness of the group of automorphisms, and then the triviality of the algebra of derivations is preserved (Example 2 (1)). This provides new examples of 8-dimensional RDA with both trivial derivations (Example 2 (1)) and non-trivial derivations (Example 2 (1), (2), (3)).

Also, there are a family of four-dimensional RDA which are neither isotopic to $\mathbb{H}$ nor third-power associative but can be embedded into an eight-dimensional RDA (Proposition 3).

Each of the known 4-dimensional RDA can be embedded into an 8-dimensional RDA. It may be conjectured that every 4-dimensional real division algebra can be embedded into an 8-dimensional RDA.
In this section, we use the well-known isotopy process and generalized Cayley-Dickson duplication to derive some results on how to embed real division algebras.

All the vector spaces in this paper are finite-dimensional over the field $\mathbb{R}$ of real numbers. A non-associative algebra is a vector space $A$ endowed with a bilinear map $A \times A \to A$ called product of $A$. The algebra $A$ is called a real division algebra (RDA) if the linear multiplications on the left and right

$L_a : A \to A \quad x \mapsto ax, \quad R_a : A \to A \quad x \mapsto xa$

are invertible for each non-zero element $a \in A$.

We denote by $\mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, the RDA of complex numbers, quaternions and octonions. If $V$ is a vector space and $f, g : V \to V$ two endomorphisms. We will denote by $(f, g)$ the linear mapping $V \times V \to V \times V \quad (x, y) \mapsto (f(x), g(y))$, by $I_V : V \to V$ the identity of $V$ and by $\text{Span}\{a_1, \ldots, a_n\}$ the vector subspace spanned by $a_1, \ldots, a_n \in V$.

Let $A$ be an algebra and $f, g, \phi, \omega : A \to A$ linear bijections and $\delta \in \mathbb{R}$. There are two processes of building two types of algebras from $A$.

1. The $(f, g)$-isotope of $A$ is the algebra we denote by $A_{f,g}$ whose underline vector space is $A$ endowed with the product $x \odot y = f(x)g(y)$. It is obvious that $A$ is a division algebra if and only if $A_{f,g}$ is a division algebra. The notion of isotopy was introduced first by Albert [1, Section 11, p. 696].

2. The duplication of $A$ via the generalized Cayley-Dickson process is the algebra whose underline vector space is $A \times A$ endowed with the product

$$(x, y) \delta * (x', y') = \left( xx' - \phi(y')y, y\omega(x') + y'x + \frac{\delta}{2}[y', y] \right).$$

We denote this algebra by $E_{\delta, \phi, \omega}(A)$. We have obviously that $i_A : A \to E_{\delta, \phi, \omega}(A), \quad x \mapsto (x, 0)$ is a monomorphism of algebras.

In the sequel $E_{\delta, \phi, \omega}(A)$ will be noted simply $E_{\delta, \phi}(A)$.

Note that the algebras $E_{0, \sigma_\mathbb{C}}(\mathbb{C}), E_{0, \sigma_\mathbb{H}}(\mathbb{H})$ are, respectively, the quaternion algebra $\mathbb{H}$ and the octonion algebra $\mathbb{O}$, where $\sigma_\mathbb{H} : x \mapsto \bar{x}$ is the standard involution of the algebra $\mathbb{H}$. Moreover, when $A = \mathbb{H}$ and $\phi = \sigma_\mathbb{H}$, $E_{\delta, \sigma_\mathbb{H}}(\mathbb{H})$ a division algebra if and only if $|\delta| < 2$ (see [8, Proposition 5.16] and [13, Remark 3.2]).

The following proposition is easy to establish but can have some important consequences.

**Proposition 1.** With the notation above, put $F = (f, \text{Id}_A)$ and $G = (g, \text{Id}_A)$. Then $i_A : A_{f,g} \to (E_{\delta, \phi, \omega}(A))_{F,G}$ is a monomorphism of algebras.

The following is an immediate consequence of Proposition 1.
Proposition 2. (1) Every two-dimensional real division algebra $A$ can be embedded into a four-dimensional real division algebra isotopic to the quaternion algebra $\mathbb{H}$.

(2) Every real division algebra isotopic to the quaternion algebra $\mathbb{H}$ can be embedded into an eight-dimensional real division algebra isotopic to the octonion algebra $\mathbb{O}$.

Proof. The second assertion is obvious and the first one is a consequence of the fact that every two-dimensional real division algebra $A$ is isotopic to $\mathbb{C}$ (see [25]). □

Recall that an algebra $A$ is said to be quadratic if it contains an unit element $e$ and $e, x, x^2$ are linearly dependent for all $x \in A$. It is well known that $A$ is obtained from a skew-commutative algebra $(V, \wedge)$ and a bilinear form $(.,.) : A \times A \to \mathbb{R}$ by endowing the vector space $\mathbb{R}e \oplus V$ with the product

$$(\alpha e + u)(\beta e + v) = (\alpha\beta + (u, v))e + (\alpha v + \beta u + u \wedge v)$$

and we write $A = (V, \wedge, (.,.))$ [30, Theorem 1]. The associated space $V$ of vectors of $A$ is given by

$$V = \{x \in A : x^2 \in \mathbb{R}e \text{ and } x \notin \mathbb{R}e - \{0\}\}$$

and denoted $\text{Im}(A)$ [24, p. 223]. A quadratic algebra $A = (V, \wedge, (.,.))$ is called a Cayley algebra if and only if it satisfies one of the following two equivalent properties:

(1) Its splitting defines an involution $\sigma : A \to A$, $\lambda e + x \mapsto \lambda e - x$.

(2) $(.,.)$ is a symmetric bilinear form.

Let $A$ be a quadratic RDA with unit 1. A linear bijection $T : A \to A$ is called planar if $T(1) = 1$, and $T(x) \in \text{Span}\{1, x\}$ for all $x \in A$ or, equivalently, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ and $\alpha \in A^*$ such that, for every $x \in A$, $T(x) = \alpha(x) + \lambda x$ with $\alpha(1) = 1 - \lambda$.

It is worth noting that if $A$ is a quadratic RDA then

$$\text{Im}(A) = \{x \in A : x^2 \in \mathbb{R}^{-1}\}. \quad (2.1)$$

An algebra $A$ is said to be flexible if it satisfies $(xy)x = x(yx)$ for all $x, y$ in $A$. It is called power-commutative if every element of $A$ generates a commutative sub-algebra of $A$.

Theorem 1. Every four-dimensional power-commutative real division algebra can be embedded into an eight-dimensional real division algebra which is also power-commutative.

Proof. Let $B$ be a 4-dimensional power-commutative RDA. It is known that $B = \mathcal{A}_T, \mathcal{T}$ where $\mathcal{A}$ is a quadratic RDA and $T : \mathcal{A} \to \mathcal{A}$ is a planar mapping (see [15]).
According to the notations in [28, Proposition 4.5], there is an embedding of $\mathcal{A}$ into the 8-dimensional quadratic RDA: $\mathcal{V}(\mathcal{A})$ (see also [30, p. 202 and Theorem 7]). Write $T(x) = \alpha(x) + \lambda x$ and consider

$$(T, \lambda I_A) : \mathcal{V}(\mathcal{A}) \to \mathcal{V}(\mathcal{A}) \quad (x, y) \mapsto (T(x), \lambda y).$$

It is obvious that $T(x, y) = \alpha(x)(1, 0) + \lambda(x, y)$ and hence $T$ is planar. So $(\mathcal{V}(\mathcal{A}))(T, \lambda I_A)$ is an eight-dimensional power-commutative RDA [15, Theorem 3.3] containing a sub-algebra isomorphic to $\mathcal{A}$. □

**Remark 1.** There are 4-dimensional quadratic flexible real division algebras which are not isotopic to the algebra $\mathbb{H}$. Indeed, for every real number $\lambda$ the mutation algebra $\mathbb{H}^{(\lambda)} := (\mathbb{H}, \cdot) \quad (x \cdot y = \lambda xy + (1 - \lambda)y)\) is a real quadratic flexible algebra with unit element 1. If, moreover, $\lambda$ is different from $\frac{1}{2}$ then $\mathbb{H}^{(\lambda)}$ is a division algebra. If, in addition, $\lambda \neq 0, 1$ then $\mathbb{H}^{(\lambda)}$ is not associative and cannot be isomorphic to the associative algebra $\mathbb{H}$ [27, p. 53]. However, the algebra $\mathbb{H}^{(\lambda)}$ is trivially embedded into the algebra $\mathcal{O}^{(\lambda)}$.

Next, we need the following preliminary result:

**Lemma 1.** Let $(\text{Im}(\mathbb{H}), \wedge, (.,.))$ be the associative Cayley real division algebra $\mathbb{H}$ and let $\psi : \mathbb{H} \to \mathbb{H}$ be a linear bijection such that

1. $\psi(1) = 1$.
2. $\psi$ leaves invariant $\text{Im}(\mathbb{H})$, and
3. $\psi$ has positiv spectrum.

Then $E_{0,\sigma_{\mathbb{H}},\psi\sigma_{\mathbb{H}}}(\mathbb{H})$ is a division algebra.

**Proof.** Let $x, y, x', y'$ be arbitrary elements of $\mathbb{H}$ with $(x, y) \neq (0, 0)$ such that $(x, y)(x', y') = (0, 0)$. We can assume, without lost of generality, that $y \neq 0$. We have

$$(x, y)(x', y') = (0, 0) \Leftrightarrow \begin{cases} xx' - y'y = 0 \quad (1) \\ y\psi(x') + y'x = 0 \quad (2) \end{cases}$$

So

$$0 = -y'(xx' - \overline{y'y}) + (y\psi(x') + y'x)x'$$

$$= |y'|^2 y + y\psi(x')x'$$

$$= y(|y'|^2 + \psi(x'))$$

and consequently

\[1\] Here the concept of Cayley-Dickson process was used from an arbitrary 4-dimensional real quadratic (not necessarily Cayley) division algebra $\mathcal{A}$. 
\[ |y'|^2 + \psi(x')x' = 0. \] (2.2)

Write \( x' = \alpha + u \), we have

\[
\psi(x')x' = (\alpha - \psi(u))(\alpha + u) = \alpha^2 + \alpha(u - \psi(u)) - \psi(u)u.
\]

As \( \psi(x')x' = -|y'|^2 \) is a scalar, its vector part \( \alpha(u - \psi(u)) - \psi(u)u \) must vanish. So \( \psi(u), u \) are linearly dependent [30, Theorem 3 (iii)], that is \( u \) is an eigenvector of \( \psi \) associated to a positive eigenvalue \( \lambda \). Thereby \( \alpha(u - \psi(u)) = 0 \) and we have:

\[
0 \leq \alpha^2 - \lambda u^2 \text{ by (2.1)} \\
= \psi(x')x' \\
= -|y'|^2 \\
\leq 0.
\]

So \( y' = 0 \) and also is \( x' \). \( \square \)

**Corollary 1.** Let \( \psi \) be an endomorphism of space \( \mathbb{H} \) whose matrix with respect to the canonical basis \( \{1, i, j, k\} \) is \( \text{diag}\{1, \alpha, \beta, \gamma\} \) where \( \alpha, \beta, \gamma \) are positive real numbers. Then \( E_{0, \sigma_\mathbb{H},\psi|\mathbb{H}}(\mathbb{H}) \) is an eight-dimensional real division algebra.

An algebra \( A \) is said to be **third-power associative** if it satisfies \( xx^2 = x^2x \) for all \( x \) in \( A \).

**Example 1.** Let \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) be non-zero real numbers and denote by \( \mathbb{H}(\alpha, \beta, \gamma, \alpha', \beta', \gamma') := (\mathbb{H}, \odot) \) the real algebra \( A \) having a basis \( \{1, i, j, k\} \) for which the multiplication is given by the table

<table>
<thead>
<tr>
<th>( \odot )</th>
<th>1</th>
<th>i</th>
<th>j</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
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<tr>
<td>i</td>
<td>i</td>
<td>-1</td>
<td>\gamma k</td>
<td>-\beta' j</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>-\gamma' k</td>
<td>-1</td>
<td>\alpha i</td>
</tr>
<tr>
<td>k</td>
<td>k</td>
<td>\beta j</td>
<td>-\alpha' i</td>
<td>-1</td>
</tr>
</tbody>
</table>

[10, (15.7), p. 179]. It is shown that if \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) are all positive and \( \alpha + \beta + \gamma - \alpha\beta\gamma = \alpha' + \beta' + \gamma' - \alpha'\beta'\gamma' \) then \( (\mathbb{H}, \odot) \) is a division algebra [10, Theorem 15A, p. 180]. Moreover, \( (\mathbb{H}, \odot) \) has an unit element 1. If, in addition, \( \gamma \neq \gamma' \) then
\[(i + j) \odot ((i + j) \odot (i + j)) = -2(i + j) + (\gamma - \gamma')(\alpha i - \beta' j)
\neq -2(i + j) + (\gamma - \gamma')(-\alpha' i + \beta j)
= ((i + j) \odot (i + j)) \odot (i + j).
\]

So \((\mathbb{H}, \odot)\) is not third-power associative and cannot be isotopic to \(\mathbb{H}\) [27, p. 53].

Now, according to the notations of Example 1, we have the following:

**Proposition 3.** Let \(\alpha\) be a positive real number different from 1. Then \(\mathbb{H}(1, \alpha, 1, 1, 1, \alpha)\) is a four-dimensional real division algebra which is neither isotopic to \(\mathbb{H}\) nor third-power associative but can be embedded into an eight-dimensional real division algebra.

**Proof.** Let \(\psi\) be the isomorphism of space \(\mathbb{H}\) whose matrix with respect to the canonical basis \(\{1, i, j, k\}\) is \(\text{diag}\{1, \alpha, 1, 1\}\). Then \(\psi\) satisfies to the hypothesis in Lemma 1 and we have an 8-dimensional real division algebra \(E_{0,\sigma_{\mathbb{R}},\psi_{\mathbb{R}}}((\mathbb{H}))\). Let \(1, i, j, k, f, if, jf, kf\) be the canonical basis of space \(E_{0,\sigma_{\mathbb{R}},\psi_{\mathbb{R}}}((\mathbb{H}))\). It is easy to see that \(\text{Span}\{1, i, f, if\}\) is a 4-dimensional sub-algebra of \(E_{0,\sigma_{\mathbb{R}},\psi_{\mathbb{R}}}((\mathbb{H}))\) isomorphic to \(\mathbb{H}(1, \alpha, 1, 1, 1, \alpha)\). Moreover, \(\mathbb{H}(1, \alpha, 1, 1, 1, \alpha)\) is neither isotopic to \(\mathbb{H}\) nor third-power associative.

\[\Box\]

### 3. On the Groups of Automorphisms of the Isootope and the Duplication of a Real Division Algebra

Let \(A\) be a real division algebra and denote by \(\text{Aut}(A)\), \(\text{Der}(A)\) its group of automorphisms and Lie algebra of derivations. It is known that \(\text{Aut}(A)\) is a compact Lie group whose Lie algebra is \(\text{Der}(A)\). The exponential \(\exp : \text{Der}(A) \rightarrow \text{Aut}(A)\) satisfies \(\exp \big(\text{Der}(A)\big) = \text{Aut}(A)^0\) where \(\text{Aut}(A)^0\) is the connected component of the identity. We have

\[\text{Aut}(\mathbb{R}) = \{I_\mathbb{R}\}, \quad \text{Aut}(\mathbb{C}) = \{I_\mathbb{C}, \sigma_\mathbb{C}\},
\]

\[\text{Aut}(\mathbb{H}) = SU(2) \quad \text{and} \quad \text{Aut}(\mathbb{O}) = G_2.\]

If \(\psi \in \text{Aut}(A)\) and \(\partial \in \text{Der}(A)\) then the eigenvalues of \(\partial\) are purely imaginary and the eigenvalues of \(\psi\) are of length one. In particular, 0 is the only possible real eigenvalue of \(\partial\) and 1, \(-1\) are the only possible eigenvalues of \(\psi\) (see [6] for more details).

Let \(f, g, \phi\) be three isomorphisms of \(A\). The group of automorphisms of \(A\) which commute with \(f\) and \(g\), noted \(\text{Aut}(A)^{f,g}\), is a compact subgroup of \(\text{Aut}(A_{f,g})\) whose Lie algebra is \(\text{Der}(A)^{f,g}\). Moreover, the map \(\phi \mapsto (\phi, \phi)\) is an embedding of \(\text{Aut}(A)^\phi := \text{Aut}(A)^{\phi,\phi}\) as a compact subgroup of \(\text{Aut}(E_{\delta,\phi}(A))\) whose Lie algebra is isomorphic to \(\text{Der}(A)^\phi\).
Proposition 4. Let $A$ be a real division algebra having an unit $1$ and let $f, g : A \to A$ be two linear bijections fixing $1$. Then:

1. $\text{Aut}(A)^{f,g} = \{ \phi \in \text{Aut}(A_{f,g}) : \phi(1) = 1 \}$. 
2. If $A_{f,g}$ has a right or a left unit then $\text{Aut}(A_{f,g}) = \text{Aut}(A)^{f,g}$. In particular, $\text{Aut}(A_{1_A}) = \text{Aut}(A)^f$ and $\text{Aut}(A_{1_A,g}) = \text{Aut}(A)^g$.
3. $\text{Der}(A)^f = \text{Der}(A_{f,1_A})$.
4. If $f$ is diagonalizable over $\mathbb{R}$ with all eigenspaces of dimension $1$ then $\text{Der}(A_{f,1_A}) = \{0\}$ and hence $\text{Aut}(A_{f,1_A})$ is finite.

Proof. (1) It is obvious that $\text{Aut}(A)^{f,g} \subset \{ \phi \in \text{Aut}(A)_{f,g} : \phi(1) = 1 \}$.

Conversely, let $\phi \in \text{Aut}(A_{f,g})$ such that $\phi(1) = 1$. Then for any $x \in A$,

$$
\phi(f(x)) = \phi(x \circ 1) = \phi(x) \circ \phi(1) = f(\phi(x)).
$$

A same argument show that $\phi \circ g = g \circ \phi$ and hence $\phi$ commutes with $f$ and $g$. Moreover, for any $x, y \in \mathbb{A}$,

$$
\phi(xy) = \phi(f^{-1}(x) \circ g^{-1}(y)) = \phi(f^{-1}(x)) \circ \phi(g^{-1}(y)) = \phi(x)\phi(y), \quad \phi \in C(f,g) \quad (\text{the commutant of } f \text{ and } g).
$$

(2) Note first that $1$ is an idempotent of $A_{f,g}$ and hence if $A_{f,g}$ admits a right unit or a left unit then it must be equal to $1$. Moreover, if $\phi$ is an automorphism of $A_{f,g}$ then $\phi(1)$ is an idempotent and hence $\phi(1) = 1$. Moreover, $1$ is right unit (resp. left unit) of $A_{1_A,g}$ (resp. $A_{f,1_A}$).

(3) It is obvious that $\text{Der}(A)^f \subset \text{Der}(A_{f,1_A})$.

Conversely, let $\partial \in \text{Der}(A_{f,1_A})$ and denote by $\circ$ the product in algebra $A_{f,1_A}$. We have

$$
\partial 1 = \partial (1 \circ 1) = (\partial 1) \circ 1 + 1 \circ \partial 1 = f(\partial 1) + \partial 1.
$$

So $f(\partial 1) = 0$ and then $\partial 1 = 0$. This allow on to show easily that $\partial$ commutes with $f$ and is a derivation of algebra $A$.

(4) Let $\partial$ be in $\text{Der}(A_{f,1_A})$ then $\partial$ is a derivation of $A$ commuting with $f$. So $\partial$ leaves invariant the eigenspaces of $f$ and hence $\partial$ is diagonalizable over $\mathbb{R}$. Therefore $\partial$ must vanishes since the only possible real eigenvalue of $\partial$ is $0$. \qed
Theorem 2. Let $\delta \in \mathbb{R}$ be such that $|\delta| < 2$ and let $g : \mathbb{H} \to \mathbb{H}$ be a linear bijection fixing 1, leaving invariant $\text{Im}(\mathbb{H})$ and such that $g-I_H$ induces a bijection from $\text{Im}(\mathbb{H})$ onto itself. Let $E_{\delta,g}(\mathbb{H})(f_H, f_H) := A$. Then

$$\text{Aut}(A) = \begin{cases} \{(\phi, \phi), \phi \in \text{Aut}(\mathbb{H})^g\} & \text{if } \delta \neq 0, \\ \{(\phi, d\phi), \phi \in \text{Aut}(\mathbb{H})^g, d \in S^3\} & \text{if } \delta = 0. \end{cases}$$

Moreover, if $\delta \neq 0$ then $\text{Der}(A)$ is isomorphic to either $su(2)$ or an abelian algebra of dimension 0 or 1. If $\delta = 0$ then $\text{Der}(A)$ is isomorphic to either $su(2) \oplus su(2)$ or $su(2) \oplus N$ where $N$ is an ideal of dimension 0 or 1.

Proof. Recall that $E_{\delta,g}(\mathbb{H})$ is the algebra $\mathbb{H} \times \mathbb{H}$ endowed with the product

$$(x, y) \ast (x', y') = \left( x x' - y y', y x' + x y' + \frac{\delta}{2} [y', y] \right).$$

It is a RDA with $(1, 0)$ as an unit. According to the assertion (2) of Proposition 4,

$\text{Aut}(A) = \text{Aut}(E_{\delta,g}(\mathbb{H}))^{(g, I_H)}.$

Let now $\Phi \in \text{Aut}(E_{\delta,g}(\mathbb{H}))$ which commutes with $(g, I_H)$. Write, for any $x, y \in \mathbb{H},$

$$\Phi(x, y) = \Phi(x, 0) + \Phi(0, y) = (\phi(x), \psi(x)) + \Phi(0, y) \quad \text{and} \quad \Phi(0, 1) = (c, d),$$

where $\phi, \psi$ are two endomorphisms of $\mathbb{H}$ and $c, d \in \mathbb{H}$. The relation $(\Phi \circ (g, I_H))(x, 0) = ((g, I_H) \circ \Phi)(x, 0)$ gives

$$\phi \circ g = g \circ \phi \quad \text{and} \quad \psi(g(x) - x) = 0, \quad x \in \mathbb{H}.$$ 

This implies that $\psi|_{\text{Im}(\mathbb{H})} = 0$. But $\Phi(1, 0) = (1, 0)$ so $\psi(1) = 0$ thus $\psi = 0$. Now, for $x \in \mathbb{H}$, $\Phi(x, 0) = (\phi(x), 0)$ which implies that $\phi$ is an automorphism of $\mathbb{H}$ and hence $\phi \in \text{Aut}(\mathbb{H})^g$. On the other hand, for any $y \in \mathbb{H},$

$$(y, 0) \ast (0, 1) = (0, y).$$

So,

$$\Phi(0, y) = \Phi(y, 0) \ast (c, d) = (\phi(y), 0) \ast (c, d) = (\phi(y)c, d\phi(y)).$$

The relation $\Phi \circ (g, I_H) = (g, I_H) \circ \Phi$ gives that for any $y \in \mathbb{H},$

$$\left(g(\phi(y)c), d\phi(y)\right) = (\phi(y)c, d\phi(y)).$$

This implies that, for any $y \in \mathbb{H}$, $\phi(y)c \in \text{ker}(g - I_H) = \mathbb{R}1$. So there exists a linear form $\alpha \in \mathbb{H}^*$ such that, for any $y \in \mathbb{H}$, $\phi(y)c = \alpha(y)$. Since $\phi$ is bijective this is impossible unless $c = 0$. So, we have shown so far that, for any $x, y \in \mathbb{H},$

$$\Phi(x, y) = (\phi(x), d\phi(y)), \quad d \neq 0.$$
The relation $\Phi((x, y)^{\delta} (x', y')) = \Phi(x, y)^{\delta} \Phi(x', y')$ is equivalent to
\[
\frac{\delta}{2} [d\phi(y'), \phi(y)] = \frac{\delta}{2} [d\phi(y'), d\phi(y)].
\]
Since $\phi$ is bijective, the first equation is equivalent to $|d| = 1$. Assume now that $\delta \neq 0$, then the second equation gives for $y' = 1$, $[d, d\phi(y)] = 0$ for any $y \in \mathbb{H}$ and hence $d$ is a scalar. The case $d = -1$ is impossible according to the second equation and hence $d = 1$. To complete the proof of the theorem we need to determine the dimension of the Lie group $\text{Aut}(\mathbb{H})^g$. Its Lie algebra is $\text{su}(2)^g$ which is a subalgebra of $\text{su}(2)$. But it is well-known that a subalgebra of $\text{su}(2)$ has dimension 0, 1 or 3. The result of Benkart-Osborn in [6, p. 1135] permits to achieve the proof of the theorem.

All the possibilities of $\text{Der}(E_{\delta, \sigma, H}(I_H, I_H), (g, I_H))$ given in Theorem 2 can be realized as shows the following examples.

**Example 2.** Put $\mathcal{D} = \text{Der}(E_{\delta, \sigma, H}(I_H, I_H), (g, I_H))$ where $g$ is as defined in Theorem 2. We assume, in addition, that $g$ is diagonalizable over $\mathbb{R}$. Let $N$ be the number of elements of the spectrum of $g$. We have

1. If $N = 4$ then $\mathcal{D} = \{0\}$ if $\delta \neq 0$ and is isomorphic to $\text{su}(2)$ if $\delta = 0$.
2. If $N = 3$ then $\mathcal{D}$ is isomorphic to $N$ if $\delta \neq 0$ and $\text{su}(2) \oplus N$ if $\delta = 0$ where $N$ is an ideal of dimension 1.
3. If $N = 2$ then $\mathcal{D}$ is isomorphic to $\text{su}(2)$ if $\delta \neq 0$ and $\text{su}(2) \oplus \text{su}(2)$ if $\delta = 0$.

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**References**


Embedding problems for real division algebras


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