Prime Gamma Rings with Centralizing and Commuting Generalized Jordan Derivations

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Abstract

Let $M$ be a prime $Γ$-ring satisfying a certain assumption and $D$ a nonzero derivation on $M$. Let $f: M \to M$ be a generalized Jordan derivation such that $f$ is centralizing and commuting on a left ideal $J$ of $M$. Then we prove that $M$ is commutative.

Keywords: Prime $Γ$-ring, Centralizing and Commuting, Derivation, Jordan derivation, Generalized derivations, Generalized Jordan derivations

Introduction

The concept of a $Γ$-ring was first introduced by Nobusawa [13] and also shown that $Γ$-rings, more general than rings. Bernes [1] weakened slightly the conditions in the definition of $Γ$-ring in the sense of Nobusawa. Bresar [2] studied centralizing mappings and derivations in prime rings. Kyuno [9], Luh [10], [11], Hoque and Paul [5], [6] and others were obtained a large numbers of important basic properties of $Γ$-rings in various ways and determined some more remarkable results of $Γ$-rings. Ceven [3] studied on Jordan left derivations on completely prime $Γ$-rings. Mayne [12] have developed some remarkable result on prime rings with commuting and centralizing. Jaya Subba Reddy et.al [8] studied centralizing and commutating left generalized derivation on prime ring is commutative. Hoque and Paul [7] studied prime gamma rings with centralizing and commuting.
generalized derivations is a commutative. In this paper, following [7], we extended some results on prime gamma rings with centralizing and commuting generalized Jordan derivations.

Let $M$ and $\Gamma$ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \rightarrow x\alpha y$ of $M \times \Gamma \times M \rightarrow M$, which satisfies the conditions

(i) $x\alpha y \in M$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha (y + z) = x\alpha y + x\alpha z$

(iii) $(x\alpha y)\beta z = x\alpha (y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is called a $\Gamma$-ring.

Every ring $M$ is a $\Gamma$-ring with $M = \Gamma$. However a $\Gamma$-ring need not be a ring. Let $M$ be a $\Gamma$-ring. Then an additive subgroup $U$ of $M$ is called a left (right) ideal of $M$ if $MTU \subset U(U \Gamma M \subset U)$.

If $U$ is both a left and a right ideal, then we say $U$ is an ideal of $M$. Suppose again that $M$ is a $\Gamma$-ring. Then $M$ is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal $P_1$ of a $\Gamma$-ring $M$ is said to be prime if for any ideals $A$ and $B$ of $M$, $AB \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal $P_2$ of a $\Gamma$-ring $M$ is said to be semiprime if for any ideal $U$ of $M$, $U \Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A $\Gamma$-ring $M$ is said to be prime if $a \Gamma M \Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a \Gamma M \Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, $M$ is said to be commutative $\Gamma$-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } y \in M \text{ and } \alpha \in \Gamma \}$ is called the centre of the $\Gamma$-ring $M$. If $M$ is a $\Gamma$-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of $x$ and $y$ with respect to $\alpha$, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha [y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha [x, z]_\beta$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:

$x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ ..........................(A).

An additive mapping $D : M \rightarrow M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. An additive mapping $D : M \rightarrow M$ is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A mapping $f$ is said to be commuting on a left ideal $J$ of $M$ if $[f(x), x]_\alpha = 0$ for all $x \in J$ and $\alpha \in \Gamma$ and $f$ is said to be centralizing if $[f(x), x]_\alpha \in Z(M)$ for all $x \in J$ and $\alpha \in \Gamma$. An additive mapping $f : M \rightarrow M$ is said to be a generalized derivation on $M$, if $f(x\alpha y) = f(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$, where $D$ is a derivation on $M$. An additive mapping $f : M \rightarrow M$ is called a generalized Jordan derivation on $M$, if $f(x\alpha x) = f(x)\alpha x + x\alpha D(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$, where $D$ is a derivation on $M$.

**Preliminaries and main results**

We have to make some use of the following well-known results:

**Remark 1:** Let $M$ be a prime $\Gamma$-ring. If $aab \in Z(M)$ with $0 \neq a \in Z(M)$, then $b \in Z(M)$.
Remark 2: Let $M$ be a prime $\Gamma$-ring and $J$ a nonzero left ideal of $M$. If $D$ is a nonzero derivation on $M$, then $D$ is also a nonzero on $J$.

Remark 3: Let $M$ be a prime $\Gamma$-ring and $J$ a nonzero left ideal of $M$. If $J$ is commutative, then $M$ is also commutative.

Lemma 1: Suppose $M$ is a prime $\Gamma$-ring satisfying the assumption (A) and $D: M \rightarrow M$ be a Jordan derivation. For an element $a \in M$, if $aaD(x) = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then either $a = 0$ or $D = 0$.

Proof: By our assumption, $aaD(x) = 0$, for all $x \in M$, and $\alpha \in \Gamma$. We replacing $x$ by $x\beta x$ in above equation then, we get $aaD(x\beta x) = 0$.

\[ aaD(x)\beta x + aax\beta D(x) = 0 \]

\[ aax\beta D(x) = 0, \text{ for all } x \in M, \text{ and, } \alpha, \beta \in \Gamma. \]

If $D$ is nonzero, that is, if $D(x) \neq 0$, for some $x \in M$. Then by definition of prime $\Gamma$-ring, $a = 0$.

Lemma 2: Suppose $M$ is a prime $\Gamma$-ring satisfying the assumption (A) and $J$ a nonzero left ideal of $M$. If $M$ has a derivation $D$ which is zero on $J$, then $D$ is zero on $M$.

Proof: By the hypothesis, $D(J) = 0$.

Replacing $J$ by $M\Gamma J$ in above equation then, we get $D(M\Gamma J) = 0$.

\[ D(M)\Gamma J + M\Gamma D(J) = 0 \]

\[ D(M)\Gamma J = 0. \]

By lemma 1, $D$ must be zero, since $J$ is nonzero.

Lemma 3 [7]: Suppose $M$ is a prime $\Gamma$-ring satisfying the assumption (A) and $J$ a nonzero left ideal of $M$. If $J$ is commutative on $M$, then $M$ is commutative.

Lemma 4: Suppose $M$ is a prime $\Gamma$-ring and $f: M \rightarrow M$ be an additive mapping. If $f$ is centralizing on a left ideal $J$ of $M$, then $f(a) \in Z(M)$, for all $a \in J \cup Z(M)$.

Proof: $f$ is centralizing on left ideal $J$ of $M$, we have $[f(a), a]_\alpha \in Z(M)$ for all $a \in J$ and $\alpha \in \Gamma$. 
By linearization, we have
\[ a, b \in J \implies a + b \in J, \text{ for all } \alpha \in \Gamma. \]
\[ [f(a + b), \alpha a + b] \in Z(M) \]

\( f \) is an additive mapping then
\[ [f(a) + f(b), a + b] \in Z(M) \]
\[ [f(a), \alpha a] + [f(a), b] \alpha_a + [f(b), a] \alpha_a + [f(b), b] \alpha_a \in Z(M) \]

\( f \) is a centralizing on left ideal \( J \) of \( M \) then, we get
\[ [f(a), b] \alpha = 0, [f(b), b] \alpha = 0 \]
\[ [f(a), b] \alpha_a + [f(b), a] \alpha_a \in Z(M), \text{ for all } a, b \in J \text{ and } \alpha \in \Gamma. \] (1)

If \( a \in Z(M) \), then equation (1) becomes
\[ [f(a), b] \alpha_a \in Z(M). \]

Replacing \( b \) by \( f(a) \beta b \) in above equation then, we get
\[ [f(a), f(a) \beta b] \alpha_a \in Z(M) \]
\[ [f(a), f(a)] \alpha_a \beta b + f(a) \beta [f(a), b] \alpha_a \in Z(M) \]
\( f(a) \beta [f(a), b] \alpha_a \in Z(M). \text{ If } [f(a), b] \alpha_a = 0. \)

Then \( f(a) \in C_{\Gamma M}(J). \)

The centralizer of \( J \) in \( M \) and hence \( f(a) \in Z(M) \). Otherwise, if \( [f(a), b] \alpha_a \neq 0 \), remark 1 follows that \( f(a) \in Z(M) \). Hence the lemma.

**Theorem 1:** Let \( M \) be a prime \( \Gamma \)-ring satisfying the assumption \( A \) and \( D \) is a nonzero derivation on \( M \). If \( f \) is a generalized Jordan derivation on a left ideal \( J \) of \( M \) such that \( f \) is commuting on \( J \), then \( M \) is commutative.

**Proof:** Since \( f \) is commuting on \( J \), we have
\[ [f(a), a] \alpha_a = 0, \text{ for all } a \in J \text{ and } \alpha \in \Gamma. \]

Replacing \( a \) by \( a + b \) in above equation, we get
\[ [f(a + b), a + b] \alpha_a = 0 \]
\[ [f(a) + f(b), a + b] \alpha_a = 0 \]
\[ [f(a), a] \alpha_a + [f(a), b] \alpha_a + [f(b), a] \alpha_a + [f(b), b] \alpha_a = 0 \]
\[ [f(a), b]_\alpha + [f(b), a]_\alpha = 0 \] (2)

Replacing \( b \) by \( a\beta a \) in equation (2), we get
\[ [f(a), a\beta a]_\alpha + [f(a\beta a), a]_\alpha = 0 \]
\[ [f(a), a]_\alpha \beta a + a\beta [f(a), a]_\alpha + [f(a)\beta a + a\beta D(a), a]_\alpha = 0 \]
\[ [f(a), a]_\alpha \beta a + a\beta [f(a), a]_\alpha + [f(a)\beta a, a]_\alpha + [a\beta D(a), a]_\alpha = 0 \]

\( f \) is centralizer, then \([f(a), a]_\alpha \beta a = 0, a\beta [f(a), a]_\alpha = 0, [f(a), a]_\alpha \beta a = 0, f(a)\beta [a, a]_\alpha = 0 \).
\[ [a\beta D(a), a]_\alpha = 0 \] (3)

Replacing \( a\beta \) by \( b\beta \) in equation (3), we get
\[ [b\beta D(a), a]_\alpha = 0 \]

Replacing \( b \) by \( r\gamma a \) in above equation, then we get
\[ [r\gamma a\beta D(a), a]_\alpha = 0 \]
\[ r\gamma a\beta [D(a), a]_\alpha + [r\gamma a, a]_\alpha \beta D(a) = 0 \]
\[ r\gamma a\beta [D(a), a]_\alpha + r[y, [a, a]_\alpha \beta D(a) + [r, a]_\alpha \gamma a\beta D(a) = 0 \]
\[ [r, a]_\alpha \gamma a\beta D(a) = 0, \text{ for all } a \in J, r \in M \text{ and } \alpha, \beta, \gamma, \in \Gamma. \]

Since \( M \) is prime \( \Gamma \)-ring, thus \([r, a]_\alpha = 0 \text{ or } D(a) = 0 \)

Since \( D \) is nonzero derivation on \( M \), then by lemma 2, \( D \) is nonzero on \( J \).

Suppose \( D(a) \neq 0 \) for some \( a \in J \), then \( a \in Z(M) \).

Let \( c \in J \) with \( c \neq Z(M) \). Then \( D(c) = 0 \) and \( a + c \in Z(M) \), that is, \( D(a + c) = 0 \) and so \( D(a) = 0 \), which is a contradiction. Thus \( c \in Z(M) \) for all \( c \in J \).

Hence \( J \) is commutative and lemma3, we get \( M \) is commutative.

**Theorem 2:** Let \( M \) be a prime \( \Gamma \)-ring satisfying the assumption \( (A) \) and \( J \) a left ideal of \( M \) with \( J \cap Z(M) \neq 0 \). If \( f \) is a generalized Jordan derivation on \( M \) with associated nonzero derivation \( D \) such that \( f \) is commuting on \( J \), then \( M \) is commutative.
Proof: we claim that, $Z(M) \neq 0$ because of $f$ is commuting on $J$ and the proof is complete.

Now from equation (1), we get

$[f(a), b]_\alpha + [f(b), a]_\alpha \in Z(M)$

We replace $a$ by $c\beta c$ with $0 \neq c \in Z(M)$, we get

$[f(c\beta c), b]_\alpha + [f(b), c\beta c]_\alpha \in Z(M)$

$[f(c)\beta c + c\beta D(c), b]_\alpha + [f(b), c]_\alpha \beta c + c\beta [f(b), c]_\alpha \in Z(M)$

$[f(c)\beta c, b]_\alpha + [c\beta D(c), b]_\alpha + [f(b), c]_\alpha \beta c + c\beta [f(b), c]_\alpha \in Z(M)$

$f(c)\beta [c, b]_\alpha + [f(c), b]_\alpha \beta c + c\beta [D(c), b]_\alpha + [c, b]_\alpha \beta D(c) + [f(b), c]_\alpha \beta c + c\beta [f(b), c]_\alpha \in Z(M)$

$\forall c \in Z(M) \Rightarrow [c, b]_\alpha = 0$, for all $b \in J$.

Since $c \in Z(M) \Rightarrow f$ is a centralizer on $J$.

$f(b) \in Z(M) \Rightarrow [f(b), c]_\alpha = 0.$

$[f(c), b]_\alpha \beta c + c\beta [D(c), b]_\alpha \in Z(M)$

From lemma 1, $f(c) \in Z(M)$ and hence $c\beta [D(c), b]_\alpha \in Z(M)$.

Replacing $b$ by $b + c$ in above equation, we get

$c\beta [D(c), b + c]_\alpha \in Z(M)$.

$c\beta [D(c), b]_\alpha + c\beta [D(c), c]_\alpha \in Z(M)$.

And consequently $c\beta [D(c), c]_\alpha \in Z(M)$.

As $c$ is nonzero, remark 1 follows that $[D(c), c]_\alpha \in Z(M)$. This implies $D$ is centralizing on $J$ and hence we conclude that $M$ is commutative.

References


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