Relationship the P-Ideal with
Other Concepts of BH-Algebra

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Abstract

In this paper, we study the notions of p-ideal of a BH-algebra and we state and prove some theorems which determine the relationships among this ideal with the intersection, union, image of function, inverse function for p-ideals of BH-algebra and also we give some properties of this ideal and link it with other types of concepts of a BH-algebra.

Keywords: BH-algebra, p-ideal, positive implicative

1. Preliminaries

In this section, is devoted to some basic ordinary concepts of BH-algebra, p-ideal and homomorphism in BH-algebra, we give some basic concepts about image of function, the inverse image, positive implicative and translation ideal of a BH-algebra with some propositions and theorems.

Definition (1.1): [7] A BH-algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following conditions:

i. \( x * x = 0, \forall x \in X \).
ii. \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y\), \(\forall\ x, y \in X\).

iii. \(x \ast 0 = x\), \(\forall\ x \in X\).

**Definition (1.2):** A nonempty subset \(I\) of a BH-algebra \(X\) is called a \textbf{P-ideal} of \(X\) if:

i. \(0 \in I\),

ii. \((x \ast z) \ast (y \ast z) \in I\) and \(y \in I\) imply \(y \in I\), \(\forall\ x, y, z \in X\).

**Remark (1.3):** Let \(X\) and \(Y\) be BH-algebras. A mapping \(f: X \rightarrow Y\) is called a \textbf{homomorphism} if \(f(x \ast y) = f(x) \ast f(y)\), \(\forall\ x, y \in X\). A homomorphism \(f\) is called a \textbf{monomorphism} (resp., \textbf{epimorphism}) if it is injective (resp., surjective). A bijective homomorphism is called an \textbf{isomorphism}. Two BH-algebras \(X\) and \(Y\) are said to be \textbf{isomorphic}, written \(X \cong Y\), if there exists an isomorphism \(f: X \rightarrow Y\). For any homomorphism \(f: X \rightarrow Y\), the set \(\{x \in X: f(x) = 0\}\) is called the \textbf{kernel} of \(f\), denoted by \(\ker(f)\), and the set \(\{f(x): x \in X\}\) is called the \textbf{image} of \(f\), denoted by \(\text{Im}(f)\). Notice that \(f(0) = 0\), \(\forall\) homomorphism \(f\).

**Remark (1.4):** Let \((X, \ast, 0)\) be a BH-algebra and let \(N\) be a normal subalgebra of \(X\). Define a relation \(\sim_N\) on \(X\) by \(a \sim_N b\) if and only if \(a \ast b \in N\) and \(b \ast a \in N\), \(\forall\ a, b \in X\). Then \(\sim_N\) is an equivalence relation on \(X\). Denote this by \([a]_N\), i.e., \([a]_N = \{b \in X | a \sim_N b\}\) and \(X/N = \{[a]_N | a \in X\}\). And define \([a]_N \oplus [b]_N = [a \ast b]_N\), then ((\(X/N\), \(\oplus\), \(\{0\}_N\))) is a BH-algebra.

**Theorem (1.5):** Let \(N\) be a normal subalgebra of BH-algebra \(X\). If \(I\) is an ideal of \(X\), then \(I/N\) is an ideal of \(X/N\).

**Remark (1.6):** Let \(A\) be a translation ideal of a BH-algebra \((X, \ast, 0)\). If we define \([a]_A \oplus [b]_A = [a \ast b]_A\) for all \(a, b \in X\), then \((X/A, \oplus, [0]_A)\) is a BH-algebra.

**Theorem (1.7):** Let \(A\) be a translation ideal of a BH-algebra \((X, \ast, 0)\). If we define \([a]_A \oplus [b]_A = [a \ast b]_A\) for all \(a, b \in X\), then \((X/A, \oplus, [0]_A)\) is a BH-algebra.

**Definition (1.8):** Let \(X\) be a BH-algebra. For a fixed \(b \in X\), we define a map \(R_b: X \rightarrow X\) such that \(R_b(x) = x \ast b\), \(\forall\ x \in X\) and call \(R_b\) a \textbf{right map} on \(X\). Symbolize the set of all right maps on \(X\) by \(R(X)\). A left map \(L_b\) is defined by a similar way, we define a map \(L_b: X \rightarrow X\) such that \(L_b(x) = b \ast x\), \(\forall\ x \in X\) and call \(L_b\) a \textbf{left map} on \(X\).
Definition (1.9): [1, 5] A BH-algebra \((X, *, 0)\) is said to be a **positive implicative** if it satisfies \(\forall a, b \text{ and } c \in X, (a*c) \ast (b*c) = (a*b) \ast c\).

Theorem (1.10): [5] If \(X\) is a positive implicative BH-algebra, then \((L(X), \oplus, L_0)\) is a positive implicative BH-algebra.

Remark (1.11): Let \(X\) be a BH-algebra and let \(I\) be a subset of \(X\). We will define to the set \(\{ L_a \in L(X) : a \in I \}\) by \(L(I)\).

Remark (1.12): [5] Suppose that \(X\) be a positive implicative BH-algebra, defined \(\oplus\) an operation in \(L(X)\) is \((L_a \oplus L_b)(x) = L_a(x) \ast L_b(x)\) and \((L_a \oplus L_b)(x) = L_{a \lor b}(x)\), \(\forall L_a, L_b \in L(X)\) and \(\forall x \in X\).

Definition (1.13): [2] A BH-algebra \(X\) is called an **associative BH-algebra** if:
\[(x \ast y) \ast z = x \ast (y \ast z), \quad \forall x, y, z \in X.\]

Theorem (1.14): [2] Let \(X\) be an associative BH-algebra. Then the following properties are hold:

i. \(0 \ast a = a\); \(\forall a \in X\)
ii. \(a \ast b = b \ast a\); \(\forall a, b \in X\)
iii. \(a \ast (a \ast b) = b\); \(\forall a, b \in X\)
iv. \((c \ast a) \ast (a \ast b) = a \ast b\); \(\forall a, b, c \in X\)
v. \(a \ast b = 0 \Rightarrow a = b\); \(\forall a, b \in X\)
vi. \((a \ast (a \ast b)) \ast b = 0\); \(\forall a, b \in X\)
vii. \((a \ast (a \ast b)) \ast c = (a \ast c) \ast b\); \(\forall a, b, c \in X\)
viii. \((a \ast c) \ast (b \ast d) = (a \ast (b \ast d)) \ast (c \ast d)\); \(\forall a, b, c, d \in X\)

Proposition (1.15): [1] Every P-ideal of a BH-algebra \(X\) is an ideal of \(X\).

2. The Relationship the P-Ideal with Other Notions

Theorem (2.1): Let \(\{I_i, i \in \Gamma\}\) be a family of p-ideals of a BH-algebra \(X\). Then \(\bigcap_{i \in \Gamma} I_i\) is a p-ideal of \(X\).

Proof: To prove \(\bigcap_{i \in \Gamma} I_i\) is a p-ideal of \(X\).

i. \(0 \in I_i, \forall i \in \Gamma\). [Since each \(I_i\) are p-ideals of \(X, \forall i \in \Gamma\). By definition (1.2)(i))]

\(\Rightarrow 0 \in \bigcap_{i \in \Gamma} I_i\).
ii. Let \( x, y, z \in X \) such that \( (x*z)\star(y*z) \in \bigcap_{i \in \Gamma} I_i \) and \( y \in \bigcap_{i \in \Gamma} I_i \)
\[ \Rightarrow (x*z)\star(y*z) \in I_i \text{ and } y \in I_i, \forall i \in \Gamma. \]
\[ \Rightarrow x \in I_i, \forall i \in \Gamma. \text{ [Since each } I_i \text{ are p-ideal of } X, \forall i \in \Gamma. \text{ By definition (1.2)(ii)]} \]
\[ \Rightarrow x \in \bigcap_{i \in \Gamma} I_i. \text{ Therefore, } \bigcap_{i \in \Gamma} I_i \text{ is a p-ideal of } X. \]

**Theorem (2.2):** Let \( \{ I_i, i \in \Gamma \} \) be a chain p-ideals of a BH-algebra \( X \). Then \( \bigcup_{i \in \Gamma} I_i \) is a p-ideal of \( X \).

**Proof:** To prove \( \bigcup_{i \in \Gamma} I_i \) is a p-ideal of \( X \).

i. \( 0 \in I_i, \forall i \in \Gamma. \) [Since each \( I_i \) is a p-ideal of \( X \), \( \forall i \in \Gamma \). By definition (1.2)(i)]
\[ \Rightarrow 0 \in \bigcup_{i \in \Gamma} I_i. \]

ii. Let \( x, y, z \in X \) such that \( (x*z)\star(y*z) \in \bigcup_{i \in \Gamma} I_i \) and \( y \in \bigcup_{i \in \Gamma} I_i \).
\[ \exists I_j, I_k \in \{ I_i \}_{i \in \Gamma}, \text{ such that } (x*z)\star(y*z) \in I_j \text{ and } y \in I_k \]
\[ \Rightarrow \text{ either } I_j \subseteq I_k \text{ or } I_k \subseteq I_j \quad \text{ [Since } \{ I_i \}_{i \in \Gamma} \text{ is a chain ]} \]
\[ \Rightarrow \text{ either } (x*z)\star(y*z) \in I_j \text{ and } y \in I_j \text{ or } (x*z)\star(y*z) \in I_k \text{ and } y \in I_k \]
\[ \Rightarrow \text{ either } x \in I_j \text{ or } x \in I_k. \text{ [Since } I_j \text{ and } I_k \text{ are p-ideals of } X. \text{ By definition (1.2)(ii)]} \]
\[ \Rightarrow x \in \bigcup_{i \in \Gamma} I_i. \text{ Therefore, } \bigcup_{i \in \Gamma} I_i \text{ is a p-ideal of } X. \]

**Proposition (2.3):** Let \( f: (X,*,0) \to (Y,*',0') \) be a BH-epimorphism. If \( I \) is a p-ideal of \( X \), then \( f(I) \) is a p-ideal of \( Y \).

**Proof:** Let \( I \) be a p-ideal of \( X \). Then

i. \( f(0) = 0'. \quad \text{[Since } f \text{ is an epimorphism. By remark (1.3)]} \]
\[ \Rightarrow 0' \in f(I) \]

ii. Let \( x, y, z \in Y \) such that \( (x*z)\star(y*z) \in f(I) \) and \( y \in f(I) \)
\[ \Rightarrow \exists a, b, c \in I \text{ such that } f(a) = x, f(b) = y \text{ and } f(c) = z \]
\[ \Rightarrow (x*z)\star(y*z) = (f(a)\star f(c))\star(f(b)\star f(c)) = f((a*c)\star(b*c)) \in f(I), \quad \text{[Since } f \text{ is an epimorphism. By remark (1.3)]} \]

\[ \Rightarrow (a^*c)^*(b^*c) \in I \quad \text{and} \quad b \in I, \quad \text{[Since } f(I) = \{f(x) : x \in I\}\] 

\[ \Rightarrow a \in I, \quad \text{[Since } I \text{ is a } \text{p-ideal of } X. \text{ By definition (1.2)(ii)\] 

\[ \Rightarrow f(a) \in f(I). \quad \text{[Since } f(I) = \{f(x) : x \in I\}\] 

Then \( f(I) \) is a \( \text{p-ideal of } Y. \) ■

**Proposition (2.4):** Let \( f: (X, *, 0) \to (Y,*', 0') \) be a BH-homomorphism. If \( I \) is a \( \text{p-ideal of } Y, \) then \( f^{-1}(I) \) is a \( \text{p-ideal of } X. \)

**Proof:** Let \( I \) be a \( \text{p-ideal of } Y. \) Then

i. \( f(0) = 0' \) \quad \text{[Since } f \text{ is a homomorphism. By remark (1.3)\] 

\[ \Rightarrow 0 \in f^{-1}(I). \]

ii. Let \( x, y, z \in X \) such that \( (x*z)*(y*z) \in f^{-1}(I) \) and \( y \in f^{-1}(I) \)

\[ \Rightarrow f((x*z)*(y*z)) \in I \quad \text{and} \quad f(y) \in I \]

\[ \Rightarrow f((x*z)*(y*z)) = f(x)^*f(z)^* = f(y)^*f(z) \in I \quad \text{and} \quad f(y) \in I, \quad \text{[Since } f \text{ is a homomorphism.]\] 

\[ \Rightarrow f(x) \in I, \quad \text{[Since } I \text{ is a } \text{p-ideal of } Y. \text{ By definition (1.2)(ii)\] 

\[ \Rightarrow x \in f^{-1}(I). \quad \text{[Since } f \text{ is a homomorphism, by Remark (1.3)\] 

Therefore, \( f^{-1}(I) \) is a \( \text{p-ideal of } X. \) ■

**Theorem (2.5):** Let \( N \) be a normal subalgebra of BH-algebra \( X. \) If \( I \) is a \( \text{p-ideal of } X, \) then \( I/N \) is a \( \text{p-ideal of } X/N. \)

**Proof:** Suppose that \( I \) is a \( \text{p-ideal of } X. \)

\[ \Rightarrow I \text{ is an ideal of } X. \quad \text{[By proposition (1.15)\] 

\[ \Rightarrow I/N \text{ is an ideal of } X/N. \quad \text{[By theorem (1.5)\] 

i. \( [0]_N \in I/N. \quad \text{[Since } 0 \in I. \text{ By definition (1.2)(i)\] 

ii. Let \( [x]_N, [y]_N, [z]_N \in X/N \) such that

\[ ([x]_N^*[z]_N)^*([y]_N^*[z]_N) \in I/N \quad \text{and} \quad [y]_N \in I/N, \]

\[ \Rightarrow [x*z]_N)^*([y*z]_N \in I/N \quad \text{and} \quad [y]_N \in I/N. \quad \text{[Since } [x]_N^*[y]_N=[x*y]_N \]

\[ \Rightarrow [(x*z)^*]_N \in I/N \quad \text{and} \quad [y]_N \in I/N, \]

\[ \Rightarrow (x*z)^* y \in I \quad \text{and} \quad y \in I. \quad \text{[Since } I/N = \{[x]_N | x \in I\}. \text{By remark (1.4)\]} 

⇒ \( x \in I \), [Since I is a p-ideal. By definition (1.2)(ii)]
⇒ \([x]_N \in I/N\). Therefore, I/N is a p-ideal of X/N. ■

**Proposition (2.6):** Let A be a translation ideal of a BH-algebra X. If I is a p-ideal of X, then I/A is a p-ideal of X/A.

**Proof:** Assume that I be a p-ideal of X.

i. \([0] \in I/A\). [By definition (1.2)(i)]

ii. Let \([x]_A, [y]_A, [z]_A \in X/A\) such that
\([x+z]_A \oplus [y+z]_A \in I/A\) and \([y]_A \in I/A\)
⇒ \([x*z]_A \oplus [y*z]_A \in I/A\) and \([y]_A \in I/A\).

[Since \([x]_A \oplus [y]_A = [x*y]_A\). By remark (1.6)]
⇒ \([x*z]_A \oplus (y*z)_A \in I/A\) and \([y]_A \in I/A\),
⇒ \([x*z]_A \oplus (y*z)_A \in I\) and \([y]_A \in I\), [Since I is a p-ideal. By definition (1.2)(ii)]
⇒ \([x]_A \in I/A\). Then, I/A is a p-ideal of X/N. ■

**Proposition (2.7):** Let X be a positive implicative BH-algebra. If I is a p-ideal of X, then L(I) is a p-ideal of \((L(X), \oplus, L_0)\).

**Proof:** Let I be a p-ideal of X.

i. \(0 \in I\) [By definition (1.2)(i)]
⇒ \(L_0 \in L(I)\)

ii. Let \(L_a, L_b, L_c \in L(I)\) such that \((L_a \oplus L_c) \oplus (L_b \oplus L_c) \in L(I)\) and \(L_a \in L(I)\).
⇒ \((a * c) \ast (b \ast c)) \ast c \in I\) and \(b \in I\), [Since \((L_a \oplus L_c) \oplus (L_b \oplus L_c) = L_{(a\ast c)+(b\ast c)} \in L(I)\)]
⇒ \(a \in I\), [By definition (1.2)(ii)]
⇒ \(L_a \in L(I)\). Therefore, L(I) is a p-ideal of \((L(X), \oplus, L_0)\). ■

**Theorem (2.8):** Let X be a positive implicative BH-algebra and let
\(H_t = \{a \in X | a \ast t = 0, t \in X\}\)
be a subset of X. If \((a \ast t) \ast (b \ast t) = a \ast t\) with \(a \ast t \neq b \ast t\), \(\forall a, b, t \in X\), then \(H_t \cup \{0\}\) is a p-ideal of X.

**Proof:** We must show that \(H_t \cup \{0\}\) is a p-ideal of X.
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i. we have $0 \in H_t \cup \{0\}$.

ii. Let $a, b, c \in X$ such that $(a*c)*(b*c) \in H_t$ and $b \in H_t$.

$\Rightarrow (a*c)*(b*c) \in H_t$ [Since $H_t \cup \{0\}$ is an ideal]

$\Rightarrow (a*c)*(b*c)*t = 0$

$\Rightarrow ((a*t)*(c*t))*((b*t)*(c*t))=0$ [ $(a*b)*t=(a*t)*(b*t)$, since $X$ be positive implicative]

$\Rightarrow ((a*t)*(c*t))*((b*t)*(c*t))=0$

$\Rightarrow a*t = 0$, then $a \in H_t$. Therefore, $H_t \cup \{0\}$ is a p-ideal of X.

**Theorem (2.9):** If $g : (X, *, 0) \rightarrow (Y, *', 0')$ be a homomorphism from an associative BH-algebra $X$ into BH-algebra $Y$, then ker($g$) is a p-ideal of $X$.

**Proof:** We must show that ker($g$) is a p-ideal of $X$.

i. $g(0)=0'$. [Since $g$ be a homomorphism]. Then $0 \in$ ker($g$).

ii. Let $(x*z)*(y*z) \in$ ker($g$) and $y \in$ ker($g$)

$\Rightarrow g((x*z)*(y*z))=0'$ and $g(y)=0'$ [By remark (1.3)]

$\Rightarrow (g(x)*g(z))*g(y)'*g(z)) = 0'$. [Since $g$ is a homomorphism.]

$\Rightarrow (g(x)*g(z))*'(0'*g(z)) = 0'$ [Since $g(y)=0'$]

$\Rightarrow g((x*z)*(0*z))=0'$ [Since $X$ is an associative.

$(a*c)*(b*d)=(a*b)*(c*d)$ ]

$\Rightarrow g((x*0)*(z*z))=0'$ [Since $X$ is BH-algebra ; $x*0=x$ and $x*x=0$]

$\Rightarrow g(x*0)=0'$ [Since $X$ is BH-algebra ; $x*0=x$ ]

$\Rightarrow g(x)=0'$ [Since $X$ is BH-algebra ; $x*0=x$ and $x*x=0$]

$\Rightarrow x \in$ ker($g$). Therefore, ker($g$) is a p-ideal of $X$.

**References**


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