International Journal of Algebra, Vol. 12, 2018, no. 1, 31 - 51
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2018.812

Hochschild Cohomology of G-Green Functors[1]

Ali N. A. Koam

Department of Mathematics, Faculty of Sciences
Jazan University, Saudi Arabia

Abstract

In this paper, our goal is to develop the equivariant version of Hochschild cohomology. Here we develop a cohomology theory for Green functors.

Mathematics Subject Classification: 16E30, 16E40

Keywords: Hochschild cohomology, G-Mackey functors, G-Green functors

1. Introduction

One of the main application of homological algebra is the classical cohomology of associative algebras invented by Hochschild [2] in 1945. It is a particular case of general machinery developed by Cartan and Eilenberg. Let A be an associative k-algebra and let M be an A-A-bimodule. The low dimensional groups ($n \leqslant 2$) have well known interpretations of classical algebraic structures such as derivations and extensions. Moreover, Gerstenhaber [1] observed that the second Hochschild cohomology group of a finite dimensional algebra $H^2(A, A)$ has a close connection to the deformation theory of A, that is, if $H^2(A, A) = 0$ then all deformations of A are trivial. By the work of A. Connes in 80's the Hochschild cohomology plays an important role in so called noncommutative differential geometry.

The theory of Mackey functors was originally initiated by Green [3] in the early 1970's and later developed by numerous authors in the last four decades (A. Dress [4], P. Webb [5]). The notion of Mackey functors associated to a finite group G are a standard tool for studying representations of a finite group and its subgroups.

[1] This research was supported by the Jazan University.

There are at least three equivalent definitions of Mackey functors for a group G. In this work we only state two definitions. The first definition which is due to Green [3] is based on the poset of subgroups of G. The second definition which is due to Dress [4] uses the category of G-sets.

Roughly speaking, a Green functors for the finite group G over the commutative ring R is a Mackey functor with a compatible ring structure. More specifically, there are two equivalent definitions of Green functors. The first definition which is due to Green [3] relies on the poset of subgroups of G. The second definition is analogous to the Dress definition of Mackey functors which is based on the category of G-sets, and is detailed in [6].

In the equivariant world there is a group G which acts on objects. The simplest naive object which can be considered is a G-algebra, that is, an associative algebra A on which G acts via algebra automorphisms. In our work we consider one more general situation. In particular, we develop a cohomology theory for G-Green functors.

We organize this paper into the following sections. In Section 2, we fix some notations for the standard chain complexes associated to groups and associative algebras. In Section 3, we provide two definitions of G-Mackey functors. In Section 4, we state two definitions of G-Green functors. In Section 5, we bulid the G-tensor products of G-Mackey functors, and in Section 6, we bulid the G-$\mathcal{HOM}(A, B)$. In Section 7, we extend the definition of Hochschild (co)homology to G-Mackey functors. In Section 8, we extend the well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras to G-Green functors. Finally, we extend the deformation theory of associative algebras to G-Green functors.

2. Preliminaries

In this section we fix some notations for the standard chain complexes associated to groups and associative algebras.

For a group G and G-module C we let $C^\bullet(G, C)$ denote the standard complex computing the group cohomology. Recall that

$$C^n(G, C) = Maps(G^n, C)$$

and the coboundary map

$$\partial : Maps(G^n, C) \longrightarrow Maps(G^{n+1}, C)$$

is given by

$$\begin{aligned}(\partial\alpha)(x_1, \cdots, x_{n+1}) &= x_1\alpha(x_2, \cdots, x_{n+1}) \\ &+ \sum_{i=1}^{n}(-1)^i\alpha(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) \\ &+ (-1)^{n+1}\alpha(x_1, \cdots, x_n).\end{aligned}$$

So by the definition

$$H^n(G,C) = H^n(C^\bullet(G,C)).$$

Let A be an associative algebra. Recall that the Hochschild cohomology of A with coefficients in a A-bimodule M is the cohomology of the following cochain complex:

$$0 \to M \xrightarrow{\delta^0} \mathrm{Hom}(A,M) \xrightarrow{\delta^1} \mathrm{Hom}(A^{\otimes 2},M) \xrightarrow{\delta^2} \cdots$$

where the coboundary map

$$\delta^n : Hom(A^{\otimes n}, M) \longrightarrow Hom(A^{\otimes n+1}, M)$$

is given by

$$\begin{aligned}\delta(f)(a_1,\cdots,a_{n+1}) &= a_1 f(a_2,\cdots,a_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f(a_1,\cdots,a_i a_{i+1},\cdots,a_{n+1}) \\ &+ (-1)^{n+1} f(a_1,\cdots,a_n)a_{n+1}.\end{aligned}$$

Hence, $H^n(A,M) = H^n(C^n(A,M))$, where $C^n(A,M) = Hom(A^{\otimes n}, M)$.

3. G-Mackey Functors

There are several equivalent definitions of G-Mackey functors for a finite group G. In this section, we will state two definitions of G-Mackey functors. The first definition is due to Green [3].

Definition 3.1. *A G-Mackey functor M consists of a collection of abelian groups $M(H)$ together with transfer maps $tr_K^H : M(K) \longrightarrow M(H)$ and restriction maps $res_K^H : M(H) \longrightarrow M(K)$ for all subgroups $K < H \leqslant G$, and conjugation maps $c_{x,H} : M(H) \longrightarrow M({}^xH)$ for $x \in G$, such that the following axioms hold:*

1. *If $T \leqslant K \leqslant H$, then $tr_K^H tr_T^K = tr_T^H$ and $res_T^K res_K^H = res_T^H$.*
2. *If $x, y \in G$ and $H \leqslant G$, then $c_{y,{}^xH} c_{x,H} = c_{yx,H}$.*
3. *If $x \in G$ and for all subgroups $K \leqslant H$, then $c_{x,H} tr_K^H = tr_{{}^xK}^{{}^xH} c_{x,K}$ and $c_{x,K} res_K^H = res_{{}^xK}^{{}^xH} c_{x,H}$. Furthermore, $c_{x,H} = Id$ if $x \in H$.*
4. *(Mackey axiom) for all subgroups $T, K \leqslant H$*

$$res_T^H tr_K^H = \sum_{x \in [T\backslash H/K]} tr_{T\cap {}^xK}^T c_{x,T^x\cap K} res_{T^x\cap K}^K.$$

Definition 3.2. [6] *A morphism f from a Mackey functor M to a Mackey functor N consists of a collection of morphism of group homomorphisms f_H :*

$M(H) \longrightarrow N(H)$, for $H \leqslant G$, such that if $K \leqslant H$ and $x \in G$, the squares

$$\begin{CD} M(H) @>{f_H}>> N(H) \\ @A{tr_K^H}AA @AA{tr_K^H}A \\ M(K) @>>{f_K}> N(K) \end{CD} \qquad \begin{CD} M(H) @>{f_H}>> N(H) \\ @V{res_K^H}VV @VV{res_K^H}V \\ M(K) @>>{f_K}> N(K) \end{CD} \qquad \begin{CD} M(H) @>{f_H}>> N(H) \\ @V{c_{x,H}}VV @VV{c_{x,H}}V \\ M({}^xH) @>>{f_{{}^xH}}> N({}^xH) \end{CD}$$

are commutative.

The second definition is given by Dress [4]. Let $\widehat{G}$ be the category of finite G-sets.

Definition 3.3. *A G-Mackey functor M for the finite group G is a pair of functors (M_*, M^*) from $\widehat{G}$ to Ab the category of abelian groups, such that the following properties hold:*

(1) *$M_*(X) = M^*(X) = M(X)$ for any G-set X.*
(2) *M_* is covariant and M^* is contravariant.*
(3) *If X and Y are finite G-sets, and if i_X and i_Y are the respective inclusion maps from X and Y to their disjoint union $X \sqcup Y$, then the maps $M^*(i_X) \oplus M^*(i_Y)$ and $M_*(i_X) \oplus M_*(i_Y)$ are mutual inverse R-module isomorphisms:*

$$M(X) \oplus M(Y) \xrightarrow{(M_*(i_X) \oplus M_*(i_Y))} M(X \sqcup Y) \xrightarrow{(M^*(i_X) \oplus M^*(i_Y))} M(X) \oplus M(Y)$$

(4) *If*

$$\begin{CD} X @>{f}>> Y \\ @V{g}VV @VV{u}V \\ S @>>{v}> T \end{CD}$$

is a pullback diagram of finite G-sets, then we have the following commutative diagram in Ab.

$$\begin{CD} M(X) @>{M_*(f)}>> M(Y) \\ @A{M^*(g)}AA @AA{M^*(u)}A \\ M(S) @>>{M_*(v)}> M(T) \end{CD}$$

For a map $f : X \longrightarrow Y$ of finite G-sets, we call $M_*(f) = f_*$ a transfer map and $M^*(f) = f^*$ a restriction map. We write $\mathcal{M}ack(G)$ for the category of G-Mackey functors.

Webb shows that these definitions are equivalent in [5].

Example 3.4. [6] *The simplest example of a G-Mackey functor is the fixed point G-Mackey functor. Let M be a group with an action of G. We will* *denote the fixed point G-Mackey functor of M by* $\underline{M}$*, and we define* $\underline{M}$ *by:*

$$\underline{M}(H) = M^H = \{m \in M | h \cdot m = m \text{ for all } h \in H\}$$
$$= \text{The subgroup of } H\text{-fixed points in } M.$$

For all subgroups K of H, the restriction map $res^H_K : M^H \longrightarrow M^K$ *is simply inclusion of fixed points, and we define the transfer map* $tr^H_K : M^K \longrightarrow M^H$ *by the formula:*

$$tr^H_K(m) = \sum_{g \in G} g \cdot m.$$

Example 3.5. [6] *The most significant example of a G-Mackey functor is the Burnside G-Mackey functor,* $\underline{B}$*. For all subgroups H of G, we define* $\underline{B}(H)$ *to be the Grothendieck group on the set of isomorphism classes of the category of finite H-sets, denoted by* $\widehat{H}$*, and therefore,*

$$\underline{B}(H) = \{[H]; \ H \in \widehat{H}\},$$

such that the addition is given by disjoint union, $[U]+[V] = [U \amalg V]$*. Moreover, for all subgroups K of H, the transfer map* $tr^H_K : \underline{B}(K) \longrightarrow \underline{B}(H)$ *is given by* $tr^H_K([V]) = [H \times_K V]$ *and the restriction map* $res^H_K : \underline{B}(H) \longrightarrow \underline{B}(K)$ *is given by* $res^H_K([U]) = [\psi_K U]$ *where* ψ_K *is the restriction functor from* $\widehat{H}$ *to* $\widehat{K}$*. The action of G is trivial.*

4. G-Green Functors

In this section, we provide two equivalent definitions of G-Green functors. The first is a constructive definition similar to definition 3.1 of a G-Mackey functor.

Definition 4.1. *A G-Mackey functor A is a G-Green functor if the following axioms hold:*

1. *A(H) is a ring for each subgroup H of G.*
2. *If* $K \leqslant H$ *are subgroups of G, and* $x \in G$*, then all restriction maps* $res^H_K : A(H) \longrightarrow A(K)$ *and all conjugation maps* $c_{x,H} : A(H) \longrightarrow A({}^xH)$ *are ring homomorphisms.*
3. *A satisfies Frobenius relations: If* $K \leqslant H$ *are subgroups of G then*
$$tr^H_K(a) \cdot b = tr^H_K(a \cdot res^H_K(b))$$
$$b \cdot tr^H_K(a) = tr^H_K(res^H_K(b) \cdot a)$$
for all $a \in A(K)$ *and* $b \in A(H)$*.*

Furthermore, a G-Green functor A is commutative if every A(H) is a commutative ring.

A morphism f from the Green functor A to the Green functor B is a morphism of Mackey functors such that, for any subgroup H of G, the morphism f_H is a morphism of rings.

The second is the category theoretic definition analogue of the Dress definitions of Mackey functors [6]. The two definitions are equivalent [6].

Definition 4.2. *Let R be a commutative ring. A G-Green functor A over R for the finite group G is a G-Mackey functor endowed for any G-sets X and Y with bilinear maps*

$$A(X) \times A(Y) \longrightarrow A(X \times Y)$$

denoted by $(a, b) \longrightarrow a \times b$, such that the following properties hold:

(1) *(Bifunctoriality) If $f : X \longrightarrow X_1$ and $g : Y \longrightarrow Y_1$ are morphisms of G-sets, then the following diagrams*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A_*(f)\times A_*(g)}\downarrow & & \downarrow{\scriptstyle A_*(f\times g)} \\ A(X_1) \times A(Y_1) & \xrightarrow[\times]{} & A(X_1 \times Y_1) \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A^*(f)\times A^*(g)}\uparrow & & \uparrow{\scriptstyle A^*(f\times g)} \\ A(X_1) \times A(Y_1) & \xrightarrow[\times]{} & A(X_1 \times Y_1) \end{array}$$

are commutative.

(2) *(Associativity) If X, Y and Z are G-sets, then the following diagram*

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{e_{A(X)}\times(\times)} & A(X) \times A(Y \times Z) \\ {\scriptstyle (\times)\times e_{A(Z)}}\downarrow & & \downarrow{\scriptstyle \times} \\ A(X \times Y) \times A(Z) & \xrightarrow[\times]{} & A(X \times Y \times Z) \end{array}$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$

(3) *(Unitarity) If $\star$ denotes the G-set with one element, then there exists an element $\tau \in A(\star)$, such that for any G-set X and for any $x \in A(X)$*

$$A_*(l_X)(x \times \tau) = x = A_*(k_X)(\tau \times x),$$

where l_X is the bijective projection from $X \times \star$ to X and k_X is the bijective projection from $\star \times X$ to X. We will write $\mathcal{G}reen(G)$ for the category of G-Green functors.

Example 4.3. [6] *A fixed point G-Green functor is a fixed point G-Mackey functor $\underline{M}$ if we can extend the group M to have a ring structure that is equipped with the action of G. In particular, we need that $g \cdot (ab) = (ga) \cdot (gb)$ and $g1 = 1$ for all $g \in G$ and $a, b \in M$.*

Example 4.4. [6] *The Burnside G-Mackey functor inherits the structure of a G-Green functor. For all subgroups H of G, we will define* $\underline{B}(H)$ *to be the Grothendieck group on the set of isomorphism classes of the category of finite H-sets, where addition is given by disjoint union,* $[U] + [V] = [U \amalg V]$ *and multiplication is given by the Cartesian product,* $[U][V] = [U \times V]$*. The multiplicative unit is the isomorphism class of the single point set [H/H]. The direct product of H-sets converts* $\underline{B}(H)$ *into a ring, such that all restriction maps are ring homomorphisms. Transfer and restriction form a G-Green functor structure on* $\underline{B}$*.*

5. G-tensor products of G-Mackey functors

In this section we construct a Mackey functor diagram $M \otimes N$ for G-Mackey functors M and N.

Definition 5.1. *Let M and N be G-Mackey functors in the sense of Green's definition. Then, we define* $M \otimes N$ *as follows. For all subgroups H of G:*

$$(M \otimes N)(H) = \bigoplus_{K \leqslant H} M(K) \otimes N(K)/\sim .$$

(1) *The* $\sim$ *is given by the following relations:*

$$a \otimes tr_L^K(y) \sim res_L^K(a) \otimes y$$

$$tr_L^K(x) \otimes b \sim x \otimes res_L^K(b)$$

for $L \leqslant K \leqslant H$, $a \in M(K)$, $b \in N(K)$, $x \in M(L)$ *and* $y \in N(L)$.

(2) *We denote the element in* $(M \otimes N)(H)$ *by the class* $[a \otimes b]$*, where* $a \otimes b \in M(K) \otimes N(K)$.

(3) *The action is given by*

$$c_{x,H}([a \otimes b]) = [c_{x,H}(a) \otimes c_{x,H}(b)]$$

where $x \in G$.

(4) *We define the restriction map* $res_K^H : (M \otimes N)(H) \longrightarrow (M \otimes N)(K)$ *by* $res_K^H([a \otimes b]) = res_K^H(a) \otimes res_K^H(b)$ *for* $a \otimes b \in M(K) \otimes N(K)$ *and for all subgroups L and K in H*

$$res_K^H([m \otimes n]) = \sum_{x \in [L \backslash H/K]} x \cdot ([m \otimes n])$$

for all $m \otimes n \in M(K) \otimes N(K)$.

(5) *We define the transfer map as follows:*

$$tr_K^H([a \otimes b]) = [a \otimes b]$$

for all $a \otimes b \in M(K) \otimes N(K)$.

6. G-$\mathcal{HOM}(A, B)$

Let $\mathcal{M}ack(G)$ be the category of G-Mackey functors. Recall that a morphism $f \in Hom_{\mathcal{M}ack(G)}(A, B)$ in $\mathcal{M}ack(G)$ consists of a family of group homomorphisms $f_H : A(H) \longrightarrow B(H)$ for all subgroups H of G, such that if $K \leqslant H$ and $x \in G$, the squares

$$\begin{array}{ccc} A(H) & \xrightarrow{f_H} & B(H) \\ {\scriptstyle tr_K^H}\uparrow & & \uparrow{\scriptstyle tr_K^H} \\ A(K) & \xrightarrow[f_K]{} & B(K) \end{array} \qquad \begin{array}{ccc} A(H) & \xrightarrow{f_H} & B(H) \\ {\scriptstyle res_K^H}\downarrow & & \downarrow{\scriptstyle res_K^H} \\ A(K) & \xrightarrow[f_K]{} & B(K) \end{array} \qquad \begin{array}{ccc} A(H) & \xrightarrow{f_H} & B(H) \\ {\scriptstyle c_{x,H}}\downarrow & & \downarrow{\scriptstyle c_{x,H}} \\ A({}^xH) & \xrightarrow[f_{{}^xH}]{} & B({}^xH) \end{array}$$

are commutative.

Definition 6.1. *Let A and B be G-Mackey functors in the sense of Green's definition. Then, we define G-$\mathcal{HOM}(A, B)$ as follows. For all subgroups H of G:*

$$\mathcal{HOM}(A, B)(H) = Hom_{\mathcal{M}ack(H)}(Res_H^G A, Res_H^G B).$$

1. *Hence any element f in $\mathcal{HOM}(A, B)(H)$ corresponds to a collection of morphisms $f_K : A(K) \longrightarrow B(K)$, for $K \leqslant H$.*
2. *The action of G on $\mathcal{HOM}(A, B)(H)$ is given by*
$$({}^xf)_K(m) = xf_K(x^{-1}m)$$
for $K \leqslant H$, $x \in G$ and $m \in A(K)$.
3. *We define the restriction map $R_K^H : \mathcal{HOM}(A, B)(H) \longrightarrow \mathcal{HOM}(A, B)(K)$ by*
$$R_K^H(f_K) = f_L$$
, for $L \leqslant K \leqslant H$.
4. *We define the transfer map $T_K^H : \mathcal{HOM}(A, B)(K) \longrightarrow \mathcal{HOM}(A, B)(H)$ to be a morphisms in $\mathcal{HOM}(A, B)(H)$. That is, let $f_L : A(L) \longrightarrow B(L)$ be a collection of morphisms in $\mathcal{HOM}(A, B)(K)$. Then from the commutativity of $\mathcal{HOM}(A, B)(H)$ we define $T_K^H(f_L)$ by*
$$\alpha_K(a) = tr_B f_L res_A(a)$$
and for all subgroups L and K in H
$$\beta_K(m) = \sum_{x \in [L \backslash H / K]} xf_L(x^{-1}m)$$
for a and $m \in A(K)$.

Remark 6.2. For example, we have

$$\begin{aligned} \mathcal{HOM}(A, B)(e) &= Hom(A(e), B(e)) \\ &= \{\text{group homomorphism from } A(e) \text{ to } B(e)\}. \end{aligned}$$

The proof of the following result is in [6].

Proposition 6.3. *Let M, N and P be Mackey functors for the group G. Then there exists an isomorphism*

$$\mathcal{HOM}(M \otimes N, P) \simeq \mathcal{HOM}(N, \mathcal{HOM}(M, P))$$

natural in M, N and P.

7. Hochschild (Co)Homology of G-Mackey Functor

In this section we extend the definition of Hochschild (co)homology to G-Mackey functors.

Definition 7.1. *Let A be a G-Green functor and M be a bimodule over the G-Green functor A. Then, for every subgroup H of G, the Hochschild homology of a G-Mackey functor which is again a G-Mackey functor is the homology of the following chain complex:*

$$M(H) \overset{d_0}{\longleftarrow} \bigoplus_{K \leqslant H} M(K) \otimes A(K)/\sim \overset{d_1}{\longleftarrow} \bigoplus_{K \leqslant H} M(K) \otimes A(K) \otimes A(K)/\sim \overset{d_2}{\longleftarrow} \cdots$$

before given the boundary map, we denote by

$$i_q : M(K) \otimes A(K)^{\otimes q} \longrightarrow M(K) \otimes A(K)^{\otimes q}/\sim$$

the canonical map where $q > 0$. The boundary map is given by

$$\begin{aligned} d_{q-1} \circ i_q(m \otimes a_1 \otimes \cdots \otimes a_q) = (m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_q) \\ + \sum_{0<i<q} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q) \\ + (-1)^q (a_q m \otimes a_1 \otimes \cdots \otimes a_{q-1}). \end{aligned}$$

Hence,

$$H_n(A, M) = H_n(C_n(A, M)),$$

where $C_n(A, M) = (M \otimes A^{\otimes n})(H)$.

Example 7.2.

(1) $d_0 \circ i_1(m \otimes a) = ma - am$.
(2) $d_1 \circ i_2(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + bm \otimes a$.
(3) $d_2 \circ i_3(m \otimes a \otimes b \otimes c) = ma \otimes b \otimes c - m \otimes ab \otimes c + m \otimes a \otimes bc - cm \otimes a \otimes b$.

Definition 7.3. *Let A be a G-Green functor and M be a bimodule over the G-Green functor A. Then, for every subgroup H of G, the Hochschild cohomology of a G-Mackey functor which is again a G-Mackey functor is the cohomology of the following of cochain complex:*

$$M(H) \overset{b_0}{\longrightarrow} Hom_{\mathcal{M}ack(H)}(Res^G_H A, Res^G_H M) \overset{b_1}{\longrightarrow} Hom_{\mathcal{M}ack(H)}(Res^G_H A \otimes A, Res^G_H M) \overset{b_2}{\longrightarrow} \cdots$$

where the coboundary map is given by

$$\begin{aligned} b_n(f_K)(a_1,\cdots,a_{n+1}) = a_1 f_K(a_2,\cdots,a_{n+1}) \\ + \sum_{0<i<n+1} (-1)^i f_K(a_1,\cdots,a_i a_{i+1},\cdots,a_{n+1}) \\ + (-1)^{n+1} f_K(a_1,\cdots,a_n)a_{n+1}. \end{aligned}$$

Hence,

$$H^n(A,M) = H^n(C^n(A,M)),$$

where $C^n(A,M) = G\text{-}\mathcal{HOM}(A^{\otimes n}, M)(H)$ *and for* $K \leqslant H$.

Example 7.4.

(1) $b_0(m)(a) = am - ma$.
(2) $b_1(f_K)(a,b) = af_K(b) - f_K(ab) + f_K(a)b$.
(3) $b_2(f_K)(a,b,c) = af_K(b,c) - f_K(ab,c) + f_K(a,bc) - f_K(a,b)c$.

8. Classification Of Singular Extension Of G-Green Functors

It is a well-known fact that the second Hochschild cohomology classifies the singular extensions of associative algebras [7]. Here we obtain a similar result for G-Green functors.

Definition 8.1. *Let A be a G-Green functor and M be an A-bimodule. A singular extension E of A by M is an exact sequence of Mackey functors*

$$E: 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0,$$

where B is a G-Green functor, j is a homomorphism of G-Green functors and i is a homomorphism of G-Mackey functors such that for all subgroups H of G the following sequences:

$$0 \to M(H) \xrightarrow{i_H} B(H) \xrightarrow{j_H} A(H) \to 0$$

are singular extensions of the ring $A(H)$ by $M(H)$.

Definition 8.2. *A singular extension $E: 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$ is called M-split if for all subgroups H of G there exist a group homomorphism:*

$$s_H = s(H): A(H) \longrightarrow B(H)$$

such that

(1) $j_H \circ s_H = id_{A(H)}$.
(2) *s_H must be compatible with transfer, restriction and conjugation maps in the following sense:*
   - $res_{B^H_K} \circ s_H = s_K \circ res_{A^H_K}$,
   - $tr_{B^H_K} \circ s_K = s_H \circ tr_{A^H_K}$,
   - $c_{B_K} \circ s_K = s_K \circ c_{A_K}$,

for $K \leqslant H$.

Definition 8.3. *Let A be a G-Green functor and M be an A-bimodule. Then, for all subgroups H of G, a H-Green 2-cocycle $\mathcal{Z}^2_H(A, M)$ of A with values in M is a collection of bilinear maps*

$$f_H : A(H) \times A(H) \longrightarrow M(H)$$

such that for $K \leqslant H$, the following conditions hold:

- $xf_H(y,z) + f_H(x,yz) = f_H(xy,z) + f_H(x,y)z$.
- $c_{M_K}(f_K(a,b)) = f_K(c_{A_K}(a), c_{A_K}(b))$.
- $res_{M^H_K}(f_H(x,y)) = f_K(res_{A^H_K}(x), res_{A^H_K}(y))$.
- $f_H(tr_{A^H_K}(a), x) = tr_{M^H_K}(f_K(a, res_{A^H_K}(x)))$.
- $f_H(x, tr_{A^H_K}(a)) = tr_{M^H_K}(f_K(res_{A^H_K}(x), a))$.

Proposition 8.4. *Let f_H be a H-Green 2-cocycle, then one can construct a G-Green functor B_{f_H} as follows. For all subgroups H of G:*

- $B_{f_H}(H) = M(H) \oplus A(H)$ *as an associative ring with multiplication*
$$(u,x)(w,y) = (uy + xw + f_H(x,y), xy).$$
- $c_{B_K}(u,x) = (c_{M_K}(u), c_{A_K}(x))$.
- $tr_{B^H_K}(u,x) = (tr_{M^H_K}(u), tr_{A^H_K}(x))$.
- $res_{B^H_K}(u,x) = (res_{M^H_K}(u), res_{A^H_K}(x))$.

for $K \leqslant H$.

Proof. We need to verify that B_{f_H} satisfies all axioms of a G-Green functor. Observe that $B_{f_H}(H)$ is an associative ring since f_H is a 2-cocycle. Next, we need to check that the conjugation map c_{B_K} is a ring homomorphism. That is, for (m,a) and (n,b) be elements in $B_{f_K}(K)$ we have

$$\begin{aligned} c_{B_K}((m,a)\cdot(n,b)) &= c_{B_K}(mb + an + f_K(a,b), ab) \\ &= (c_{M_K}(m)\cdot c_{A_K}(b) + c_{A_K}(a)\cdot c_{M_K}(n) \\ &+ c_{M_K}(f_K(a,b)), c_{A_K}(a)\cdot c_{A_K}(b)). \end{aligned}$$

We also have

$$\begin{aligned} c_{B_K}(m,a)\cdot c_{B_K}(n,b) &= (c_{M_K}(m), c_{A_K}(a))\cdot(c_{M_K}(n), c_{A_K}(b)) \\ &= (c_{M_K}(m)\cdot c_{A_K}(b) + c_{A_K}(a)\cdot c_{M_K}(n) \\ &+ f_K(c_{A_K}(a), c_{A_K}(b)), c_{A_K}(a)\cdot c_{A_K}(b)). \end{aligned}$$

Hence, from condition 2 in definition 8.3 it follows that c_{B_K} is a ring homomorphism. Similarly, we need to check that the restriction map $res_{B^H_K}$ is ring homomorphism. That is, for (u,x) and (v,y) be elements in $B_{f_H}(H)$ we have

$$\begin{aligned} res_{B^H_K}((u,x)\cdot(v,y)) =& res_{B^H_K}(uy + xv + f_H(x,y), xy) \\ =& (res_{M^H_K}(u)\cdot res_{A^H_K}(y) + res_{A^H_K}(x)\cdot res_{M^H_K}(v) \\ &+ res_{M^H_K}(f_H(x,y)), res_{A^H_K}(x)\cdot res_{A^H_K}(y)). \end{aligned}$$

We also have

$$\begin{aligned} res_{B^H_K}(u,x)\cdot res_{B^H_K}(v,y) =&(res_{M^H_K}(u), res_{A^H_K}(x))\cdot(res_{M^H_K}(v), res_{A^H_K}(y)) \\ =&(res_{M^H_K}(u)\cdot res_{A^H_K}(y) + res_{A^H_K}(x)\cdot res_{M^H_K}(v) \\ &+ f_K(res_{A^H_K}(x), res_{A^H_K}(y)), res_{A^H_K}(x)\cdot res_{A^H_K}(y)). \end{aligned}$$

Therefore, from condition 3 in definition 8.3 it follows that $res_{B^H_K}$ is a ring homomorphism.

Finally, we need to check that B_{f_H} satisfies the Frobenius relations. That is, for (m,a) be an element in $B_{f_K}(K)$ and (u,x) be an element in $B_{f_H}(H)$ we have

$$\begin{aligned} tr_{B^H_K}(m,a)\cdot(u,x) &= (tr_{M^H_K}(m), tr_{A^H_K}(a))\cdot(u,x) \\ &= (tr_{M^H_K}(m)\cdot x + tr_{A^H_K}(a)\cdot u + f_H(tr_{A^H_K}(a), x), tr_{A^H_K}(a)\cdot x). \end{aligned}$$

We also have

$$\begin{aligned} tr_{B^H_K}((m,a)\cdot res_{B^H_K}(u,x)) &= tr_{B^H_K}((m,a)\cdot(res_{M^H_K}(u), res_{A^H_K}(x))) \\ &= tr_{B^H_K}(m\cdot res_{A^H_K}(x) + a\cdot res_{M^H_K}(u) \\ &+ f_K(a, res_{A^H_K}(x)), a\cdot res_{A^H_K}(x)) \\ &= (tr_{M^H_K}(m\cdot res_{A^H_K}(x)) + tr_{A^H_K}(a\cdot res_{M^H_K}(u)) \\ &+ tr_{M^H_K}(f_K(a, res_{A^H_K}(x))), tr_{A^H_K}(a\cdot res_{A^H_K}(x))). \end{aligned}$$

Hence, from the definition of G-Green functors, definition of modules over G-Green functors and condition 4 in definition 8.3 it follows that:

$$tr_{B^H_K}(m,a)\cdot(u,x) = tr_{B^H_K}((m,a)\cdot res_{B^H_K}(u,x)).$$

Likewise, from the definition of G-Green functors, definition of modules over G-Green functors and condition 5 in definition 8.3 it follows that:

$$(u,x)\cdot tr_{B^H_K}(m,a) = tr_{B^H_K}(res_{B^H_K}(u,x)\cdot(m,a)).$$

□

Definition 8.5. *Let A be a G-Green functor and M be an A-bimodule. We define $Ext(A,M)$ to be the set of equivalence classes of M-split extensions of A by M.*

Definition 8.6. *Let A be a G-Green functor and M be an A-bimodule. For all subgroups H of G we define*

$$C^1_H(A,M) = \left\{ h_H \;\middle|\; \begin{array}{r} \forall x\in A(H),\;\; res_{M^H_K}(h_H(x)) = h_K(res_{A^H_K}(x)) \\ \forall a\in A(K),\;\; c_{M_K}(h_K(a)) = h_K(c_{A_K}(a)) \\ \forall a\in A(K),\;\; tr_{M^H_K}(h_K(a)) = h_H(tr_{A^H_K}(a)) \end{array} \right\},$$

where $h_H : A(H)\longrightarrow M(H)$ and $K\leqslant H$. Moreover, there exists a map

$$\partial : C^1_H(A,M)\longrightarrow \mathcal{Z}^2_H(A,M)$$

such that $\partial(h_H) = (\delta h_H(x,y))$, where

$$\delta h_H(x,y) = xh_H(y) - h_H(xy) + h_H(x)y.$$

Definition 8.7. *Let A be G-Green functor and M be an A-bimodule. Then, for all subgroups H of G, we define the second cohomology by*

$$H^2_H(A, M) = coker\ \partial.$$

Theorem 8.8. *Let A be a G-Green functor, M be an A-bimodule and $Ext(A, M)$ be the set of equivalence classes of M-split extensions A by M. There is a one-to-one correspondence between the elements of $Ext(A, M)$ and those of $H^2_H(A, M)$.*

Proof. To prove the theorem, we are going to follow these steps.
Step 1. Show that there is a well-defined map from $Ext(A, M)$ to $H^2_H(A, M)$.
Step 2. Show that there is a well-defined map from $H^2_H(A, M)$ to $Ext(A, M)$.
Step 3. Show that these two maps are inverse to each other.
Step 1. Consider a singular extension

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

and let

$$s_H : A(H) \longrightarrow B(H)$$

be an abelian group homomorphism such that

$$j_H \circ s_H = id_{A(H)}.$$

Then, for every $x, y \in A(H)$, there exists a uniquely determined element $f_H(x, y) \in M(H)$ such that

$$s_H(x)s_H(y) = s_H(xy) + i_H f_H(x, y). \tag{8.1}$$

For $x, y, z \in A(H)$,

$$\begin{aligned} s_H(x)(s_H(y)s_H(z)) &= s_H(x)(s_H(yz) + i_H f_H(y, z)) \\ &= s_H(x)s_H(yz) + s_H(x)i_H f_H(y, z) \\ &= s_H(xyz) + i_H f_H(x, yz) + s_H(x)i_H f_H(y, z). \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} (s_H(x)s_H(y))s_H(z) &= (s_H(xy) + i_H f_H(x, y))s_H(z) \\ &= s_H(xy)s_H(z) + i_H f_H(x, y)s_H(z) \\ &= s_H(xyz) + i_H f_H(xy, z) + i_H f_H(x, y)s_H(z). \end{aligned} \tag{8.3}$$

Thus, multiplication in $B(H)$ is associative which follows from (8.2) and (8.3) that:

$$x f_H(y, z) - f_H(xy, z) + f_H(x, yz) - f_H(x, y)z = 0$$

showing that f_H is a H-Green 2-cocycle. Next, we know that the conjugation map $c_{B_K} : B(K) \longrightarrow B(K)$ is a ring homomorphism and by applying c_{B_K} to equation (5.1) we have

$$\begin{aligned} c_{B_K}(s_K(a)s_K(b)) &= c_{B_K}(s_K(ab) + i_K f_K(a, b)) \\ \Rightarrow c_{B_K}(s_K(a))c_{B_K}(s_K(b)) &= c_{B_K}(s_K(ab)) + c_{B_K}(i_K f_K(a, b)) \end{aligned}$$

$$\Rightarrow s_K(c_{A_K}(a))s_K(c_{A_K}(b)) = s_K(c_{A_K}(ab))) + i_K(c_{M_K}(f_K(a,b)))$$
$$\Rightarrow \cancel{s_K(c_{A_K}(a)c_{A_K}(b))} + \cancel{i_K} f_K(c_{A_K}(a), c_{A_K}(b)) = \cancel{s_K(c_{A_K}(a)c_{A_K}(b))} + \cancel{i_K}(c_{M_K}(f_K(a,b)))$$
$$\Rightarrow f_K(c_{A_K}(a), c_{A_K}(b)) = c_{M_K}(f_K(a,b)).$$

Similarly, we know that the restriction map $res_{B^H_K} : B(H) \longrightarrow B(K)$ is a ring homomorphism and by applying $res_{B^H_K}$ to equation (8.1) we have

$$res_{B^H_K}(s_H(x)s_H(y)) = res_{B^H_K}(s_H(xy) + i_H f_H(x,y))$$
$$\Rightarrow res_{B^H_K}(s_H(x))res_{B^H_K}(s_H(y)) = res_{B^H_K}(s_H(xy)) + res_{B^H_K}(i_H f_H(x,y))$$
$$\Rightarrow s_K(res_{A^H_K}(x))s_K(res_{A^H_K}(y)) = s_K(res_{A^H_K}(xy)) + i_K(res_{M^H_K}(f_H(x,y)))$$
$$\Rightarrow \cancel{s_K(res_{A^H_K}(x)res_{A^H_K}(y))} + \cancel{i_K}(f_K(res_{A^H_K}(x), res_{A^H_K}(y)))$$
$$= \cancel{s_K(res_{A^H_K}(x)res_{A^H_K}(y))}) + \cancel{i_K}(res_{M^H_K}(f_H(x,y)))$$
$$\Rightarrow f_K(res_{A^H_K}(x), res_{A^H_K}(y)) = res_{M^H_K}(f_H(x,y)).$$

Furthermore, from (8.1) we have

$$s_H(y)s_H(x) = s_H(yx) + i_H f_H(y,x)$$
$$\Rightarrow i_H f_H(y,x) = s_H(y)s_H(x) - s_H(yx).$$

Now, by substituting $y = tr_{A^H_K}(a)$ in the above equation we have

$$i_H f_H(tr_{A^H_K}(a), x) = s_H(tr_{A^H_K}(a))s_H(x) - s_H(tr_{A^H_K}(a) \cdot x)$$
$$\Rightarrow i_H f_H(tr_{A^H_K}(a), x) = tr_{B^H_K}(s_K(a))s_H(x) - s_H(\overbrace{tr_{A^H_K}(a \cdot res_{A^H_K}(x))}^{\text{Frobenius relation}})$$
$$\Rightarrow i_H f_H(tr_{A^H_K}(a), x) = \overbrace{tr_{B^H_K}(s_K(a) \cdot res_{B^H_K}(s_H(x)))}^{\text{Frobenius relation}} - tr_{B^H_K}(s_K(a \cdot res_{A^H_K}(x)))$$
$$\Rightarrow i_H f_H(tr_{A^H_K}(a), x) = tr_{B^H_K}(s_K(a) \cdot s_K(res_{A^H_K}(x))) - tr_{B^H_K}(s_K(a \cdot res_{A^H_K}(x)))$$
$$\Rightarrow i_H f_H(tr_{A^H_K}(a), x) = tr_{B^H_K}\overbrace{(s_K(a \cdot res_{A^H_K}(x)) + i_K f_K(a, res_{A^H_K}(x)))}^{\text{from (8.1)}} - tr_{B^H_K}(s_K(a \cdot res_{A^H_K}(x)))$$
$$\Rightarrow i_H f_H(tr_{A^H_K}(a), x) = \cancel{tr_{B^H_K}(s_K(a \cdot res_{A^H_K}(x)))} + tr_{B^H_K}(i_K f_K(a, res_{A^H_K}(x)))$$
$$- \cancel{tr_{B^H_K}(s_K(a \cdot res_{A^H_K}(x)))}$$
$$\Rightarrow \cancel{i_H} f_H(tr_{A^H_K}(a), x) = \cancel{i_H}(tr_{M^H_K}(f_K(a, res_{A^H_K}(x))))$$
$$\Rightarrow f_H(tr_{A^H_K}(a), x) = tr_{M^H_K}(f_K(a, res_{A^H_K}(x))).$$

Similarly, from (8.1) we have

$$\Rightarrow i_H f_H(x,y) = s_H(x)s_H(y) - s_H(xy).$$

Now, by substituting $y = tr_{A^H_K}(a)$ in the above equation we have

$$i_H f_H(x, tr_{A^H_K}(a)) = s_H(x)s_H(tr_{A^H_K}(a)) - s_H(x \cdot tr_{A^H_K}(a))$$

$$\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) = s_H(x) tr_{B_K^H}(s_K(a)) - s_H \overbrace{(tr_{A_K^H}(res_{A_K^H}(x) \cdot a))}^{\text{Frobenius relation}}$$

$$\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) = \overbrace{tr_{B_K^H}(res_{B_K^H}(s_H(x) \cdot s_K(a)))}^{\text{Frobenius relation}} - tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot a))$$

$$\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) = tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot s_K(a))) - tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot a))$$

$$\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) = tr_{B_K^H} \overbrace{(s_K(res_{A_K^H}(x) \cdot a)) + i_K f_K(res_{A_K^H}(x), a)}^{\text{from (8.1)}} - tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot a))$$

$$\Rightarrow i_H f_H(x, tr_{A_K^H}(a)) = \cancel{tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot a))} + tr_{B_K^H}(i_K f_K(res_{A_K^H}(x), a)) - \cancel{tr_{B_K^H}(s_K(res_{A_K^H}(x) \cdot a))}$$

$$\Rightarrow \cancel{i_H} f_H(x, tr_{A_K^H}(a)) = \cancel{i_H}(tr_{M_K^H}(f_K(res_{A_K^H}(x), a)))$$

$$\Rightarrow f_H(x, tr_{A_K^H}(a)) = tr_{M_K^H}(f_K(res_{A_K^H}(x), a)).$$

Hence, $f_H \in \mathcal{Z}_H^2(A, M)$ satisfy all conditions in definition 8.3.

Now, let

$$s'_H : A(H) \longrightarrow B(H)$$

be an abelian homomorphism and let

$$g_H : A(H) \times A(H) \longrightarrow M(H)$$

be the 2-cocycle corresponding to choices of s'_H. Then,

$$j_H \circ s_H(x) = x = j_H \circ s'_H(x)$$

for every $x \in A(H)$, and so there exists $C_H^1(A, M) \xrightarrow{\partial} \mathcal{Z}_H^2(A, M)$ such that

$$s'_H(x) = i_H h_H(x) + s_H(x) \tag{8.4}$$

where $h_H : A(H) \longrightarrow M(H)$ and $x \in A(H)$. For $x, y \in A(H)$ and by substituting (8.4) in (8.1) we have

$$i_H f_H(x, y) + s'_H(xy) - i_H h_H(xy) = (s'_H(x) - i_H h_H(x))(s'_H(y) - i_H h_H(y))$$

$$\Rightarrow i_H f_H(x, y) + s'_H(xy) - i_H h_H(xy) = s'_H(x) s'_H(y) - s'_H(x) i_H h_H(y) - i_H h_H(x) s'_H(y) + \overbrace{i_H h_H(x) i_H h_H(y)}^{=0}$$

$$\Rightarrow i_H f_H(x, y) + \cancel{s'_H(xy)} - i_H h_H(xy) = \cancel{s'_H(xy)} + i_H g_H(x, y) - s'_H(x) i_H h_H(y) - i_H h_H(x) s'_H(y)$$

$$\Rightarrow \delta h_H(x, y) = g_H(x, y) - f_H(x, y) = x h_H(y) - h_H(xy) + h_H(x) y$$

so that f_H and g_H differ by a 2-coboundary. Therefore, we show that there exists a well-defined map from $Ext(A, M)$ to $H_H^2(A, M)$.

Step 2. Let $[f_H] \in H_H^2(A, M)$, where $f_H \in \mathcal{Z}_H^2(A, M)$. Then, we define the H-Green functor B_{f_H} as in Proposition 8.4. Therefore, the extension associated to f_H is the extension

$$E_{f_H} : 0 \to M \xrightarrow{i} B_{f_H} \xrightarrow{j} A \to 0,$$

where j is a homomorphism of H-Green functors and i is a homomorphism of H-Mackey functors. Now, we need to show that $[E_{f_H}]$ is independent of the choices of f_H. In other words, if $[f_H] = [g_H] \Leftrightarrow f_H = g_H + \delta h_H$.
Two extensions E_{f_H} and E_{g_H} are equivalent if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(H) & \xrightarrow{i_H} & B_{f_H}(H) & \xrightarrow{j_H} & A(H) & \longrightarrow & 0 \\
 & & \| & & \downarrow \alpha_H & & \| & & \\
0 & \longrightarrow & M(H) & \xrightarrow[i'_H]{} & B_{g_H}(H) & \xrightarrow[j'_H]{} & A(H) & \longrightarrow & 0
\end{array}$$

with α_H a homomorphism of rings. The commutativity of this diagram implies that

$$\alpha_H(u, x) = (u + h_H(x), x)$$

for $h_H \in C^1_H(A,M)$. The fact that α_H is a ring homomorphism gives the following equation,

$$\begin{aligned} \alpha_H((u, x)(v, y)) &= \alpha_H(uy + xv + f_H(x, y), xy) \\ &= uy + xv + f_H(x, y) + h_H(x, y), xy) \end{aligned} \tag{8.5}$$

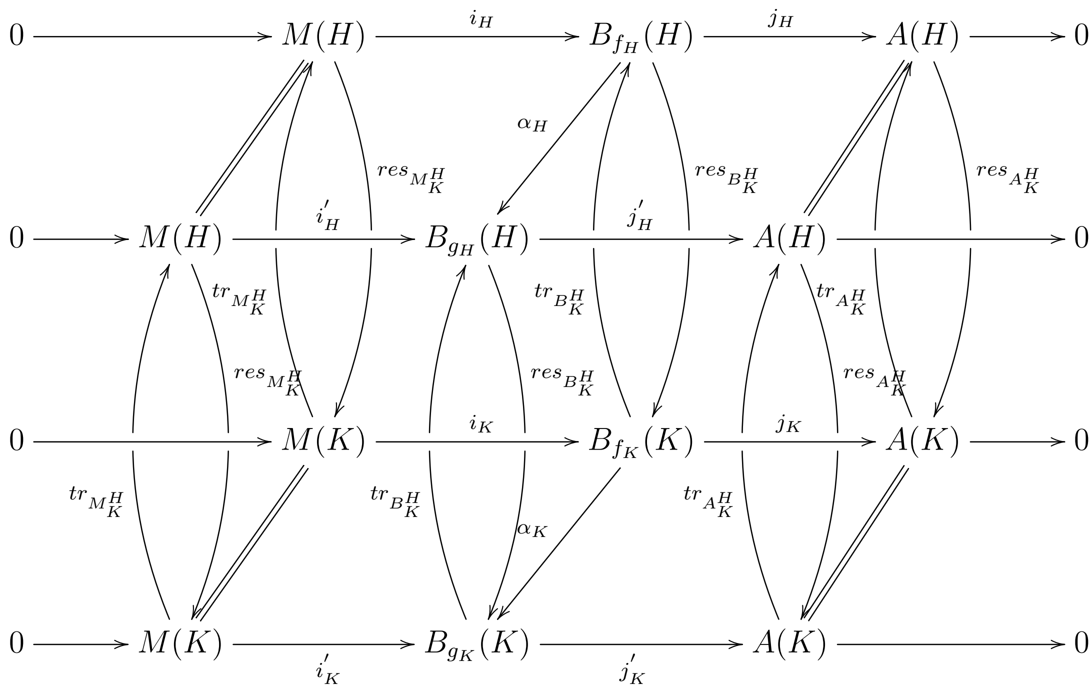
and

$$\begin{aligned} \alpha_H(u, x)\alpha_H(v, y) &= (u + h_H(x), x)(v + h_H(y), y) \\ &= uy + h_H(x)y + xv + xh_H(y) + g_H(x, y), xy) \end{aligned} \tag{8.6}$$

Hence, from (8.5) and (8.6) we obtain

$$f_H(x, y) - g_H(x, y) = xh_H(y) - h_H(x, y) + h_H(x)y = \delta h_H(x, y)$$

that is, $f_H - g_H$ is a 2-coboundary. Conversely, if $f_H - g_H$ is a 2-coboundary, then E_{f_H} and E_{g_H} are equivalent. Moreover, we need to check that the following diagram commutes:



It suffices to check that $res_{B^H_K} \circ \alpha_H(u, x) = \alpha_K \circ res_{B^H_K}(u, x)$. We have

$res_{B^H_K} \circ \alpha_H(u, x) = res_{B^H_K}(u + h_H(x), x) = (res_{M^H_K}(u) + res_{M^H_K}(h_H(x)), res_{A^H_K}(x))$

and

$\alpha_K \circ res_{B^H_K}(u, x) = \alpha_K(res_{M^H_K}(u), res_{A^H_K}(x)) = (res_{M^H_K}(u) + h_K(res_{A^H_K}(x)), res_{A^H_K}(x))$.

Thus, from definition 8.6 it follows that: $res_{B^H_K} \circ \alpha_H(u, x) = \alpha_K \circ res_{B^H_K}(u, x)$. Therefore, we show that there exists a well-defined map from $H^2_H(A, M)$ to $Ext(A, M)$.

Step 3. Let f_H be 2-cocycle. Then, we define the multiplications on $B_{f_H}(H)$ as follows:

$$(u, x)(v, y) = (uy + xv + f_H(x, y), xy)$$

where $u, v \in M(H)$ and $x, y \in A(H)$. The 2-cocycle property of f_H show that the multiplication on $B_{f_H}(H)$ is associative. Thus, $B_{f_H}(H)$ is an associative ring. We define the maps

$$i_H : M(H) \longrightarrow B_{f_H}(H)$$
$$j_H : B_{f_H}(H) \longrightarrow A(H)$$

as follows:

$$i_H(u) = (u, 0)$$
$$j_H(u, x) = x$$

where i_H is homomorphisms of H-Mackey functors and j_H is homomorphisms of H-Green functors and the sequence

$$E_{f_H} : 0 \to M \xrightarrow{i} B \underset{\text{s}}{\overset{\text{j}}{\rightleftarrows}} A \to 0$$

is exact. For $x \in A(H)$, choose $s_H(x) = (0, x)$. Then, for $x, y \in A(H)$,

$$\begin{aligned} s_H(x)s_H(y) = (0, x)(0, y) &= (f_H(x, y), xy) \\ &= (f_H(x, y), 0) + (0, xy) \\ &= i_H(f_H(x, y)) + s_H(xy). \end{aligned}$$

The choice s_{C_p} thus give the 2-cocycle f_H.

Conversely, suppose that

$$E : 0 \to M \xrightarrow{i} B \xrightarrow{j} A \to 0$$

is an extension and let f_H be the 2-cocycle obtained from this extension. We must show that the extension

$$E_{f_H} : 0 \to M \xrightarrow{i} B_{f_H} \xrightarrow{j} A \to 0$$

associated to f_H is equivalent to the given one. Indeed, E and E_{f_H} are equivalent if there exists a homomorphism $\theta_{f_H} : B_{f_H} \longrightarrow B$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & B & \underset{j}{\overset{s}{\leftrightarrows}} & A & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \theta_{f_H}} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow[i]{} & B_{f_H} & \xrightarrow[j]{} & A & \longrightarrow & 0 \end{array}$$

Now, the commutativity of this diagram implies that

$$\theta_{f_H}(u, x) = i_H(u) + s_H(x)$$

where $u \in M(H)$ and $x \in A(H)$. Therefore, it remains to check that θ_{f_H} is ring homomorphism. Let (u, x) and (v, y) be elements in $B(H)$,

$$\begin{aligned} \theta_{f_H}((u, x) \cdot (v, y)) &= \theta_{f_H}(uy + xv + f_H(x, y), xy) \\ &= i_H(uy) + i_H(vx) + i_{C_p}(f_H(x, y)) + s_H(xy) \\ &= i_H(u)s_H(y) + s_H(x)i_H(v) + s_H(x)s_H(y) \end{aligned}$$

and

$$\begin{aligned} \theta_{f_H}(u, x) \cdot \theta_{f_H}(v, y)) &= (i_H(u) + s_H(x))(i_H(v) + s_H(y)) \\ &= \overbrace{i_H(u)i_H(v)}^{=0} + i_H(u)s_H(y) + s_H(x)i_H(v) + s_H(x)s_H(y). \end{aligned}$$

Hence, θ_{f_H} is a ring homomorphism. This proves the theorem. $\square$

9. Deformation of G-Green Functors

The aim of this section is to extend the deformation theory of associative algebras due to Gerstenhaber [1] to obtain a similar result for G-Green functors.

Definition 9.1. *Let A be a G-Green functors. For all subgroups H of G, a one parameter formal deformation of A is a collection $(\Psi_H, \Phi_K, R_K^H, T_K^H)$, where*

$$\Psi_H = \sum_{i=0}^{\infty} \psi_{i_H} t^i,$$

$$\Phi_K = \sum_{i=0}^{\infty} c_K(a_i) t^i,$$

$$R_K^H = \sum_{i=0}^{\infty} res_K^H(a_i') t^i,$$

$$T_K^H = \sum_{i=0}^{\infty} tr_K^H(a_i) t^i,$$

are formal power series with $\psi_{n_H} \in Hom(A(H) \otimes A(H), A(H))$, $c_K : A(K) \longrightarrow A(K)$, $res_K^H : A(H) \longrightarrow A(K)$ and $tr_K^H : A(K) \longrightarrow A(H)$, for $K \leqslant H$.

One requires that for all $n \geqslant 0$ the following identities hold

(i) $\psi_{0_H}(a', b') = a'b'$,
(ii) $\sum_{i_H + j_H = n_H} \psi_{i_H}(\psi_{j_H}(a', b'), c') = \sum_{i_H + j_H = n_H} \psi_{i_H}(a', \psi_{j_H}(b', c'))$,
(iii) $c_K(\psi_{n_K}(a, b)) = \psi_{n_K}(c_K(a), c_K(b))$,
(iv) $res_K^H(\psi_{n_H}(a', b')) = \psi_{n_K}(res_K^H(a'), res_K^H(b'))$,
(v) $\psi_{n_H}(tr_K^H(a), b') = tr_K^H(\psi_{n_K}(a, res_K^H(b')))$,
(vi) $\psi_{n_H}(b', tr_K^H(a)) = tr_K^H(\psi_{n_K}(res_K^H(b'), a))$,

Here $a, b \in A(K)$ and $a', b', c' \in A(H)$. The last five identities can be expressed as

$$\Psi_H(\Psi_H(a', b'), c') = \Psi_H(a', \Psi_H(b', c')),$$

$$\Phi_K(\Psi_K(a, b)) = \Psi_K(\Phi_K(a), \Phi_K(b)),$$

$$R_K^H(\Psi_H(a', b')) = \Psi_K(R_K^H(a'), R_K^H(b')),$$

$$\Psi_H(T_K^H(a), b') = T_K^H(\Psi_K(a, R_K^H(b'))),$$

$$\Psi_H(b', T_K^H(a)) = T_K^H(\Psi_K(R_K^H(b'), a)),$$

which shows that $A[[t]]$ becomes a $k[[t]]$-Green functors. If for fixed $m \geqslant 1$ there are given $\psi_{n_H} \in Hom(A(H) \otimes A(H), A(H))$, $c_K : A(K) \longrightarrow A(K)$, $res_K^H : A(H) \longrightarrow A(K)$ and $tr_K^H : A(K) \longrightarrow A(H)$ for $n = 0, \cdots, m$ satisfying above identities for $n = 0, \cdots, m$, then we say that there is given an *m-deformation*. For $m = 1$, one also says that there is an *infinitesimal deformation*.

Definition 9.2. *Two deformations* $(\Psi_H, \Phi_K, R^H_K, T^H_K)$ *and* $(\Psi'_H, \Phi'_K, R'^H_K, T'^H_K)$ *are equivalent if there exists a formal power series*

$$\Omega_H = \sum_{n=0}^{\infty} \omega_{n_H} t^n,$$

with properties

(i) $\omega_{n_H} \in Hom(A(H), A(H)),\ n \geqslant 0,$
(ii) $\omega_{0_H}(a') = a',\ a' \in A(H),$
(iii) $\sum_{i_H+j_H=n_H} \omega_{i_H}(\psi'_{j_H}(a', b')) = \sum_{i_H+j_H+k_H=n_H} \psi_{i_H}(\omega_{j_H}(a'), \omega_{k_H}(b')).$

Here $n \geqslant 0$, $a' \in A(H)$ and $b' \in B(H)$. The last equation can be expressed also as

$$\Omega_H(\Psi'_H(a', b')) = \Psi_H(\Omega_H(a'), \Omega_H(b')).$$

In other words, Ω_H defines an isomorphism of $k[[t]]$-Green functors

$$(\Psi_H, \Phi_K, R^H_K, T^H_K) \to (\Psi'_H, \Phi'_K, R'^H_K, T'^H_K).$$

In a same way one can define under what condition two m-deformations are equivalent.

Corollary 9.3. *i) Let* $(\Psi_H, \Phi_K, R^H_K, T^H_K)$ *be a one parameter formal deformation of a G-Green functors A. Assume* $n > 0$ *is a natural number such that*

$$\psi_{i_H} = 0,\ for\ 0 < i < n.$$

Then ψ_{n_H} *is a H-Green 2-cocycle in* $C^n(A, A)$. *In particular* ψ_{1_H} *is a H-Green 2-cocycle in* $C^n(A, A)$.

ii) There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of a G-Green functors A and $H^2_H(A, A)$.

Proof. The part ii) easily follows from i). To prove i), we observe that these equations gives

$$\psi_{n_H}(a, b)c + \psi_{n_H}(ab, c) = a\psi_{n_H}(b, c) + \psi_{n_H}(a, bc),$$
$$c_K(\psi_{n_K}(a, b)) = \psi_{n_K}(c_K(a), c_K(b)),$$
$$res^H_K(\psi_{n_H}(a', b')) = \psi_{n_K}(res^H_K(a'), res^H_K(b')),$$
$$\psi_{n_H}(tr^H_K(a), b') = tr^H_K(\psi_{n_K}(a, res^H_K(b'))),$$
$$\psi_{n_H}(b', tr^H_K(a)) = tr^H_K(\psi_{n_K}(res^H_K(b'), a)).$$

Hence, ψ_{n_H} is a H-Green 2-cocycle in $C^n(A, A)$. □

10. Results and discussion

In this work we proved several important results on such cohomologies. The main result of this paper is Theorem 8.8, which says that the second Hochschild cohomology classifies the singular extensions of G-Green functors.

11. Conclusions

The results stated here are new and potentially useful. Since the research field here and its related ones are consistent, the content of this paper may attract interested readers who have been interested in this and related research fields.

Competing interests. The author declares that they has no competing interests.

Author's contributions. Only the author contributed in writing this paper.

Acknowledgements. This work was supported by the Jazan Uinversity.

References

[1] M. Gerstenhaber, On the Deformation of Rings and Algebras, *The Annals of Mathematics*, **79** (1964), 59-103. https://doi.org/10.2307/1970484

[2] G. Hochschild, On the cohomology groups of an associative algebra, *Ann. of Math.*, **46** (1945), 58-67. https://doi.org/10.2307/1969145

[3] J.A. Green, Axiomatic representation theory for finite groups, *Journal of Pure and Applied Algebra*, **1** (1971), no. 1, 41-77. https://doi.org/10.1016/0022-4049(71)90011-9

[4] A. Dress, Contributions to the theory of induced representations, Chapter in *Classical Algebraic K-Theory, and Connections with Arithmetic*, Vol. 342, Springer, 1973, 181-240. https://doi.org/10.1007/bfb0073727

[5] P. Webb, A guide to Mackey functors, Chapter in *Handbook of Algebra*, Vol. 2, Elsevier, 2000, 805-836. https://doi.org/10.1016/s1570-7954(00)80044-3

[6] S. Bouc, *Green Functors and G-Sets*, Springer, Berlin, Heidelberg, 1997. https://doi.org/10.1007/bfb0095821

[7] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1995.

Received: January 19, 2018; Published: February 6, 2018