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On Infinite Abelian Group fromN-nomial Set

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Abstract

In this paper, he look into the collection of n-nomial to form a set. Specifically, he characterizes the set under the binary operation. From such characterization an invariant subgroups of infinite abelian group and infinite quotient groups emerged.

Keywords: N-nomial set, Infinite Abelian Groups, Invariant Subgroups, Quotient Groups

1 Introduction

An abelian groups are considered very interesting and complicated algebraic structure, those that are infinite as well as those infinitely generated. Most of these are evident in [1], [2], [3], [4] being studied as a group in the complex number $\mathbb{C}$, real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$, integers $\mathbb{Z}$, including the $\mathbb{Q}/\mathbb{Z}$ quotient group, $\mathbb{C}^{\#}$ of all non-zero complex numbers, $\mathbb{R}^{\#}$ non-zero real numbers, $\mathbb{R}^{+}$ positive reals, $\mathbb{Q}^{\#}$ non-zero rationals and $\mathbb{Q}^{+}$ positive rationals. The results obtained from these infinite abelian groups penetrate other branches of Mathematics.

This paper presents the construction of infinite abelian groups from collection of n-nomial. This collection is denoted by $N=\{(x_1+x_2)^n,(x_1+x_2+\mathrm{x}_3)^n, ,...,(x_1+x_2+\mathrm{x}_3+...+x_m)^n\}, \quad \forall n\in Z$ and $\forall m\in Z^+$.A set N is an infinite set such that $N\sim Z$ and every element G_i of N is a group generated by $\left\langle g_i^{-1},g_i^{1}\right\rangle$ for every $i\in I$, G_i is an infinite abelian group , each subgroup of G_i is generated by either $\left\langle g_i^{2j+1},g_i^{2r-1}\right\rangle$ and $\left\langle g_i^{2j},g_i^{2r}\right\rangle$ $\forall j\in Z^- and\ \forall r\in Z^+$ with corresponding odd and even positive and negative integral exponents respectively. The two sets of distinct proper invariant subgroups $\{H_i\}$ and $\{S_i\}$ of group G_i are also infinite cyclic subgroup. The normality of $\{H_i\}$ and $\{S_i\}$ in G_i formed quotient groups $\{G_i / H_i\}$ and $\{G_i / S_i\}$.

2 Infinite Set

Definition. The n-nomial set is a collection of sum of the two terms, three terms up to m-terms of the power n, for every n∈ Z.

Proposition 1.

A collection of n-nomial is an infinite set.

Proof:
Let N be a collection of n-nomial such that every element is a power of $n \in Z$, that is, $N = \left\{(x_1 + x_2)^n, (x_1 + x_2 + x_3)^n, ..., (x_1 + x_2 + x_3 + ... + x_m)^n\right\}$. Consider a map, say $\phi : N \to Z$. For any elements $\left(x_1 + ... + x_{m-1}\right)^n$ and $\left(x_1 + ... + x_{m-1}\right)^{n-1}$ $\forall m \geq 3$ there exists a corresponding distinct images $\phi\left((x_1 + ... + x_{m-1})^n\right)$ and $\phi\left((x_1 + ... + x_{m-1})^{n-1}\right)$ in Z such that $\phi\left((x_1 + ... + x_{m-1})^n\right) \neq \phi\left((x_1 + ... + x_{m-1})^{n-1}\right)$, $\left(x_1 + ... + x_{m-1}\right)^n \neq \left(x_1 + ... + x_{m-1}\right)^{n-1}$. ϕ is one to one. Now for every $z \in Z$, there is at least one element $\left(x_1 + ... + x_{m-1}\right)^n$ in N satisfying $\phi\left((x_1 + ... + x_{m-1})^n\right) = z$. This indicates that, ϕ is onto. Hence ϕ is bijective, implies that $N \,\square\, Z$. Thus N is an infinite set. □

3. Infinite Abelian Group

Proposition 2.

If N is a collection of n-nomial then every element of N formed an infinite Abelian group G_i , $\forall i \in I$.

Proof:
Let N be a collection of n-nomial such that for every element G_i of N is a power of $n \in Z$. Now take any element of G_i, say $\left(x_1 + ... + x_{m-1}\right)^n, m \geq 3$, we can define a group G_i whose elements are of the form $\left(x_1 + ... + x_{m-1}\right)^n$ with even and odd positive and negative integral exponents. Let $\left(x_1 + ... + x_{m-1}\right)^n = g_i^n \in G_i$, for any two

positive and negative integral exponents. Let $\left(x_1+...+x_{m-1}\right)^n = g_i^n \in G_i$, for any two distinct numbers $r, j \in Z$, we have $g_i^j\ g_i^r = g_i^{j+r} = g_i^0$ and $\frac{g_i^r}{g_i^j} = g_i^1$, these indicate that $r \in Z^+$ and $j \in Z^-$. Let $r=1$, $g_i^r = g_i^1 \in G_i$ and assume that $r=k$ then for $r=k+1$, we have $g_i^j g_i^r = g_i^j g_i^{k+1} = (g_i^j g_i^k)g_i = (g_i^{j+k})g_i = g_i^{j+k+1}$ $= g_i^{(k+1)+\mathrm{j}} = g_i^{\mathrm{i+j}} = g_i^r g_i^j$. Clearly r generates all distinct power of $g_i^n = \left(x_1+...+x_{m-1}\right)^n$ for every $r \in Z^+$.Now consider $j=-1$ implies that $g_i^{-1} \in G_i$. Assume $j=-k$, then for $j=-(k+1)$, we have $g_i^j g_i^r = g_i^{-(k+1)} g_i^r = (g_i^{-k} g_i^r) g_i^{-1}$ $= (g_i^{r-k})g_i^{-1} = g_i^{r-k-1} = g_i^{r-(k+1)} = g_i^r g_i^j$ .This also indicates that j generates all distinct power of $g_i^n = \left(x_1+...+x_{m-1}\right)^n$ for all $j \in Z^-$. Hence $\left\langle g_i^r, g_i^j \right\rangle = \left\langle \left(x_1+...+x_{m-1}\right)^r, \left(x_1+...+x_{m-1}\right)^j \right\rangle$ is a generator of group G_i. Thus G_i is an infinite Abelian group. □

4. Subgroups of Group G_i

Proposition 3.
Every subgroup of a group G_i is generated by positive and negative integral exponent whose sum is zero.

Proof:
Let H_i be any subgroup of group G_i. Now consider two distinct numbers, say $t, s \in Z$ such that $t+s=0$ where s is an additive inverse of t, then there exist two possibilities for r and j values. For case one $t=2r$ and $s=2j$ for even positive and negative integral exponent respectively, such that when $r=1$ implies $t=2$, we have $g_i^2 \in G_i$. Assume that it holds true for $t=2(k)$ then $t=2(k+1)$, that is, $g_i^{2j} g_i^{2r} = g_i^{2j} g_i^{2(k+1)} = g_i^{2j} g_i^{2k+2} = (g_i^{2j} g_i^{2k}) g_i^2 = g_i^{2(j+k+1)} = g_i^{2(j+(k+1))} = g_i^{2((k+1)+j)}$ $= g_i^{2(k+1)} g_i^{2j} = g_i^{2r} g_i^{2j}$, $t=2r$ holds true for any $r \in Z^+$ implies that $\left\langle g_i^{2r} \right\rangle$ generates all subgroup of group G_i with even positive integral exponent. Now consider the negative even integral exponent $s=2j$ by induction on j.Let $j=-1$, $g_i^{-2} \in G_i$ and assume the $s=2(-k)$ then $s=2(-(k+1))$, $g_i^{2r} g_i^{2j} = g_i^{2r} g_i^{2(-(k+1))} = g_i^{2r} g_i^{-(2k+2)} = (g_i^{2r} g_i^{-2k}) g_i^{-2} = g_i^{2(r-k-1)} = g_i^{2(r-(k+1))}$ $= g_i^{2(-(k+1)+r)} = g_i^{2j} g_i^{2r} = g_i^{2r} g_i^{2j}$.We have $g_i^s = g_i^{2j}$ generates all subgroup of G_i with negative integral exponent. Hence $\left\langle g_i^{2r}, g_i^{2j} \right\rangle$ is the generator of all subgroups of even integral exponent of group G_i.

For second case: Take $t = 2r - 1$ and $s = 2j + 1$ for any positive and negative odd integral exponent of $g_i = (x_1 + ... + x_{m-1})^n \in G_i$, respectively. Let $r = 1$ implies the $t = 1$ and $g_i^1 = \left(x_1 + ... + x_{m-1}\right)^1 \in G_i$. Suppose g_i^{2k-1} holds true then $g_i^{2(k+1)-1}$ implies that $g_i^{2j+1} g_i^{2r-1} = (g_i^{2j+1} g_i^{2(k+1)-1}) = (g_i^{2j+1+2(k+1)}) g_i^{-1}$ $= (g_i^{2j+2k+2}) g_i g_i^{-1}$ $= g_i^{2k+2} (g_i^{-1} g_i^{2j}) g_i^{+1}$ $= (g_i^{2k+2} g_i^{-1})(g_i^{2j} g_i^{+1})$ $= g_i^{2k+2-1} (g_i^{2j} g_i^{+1})$ $= (g_i^{2(k+1)-1})(g_i^{2j} g_i^{-1}) = g_i^{2(k+1)-1} g_i^{2j+1} = g_i^{2r-1} g_i^{2j+1}$. It follows that it holds for all $r \in Z^+$ and all power of g_i are distinct odd positive integers. Now consider $s = 2j + 1$ and let $j = -1$ implies $s = 2(-1) + 1 = -1$ and $g_i^{-1} = \left(x_1 + ... + x_{m-1}\right)^{-1} \in G_i$. Suppose g_i^{2k+1} holds true then $g_i^{2(k+1)-1}$ implies that $g_i^{2r-1} g_i^{2j+1} = (g_i^{2r-1} g_i^{2(k+1)+1})$ $= (g_i^{2r-1+2(k+1)}) g_i$ $= (g_i^{2r+2k+2}) g_i g_i^{-1}$ $= g_i^{2k+2} (g_i g_i^{2r}) g_i^{-1}$ $= (g_i^{2k+2} g_i)(g_i^{2r} g_i^{-1})$ $= g_i^{2k+2+1} (g_i^{2j} g_i^{-1}) = (g_i^{2(k+1)+1})(g_i^{2j-1}) = g_i^{2(k+1)+1} g_i^{2r-1} = g_i^{2j+1} g_i^{2r-1}$. It has been shown that for every $j \in Z^-$ generates all odd negative integral exponent of g_i. These are $(2j+1) + (2r-1) = 0$ for odd integral exponent and for even integral exponents respectively. □

5. Invariant Subgroup of G_i

Proposition 4.
If $\{H_i\}$ and $\{S_i\}$ are two distinct subgroups of group $G_i \in N$ with even negative and positive and odd negative and positive integral exponent respectively then $\{H_i\}$ and $\{S_i\}$ are sets of normal subgroups of G_i for every $i \in I$.

Proof:
Let $\{H_i\}$ and $\{S_i\}$ are non-trivial subgroups of $G_i \in N$ for every $i \in I$ such that $\{H_i\}$ generated by $\left\langle g_i^{2j}, g_i^{2r} \right\rangle$ and $\left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$ generates $\{S_i\}$, $\{H_i\} = \left\langle g_i^{2j}, g_i^{2r} \right\rangle$ and $\{S_i\} = \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$.For any distinct $g_i^t, g_i^s \in G_i$ there exists $g_i^{-t}, g_i^{-s} \in G_i$ such that $g_i^{-t} \left\langle g_i^{2j}, g_i^{2r} \right\rangle g_i^t = \left\langle g_i^{2j}, g_i^{2r} \right\rangle$, $g_i^t g_i^{-t} \left\langle g_i^{2j}, g_i^{2r} \right\rangle g_i^t = g_i^t \left\langle g_i^{2j}, g_i^{2r} \right\rangle$, $e\left\langle g_i^{2j}, g_i^{2r} \right\rangle g_i^t =$ $g_i^t \left\langle g_i^{2j}, g_i^{2r} \right\rangle$, $\left\langle g_i^{2j}, g_i^{2r} \right\rangle g_i^t = g_i^t \left\langle g_i^{2j}, g_i^{2r} \right\rangle$. So $H_i g_i^t = g_i^t H_i$. The right coset of H_i in G_i coincide to the left coset. Now consider $g_i^s \in G_i$ and take the conjugacy of subgroup $\{S_i\}$ that is, $g_i^{-s} \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle g_i^s = \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$, $g_i^s g_i^{-s} \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle g_i^s =$ $g_i^s \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$, $e\left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle g_i^s = g_i^s \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$ so $\left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle g_i^s = g_i^s \left\langle g_i^{2j+1}, g_i^{2r-1} \right\rangle$ .Hence $S_i g_i^s = g_i^s S_i$. S_i is normal subgroup of G_i .We have $\{H_i\}$ and $\{S_i\}$ are normal subgroups of group G_i . □

6. Quotient Group

Proposition 5.
If $\{X_i\}$ is any sets of normal subgroups of G_i then G_i / X_i, $\forall i \in I$.

Proof:
Let $X_i \lhd G_i$ such that $\{g_i^{2r}\}, \{g_i^{2j}\}, \{g_i^{2j+1}\}$ *and* $\{g_i^{2r-1}\}$ are sets of distinct elements of group G_i . The sets of right cosets of X_i in G_i are $X_i\{g_i^{2r}\}, X_i\{g_i^{2j}\}, X_i\{g_i^{2j+1}\}$ and $X_i\{g_i^{2j-r}\}$ Then

$$G_i = X_i\{g_i^{2r}\} \cup X_i\{g_i^{2j}\} \cup X_i\{g_i^{2j+1}\} \cup X_i\{g_i^{2r-1}\}$$

$$G_i = X_i\{g_i^{2r}\} \cup X_i\{g_i^{2j}\} \cup X_i\{g_i^{2j+1}\} \cup X_i\{g_i^{2r-1}\}$$

$$|G_i| = \left|X_i\{g_i^{2r}\}\right| \cup \left|X_i\{g_i^{2j}\}\right| \cup \left|X_i\{g_i^{2j+1}\}\right| \cup \left|X_i\{g_i^{2r-1}\}\right|$$

$$|G_i| = \left|X_i\{g_i^{2r}\}\right| + \left|X_i\{g_i^{2j}\}\right| + \left|X_i\{g_i^{2j+1}\}\right| + \left|X_i\{g_i^{2r-1}\}\right|$$

$|G_i| = [X_i : G_i]$ is the number of sets of distinct right cosets implies that

$$\frac{G_i}{X_i} = \left\{X_i\{g_i^{2r}\},\ X_i\{g_i^{2j}\}, X_i\{g_i^{2r-1}\}, X_i\{g_i^{2j+1}\}\right\}. \ \square$$

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