A Note on Linearly Independence over the Symmetrized Max-Plus Algebra

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Abstract
The symmetrized max-plus algebra is an algebraic structure which is a commutative semiring, has a zero element $\epsilon = -\infty$, the identity element $e = 0$, and an additively idempotent. Motivated by the previous study as in conventional linear algebra, in this paper will be described the necessary and sufficient condition of linear independent over the symmetrized max-plus algebra. We show that a columns of a matrix over the symmetrized max-plus algebra are linear dependent if and only if the determinat of that matrix is $\epsilon$.

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1 Introduction

The system max-plus algebra lacks an additive inverse. Therefore, some equations do not have a solution. For example, the equation $3\oplus x = 2$ has no solution since there is no $x$ such that $\max(3,x) = 2$. In De Schutter [3] and Singh [6] state that one way of trying to solve this problem is to extend the max-plus algebra to a larger system which will include and additive inverse in the same way that the natural numbers were extended to the larger system of integers. Therefore, we have that the system $(S, \oplus, \otimes)$ is called the symmetrized max-plus algebra and $S = \mathbb{R}_+^2 / \mathcal{B}$ with $\mathcal{B}$ is an equivalence relation. As in conventional linear algebra we can define the linearly dependence and independence of vectors in the max-plus sense [4]. However the definitions are a little more complex. Akian et al. [1] and Farlow [4] state that a family $m_1, m_2, ..., m_k$ of elements of a semimodule $M$ over a semiring $S$ is linearly dependent in the Gondran-Minoux sense if there exist two subset $I, J \subseteq K := 1, 2, ..., k, I \cap J = \emptyset, I \cup J = K$, and scalars $\alpha_1, \alpha_2, ..., \alpha_k \in S$, not all equal to 0, such that $\sum_{i \in I} \alpha_i m_i = \sum_{j \in J} \alpha_j m_j$. Motivated by linearly dependence and independence over the max-plus algebra, in this paper will be described linearly independence over the symmetrized max-plus algebra.

In Section 2, we will review some basic facts for the symmetrized max-plus algebra and a matrix over the symmetrized max-plus algebra, following [3] and [6].

2 Some Preliminaries on The Symmetrized Max-Plus Algebra

In this section, we review some basic facts for the symmetrized max-plus algebra and a matrix over the symmetrized max-plus algebra, following [3] and [6].
2.1 The Symmetrized Max-Plus Algebra

Let the set of all real numbers $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$ with $\epsilon := -\infty$ and $\epsilon := 0$. For all $a, b \in \mathbb{R}_\epsilon$, the operations $\oplus$ and $\otimes$ are defined as $a \oplus b = \max (a, b)$ and $a \otimes b = a + b$. Then, $(\mathbb{R}_\epsilon, \oplus, \otimes)$ is called the max-plus algebra.

**Definition 2.1.** [3] Let $u = (x, y), v = (w, z) \in \mathbb{R}_\epsilon^2$.

1. Two unary operators $\ominus$ and $(.)^\bullet$ are defined as $\ominus u = (y, x)$ and $u^\bullet = u \oplus (\ominus u)$.

2. An element $u$ is called balances with $v$, denoted by $u \nabla v$, if $x \oplus z = y \oplus w$.

3. A relation $\mathcal{B}$ is defined as follows:

\[
(x, y) \mathcal{B}(w, z) \text{ if } \begin{cases} 
(x, y) \nabla (w, z), & \text{if } x \neq y \text{ and } w \neq z \\
(x, y) = (w, z), & \text{otherwise}
\end{cases}
\]

Because $\mathcal{B}$ is an equivalence relation, we have the set of factor $\mathbb{S} = \mathbb{R}_\epsilon^2 / \mathcal{B}$ and the system $(\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max-plus algebra, with the operations of addition and multiplication on $\mathbb{S}$ is defined as follows

\[
(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)
\]

\[
(a, b) \otimes (c, d) = (a \odot c \oplus b \odot d, a \otimes d \odot b \otimes c)
\]

for $(a, b), (c, d) \in \mathbb{S}$. The system $(\mathbb{S}, \oplus, \otimes)$ is a semiring, because $(\mathbb{S}, \oplus)$ is associative, $(\mathbb{S}, \otimes)$ is associative, and $(\mathbb{S}, \oplus, \otimes)$ satisfies both the left and right distributive.

**Lemma 2.2.** [2] Let $(\mathbb{S}, \oplus, \otimes)$ be the symmetrized max-plus algebra. Then the following statements holds.

1. $(\mathbb{S}, \oplus, \otimes)$ is commutative.

2. An element $(\epsilon, \epsilon)$ is a zero element and an absorbent element.

3. An element $(\epsilon, \epsilon)$ is an identity element.

4. $(\mathbb{S}, \oplus, \otimes)$ is an additively idempotent.

The system $\mathbb{S}$ is divided into three classes, there are $\mathbb{S}^\omega$ consists of all positive elements or $\mathbb{S}^\omega = \{(t, \epsilon) | t \in \mathbb{R}_\epsilon \}$ with $\{(t, x) \in \mathbb{R}_\epsilon^2 | x < t\}$, $\mathbb{S}^\ominus$ consists of all negative elements or $\mathbb{S}^\ominus = \{(\epsilon, t) | t \in \mathbb{R}_\epsilon \}$ with $\{(\epsilon, x) \in \mathbb{R}_\epsilon^2 | x < t\}$, and $\mathbb{S}^\bullet$ consists of all balanced elements or $\mathbb{S}^\bullet = \{(t, t) | t \in \mathbb{R}_\epsilon \}$ with $\{(t, t) \in \mathbb{R}_\epsilon^2 \}$. Because $\mathbb{S}^\ominus$ isomorphic with $\mathbb{R}_\epsilon$, so it will be shown that for $a \in \mathbb{R}_\epsilon$, can be expressed by $(a, \epsilon) \in \mathbb{S}^\ominus$.

Furthermore, it is easy to verify that for $a \in \mathbb{R}_\epsilon$, we have that $a = \overline{(a, \epsilon)}$ with $(a, \epsilon) \in \mathbb{S}^\ominus$, $\ominus a = \ominus (a, \epsilon) = \ominus (a, \epsilon) = \ominus (a, \epsilon)$ with $(\epsilon, a) \in \mathbb{S}^\ominus$, and $a^\bullet = a \oplus a = (a, \epsilon) \ominus (a, \epsilon) = (a, \epsilon) \oplus (\epsilon, a) = (a, a) \in \mathbb{S}^\bullet$. 

Linearly independence over the symmetrized max-plus algebra
Lemma 2.3. For $a, b \in \mathbb{R}_{\epsilon}, a \ominus b = (a, b)$.

Proof. 

\[ a \ominus b = (a, \epsilon) \ominus (b, \epsilon) = (a, \epsilon) \oplus (\epsilon, b) = (a, b). \]

\[ \square \]

Lemma 2.4. For $(a, b) \in S$ with $a, b \in \mathbb{R}_{\epsilon}$, the following statements hold:

1. If $a > b$ then $(a, b) = (a, \epsilon)$.
2. If $a < b$ then $(a, b) = (\epsilon, b)$.
3. If $a = b$ then $(a, b) = (a, a)$ or $(a, b) = (b, b)$.

Proof. 

1. For $a > b$ we have that $a \oplus b = a$. In other words, $a \oplus \epsilon = a \oplus b$.
   The result that $(a, b) \triangledown (a, \epsilon)$. So it follows that $(a, b) B (a, \epsilon)$.
   Therefore $(a, b) = (a, \epsilon)$.

2. For $a < b$ we have that $a \oplus b = b$. In other words, $a \oplus \epsilon = b \oplus \epsilon$.
   The result that $(a, b) \triangledown (\epsilon, b)$. So it follows that $(a, b) B (\epsilon, b)$.
   Therefore $(a, b) = (\epsilon, b)$.

\[ \square \]

Corollary 2.5. For $a, b \in \mathbb{R}_{\epsilon}, a \ominus b = \begin{cases} a, & \text{if } a > b \\ \ominus b, & \text{if } a < b \\ a^*, & \text{if } a = b \end{cases}$

2.2 Matrices over The Symmetrized Max-Plus Algebra

Let $S$ the symmetrized max-plus algebra, $n$ a positive integer greater than 1 and $M_n(S)$ is the set of all $nxn$ matrices over $S$. Operation $\oplus$ and $\otimes$ for matrices over the symmetrized max-plus algebra are defined as $C = A \oplus B$ where $c_{ij} = a_{ij} \oplus b_{ij}$ and $C = A \otimes B$ where $c_{ij} = \bigoplus_{l} a_{il} \otimes b_{lj}$. Zero matrix $nxn$ over $S$ is $\epsilon_n$ with $(\epsilon_n)_{ij} = \epsilon$ and identity matrix $nxn$ over $S$ is $E_n$ with $[E_n]_{ij} = \begin{cases} e, & \text{if } i = j \\ e, & \text{if } i \neq j \end{cases}$.

Definition 2.6. We say that the matrix $A \in M_n(S)$ is invertible over $S$ if $A \otimes B \triangledown E_n$ dan $B \otimes A \triangledown E_n$ for any $B \in M_n(S)$.

Definition 2.7. [5] Let a matrix $A \in M_n(S)$. The determinant of $A$ is defined by $\det A = \bigoplus_{\sigma \in S_n} \text{sgn}(\sigma) \otimes (\bigotimes_{i=1}^{n} A_{i\sigma(i)})$ with $S_n$ is the set of all permutations of $\{1, 2, ..., n\}$, and $\text{sgn}(\sigma) = \begin{cases} 0, & \text{if } \sigma \text{ is even permutation} \\ \ominus 0, & \text{if } \sigma \text{ is odd permutation} \end{cases}$.

Note that the operator “$\triangledown$” and the systems of max-linear balances hold:
Lemma 2.8. [2]
1. \( \forall a, b, c \in S, a \ominus c \nabla b \Leftrightarrow a \nabla b \oplus c \)
2. \( \forall a, b \in S^\oplus \cup S^\ominus, a \nabla b \Rightarrow a = b \)
3. Let \( A \in M_n(S) \). The homogeneous linear balance \( A \otimes x \nabla \epsilon_{nx1} \) has a non trivial solution in \( S^\oplus \) or \( S^\ominus \) if and only if \( \det(A) \nabla \epsilon \).

3 Linear Independence over The Symmetrized Max-Plus Algebra

The symmetrized max-plus algebra is an idempotent semiring, so in order to define rank, linear combination, linear dependence, and independence we need definition of a semimodule. A semimodule is essentially a linear space over a semiring.

Let \( S \) be a semiring with \( \epsilon \) as a zero. A (left) semimodule \( M_{nx1}(S) \) over \( S \) is a commutative monoid \((S, \oplus)\) with zero element \( \epsilon \in M_{nx1}(S) \), together with an \( S \)–multiplication

\[
S \times M_{nx1}(S) \rightarrow M_{nx1}(S), (r, x) \rightarrow r \otimes x
\]

such that, for all \( r, s \in S \) and \( x, y \in M_{nx1}(S) \), we have

1. \( r \otimes (s \otimes x) = (r \otimes s) \otimes x \)
2. \( (r \oplus s) \otimes x = r \otimes x \oplus s \otimes x \)
3. \( e \otimes x = x \)
4. \( r \otimes \epsilon = \epsilon \)
5. \( r \otimes (x \oplus y) = r \otimes x \oplus r \otimes y \)

In a similar way, a right semimodule can be defined. The rank, linear combination, and linear independent are given in the next definitions.

**Definition 3.1.** [3] Let \( A \in M_{mxn}(S) \). The max-algebraic minor rank of \( A \), \( \text{rank}_\oplus(A) \), is the dimension of the largest square submatrix of \( A \) the max-algebraic determinant of which is not balanced.

**Definition 3.2.** Let \( a_1, a_2, ..., a_n \in M_{nx1}(S) \) and \( \alpha_1, \alpha_2, ..., \alpha_m \in S \). The expression \( \bigoplus_{i=1}^{m} \alpha_i \otimes a_i \) is called a linear combination of \( \{a_1, a_2, ..., a_n\} \).

**Definition 3.3.** A set of vectors \( \{a_i \in M_{nx1}(S)\}_{i=1}^{m} \) is said to be a linear independent set whenever the only solution for the scalars \( \alpha_i \) in \( \bigoplus_{i=1}^{m} \alpha_i \otimes a_i \nabla \epsilon_{nx1} \) is the trivial solution \( \alpha_i = \epsilon \).
The relation between linear dependent and linear combination are given in
the following theorem.

**Lemma 3.4.** If the set of vectors \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) is linear dependent then one of the vectors can be presented as a linear combination of the other vectors in the set.

**Proof.** Let \( \alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \epsilon \).

Because \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) is a linear dependent set, without loss of generality, we can take \( \alpha_1 \neq \epsilon \). So, there is a scalar \( \alpha_1^{-1} \) such that

\[
\alpha_1^{-1} \otimes (\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n) \nabla a_1^{-1} \otimes \epsilon
\]

\[
a_1 \oplus \alpha_1^{-1} a_2 \oplus \ldots \oplus \alpha_1^{-1} \otimes \alpha_n \otimes a_n \nabla \epsilon
\]

\[
a_1 \nabla \beta_2 \otimes a_2 + \beta_3 \otimes a_3 \oplus \ldots \oplus \beta_n \otimes a_n \quad \text{or} \quad a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i
\]

This leads to a characterization of linear dependent in term of determinants.

**Theorem 3.5.** Let \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) be a vector set. Construct a matrix \( A \) such that \( A = \left( \begin{array}{ccc} a_1 & a_2 & \ldots & a_n \end{array} \right) \). The set of vectors \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) are linear dependent if and only if \( \det(A) \nabla \epsilon \).

**Proof.** (\( \Rightarrow \)) Because \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) is a linear dependent set, this implies that one of the vectors (without loss of generality, let’s say its the vector \( a_1 \)) can be presented as a linear combination of the other vectors in the set. Or,

\[
a_1 \nabla \beta_2 \otimes a_2 + \beta_3 \otimes a_3 \oplus \ldots \oplus \beta_n \otimes a_n \quad \text{or} \quad a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i
\]

Next, take matrix \( A \) and subtract \( a_1 \) with \( \bigoplus_{i=2}^{n} \beta_i \otimes a_i \). This results in another matrix (say \( A' \)) whose last column is a \( \epsilon \) vector.

\[
A = \left( \begin{array}{ccc} a_1 \cdot a_2 & \ldots & a_n \end{array} \right) \nabla \left( \begin{array}{c} \epsilon \quad a_2 \quad \ldots \quad a_n \end{array} \right)
\]

Because \( \det \left( \begin{array}{ccc} \epsilon & a_2 & \ldots & a_n \end{array} \right) \nabla \epsilon \) so, we now have that \( \det(A') \nabla \epsilon \).

It follows from the fact that one of the columns of the matrix being \( \epsilon \), that \( \det(A) \nabla \epsilon \).

(\( \Leftarrow \)) Let \( \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\} \) is a linear independent. We can show that \( \det(A) \) not balanced with \( \epsilon \).

Construct

\[
\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \epsilon.
\]
We have that
\[ a_1 \otimes \alpha_1 \oplus a_2 \otimes \alpha_2 \oplus \ldots \oplus a_n \otimes \alpha_n \nabla \epsilon. \]

Consequently, \((a_1 \ a_2 \ \ldots \ a_n) \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \epsilon.\]

Because \(\{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\}\) is a linear independent, so we have
\[ \alpha_1 = \alpha_2 = \ldots = \alpha_n = \epsilon \]

We can see, that homogenous linear balance \(A \otimes x \nabla \epsilon\) has a trivial solution \(x = \epsilon\). Since it follows from lemma 2.8 that the homogeneous linear balance \(A \otimes x \nabla \epsilon\) has a non trivial signed solution if and only if \(\text{det}A \nabla \epsilon\), so we have \(\text{det}(A)\) not balanced with \(\epsilon\).

**Example 3.6.** Let \(\{v_1, v_2, v_3, v_4\} \subseteq M_{4\times1}(S)\) with

\[ v_1 = \begin{pmatrix} 0 \\ \ominus7 \\ \epsilon \\ \ominus2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \ominus2 \\ 9 \\ 5 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} \epsilon \\ 3 \\ 6 \\ 0 \end{pmatrix}, \quad \text{and} \quad v_4 = \begin{pmatrix} 0 \\ \ominus7 \\ 9 \\ 3 \end{pmatrix} \]

We have, \(v_2\) and \(v_4\) are linear combinations from \(\{v_1, v_3\}\) such that
\[ v_2 = \ominus2 \otimes v_1 \oplus (-1) \otimes v_3 \]

and
\[ v_4 = v_1 \oplus 3 \otimes v_3. \]

We have \(\text{det} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \text{det} \begin{pmatrix} 0 & \ominus2 & \epsilon & 0 \\ \ominus7 & 9 & 3 & \ominus7 \\ \epsilon & 5 & 6 & 9 \\ \ominus2 & 4 & 0 & 3 \end{pmatrix} = (18 \oplus 16 \oplus \epsilon \ominus 12) \ominus (\ominus 10 \ominus 17 \oplus 18 \oplus \epsilon) = 18 \ominus 18 = 18 \nabla \epsilon. \]

From Theorem 3.5, we have that \(\{v_1, v_2, v_3, v_4\}\) is linearly dependent.

The following theorem shows that relation between non trivial solution and the rank of matrix in the homogeneous linear balance.

**Theorem 3.7.** Let \(A \in M_{mxn}(S)\).
The homogeneous linear balance \(A \otimes x \nabla \epsilon\) has a non trivial signed solution (i.e. \(x \notin M_{nx1}(S^*)\)) if and only if \(\text{rank}_{\oplus}(A) < n\).

**Proof.** \((\Rightarrow)\) Let \(\text{rank}(A) = n = r\). We have \(A_r \otimes x \nabla \epsilon\) can be represented as
\[ A_r \otimes x = \begin{pmatrix} E_r \\ \epsilon \end{pmatrix} \nabla \epsilon. \]
So, the only solution of $A_r \otimes x \nabla \epsilon$ is $x \nabla \epsilon$.

($\Leftarrow$) Let $A \in M_{m \times n}(S)$ and $\text{rank}(A) = r < n$.
Suppose $A_r$, the row echelon form of $A$ can be represented as a form

$$A_r = \begin{pmatrix}
  e & e & \ldots & e & \ast & \ldots & \ast \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  e & e & \ldots & e & \ast & \ldots & \ast \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  e & e & \ldots & e & \ast & \ldots & \ast \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  e & e & \ldots & e & \ast & \ldots & \ast \\
\end{pmatrix} \otimes \begin{pmatrix} E_r \\ C \\ \epsilon \end{pmatrix} = \begin{pmatrix} E_r \\ C \\ \epsilon \end{pmatrix}$$

Without loss of generality, the rank $A$ is $r$ and there is at least $r + 1$ columns.
Therefore, $C$ has at least one columns and $A_r \otimes x = \begin{pmatrix} E_r \\ C \\ \epsilon \end{pmatrix} \otimes x = \begin{pmatrix} E_r \\ C \\ \epsilon \end{pmatrix} \otimes \begin{pmatrix} z \\ y \end{pmatrix} \nabla \epsilon$. Therefore, $E_r \otimes z \oplus C \otimes y \nabla \epsilon$ or $z = \ominus C \otimes y$. We have shown that $x = \begin{pmatrix} \ominus C \otimes y \\ \epsilon \end{pmatrix}$ is non trivial solution for $y$ not balanced with $\epsilon$.

**Example 3.8.** Let $A \otimes x \nabla \epsilon$ with $A = \begin{pmatrix} \ominus 2 & 1 & \ast & \epsilon \\ \epsilon & 0 & \ominus 0 & \epsilon \\ 1 & 0 & \epsilon & 1 \end{pmatrix}$. With row echelon form, we have $\text{rank}(A) = 3 < 4$ and $C = \begin{pmatrix} \ominus (2) \\ \ominus (1) \\ \ominus (2) \end{pmatrix}$. We can show that for $y = 1$, we have $x = \begin{pmatrix} -1 \\ 0 \\ (-1) \ast \\ 1 \end{pmatrix}$ is nontrivial solution.

**Theorem 3.9.** Let the set of vectors $S = \{a_i \in M_{n \times 1}(S) | i = 1, 2, \ldots, n\}$. Construct a matrix $A$ such that $A = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$ with $m \times n$.
If $n > m$ then $S$ is linear dependent.

**Proof.** Let $\{a_i \in M_{n \times 1}(S) | i = 1, 2, \ldots, n\}$ and construct $\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \epsilon$. We have that $a_1 \otimes \alpha_1 \oplus a_2 \otimes \alpha_2 \oplus \ldots \oplus a_n \otimes \alpha_n \nabla \epsilon$. So, $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \epsilon$. Because $A = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$ with $m \times n$, $m < n$ and $\text{rank}(A) \leq n$, this implies that $\text{rank}(A) \leq m < n$. 
Since it follows from Theorem 3.7, so \((a_1 \ a_2 \ \ldots \ a_n) \otimes (\alpha_1 \ \alpha_2 \ \vdots \ \alpha_n)\) has a non trivial solution. Therefore, there is \(\alpha_i \neq \epsilon\). This means that \(S = \{a_i \in M_{nx1}(S)|i = 1, 2, \ldots, n\}\) is linearly dependent.

\[\square\]

**References**


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