International Journal of Algebra, Vol. 12, 2018, no. 3, 141 - 148
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2018.8515

On Description of Coherent Sheaves using Derived Geometric Methods

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Abstract

This work presents an investigation of characterizing coherent sheaves on smooth projective varieties and their complexes up to quasi-isomorphism using standard homological methods involving derived functors and Fourier-Mukai technique as pioneered by Grothendieck-Verdier and Shigeru Mukai respectively followed by the works of Bondal and Orlov.

Mathematics Subject Classification: 13D09, 14F05, 32C37

Keywords: Derived Category, Triangulated Category, Smooth Projective Variety, Fourier-Mukai Transform, Coherent Sheaf, Scheme

1 Introduction

Here from the outset, we assume familiarity with the basics of both triangulated categories and derived categories. A good detailed introduction can be found in [14] and [16]. However, one can also consider [7] (ch. XI), [8],[9], [15] and the appendix of [6] for a quick tour. Since the algebraic varieties we are considering in this paper are all smooth and projective over some algebraically closed field k, the derived categories we will be considering are all modelled on the abelian category of coherent sheaves on X that we denote by $D^b(X)$, where X is the corresponding smooth projective variety which is a separated scheme of finite type over k (although Hartshorne assumes integrality as well (cf. [13] ch. 2)) and b in the superscript corresponds to the maximum length of the complexes in $\mathrm{D}(\mathrm{Coh}(X))$ (i.e. the derived category of the abelian category of coherent sheaves on X) which we can take to be finite (cf. [9]).

Historically speaking, $D^b(X)$ -as a special case of $D^*(\mathcal{A})$ (with $*$ may denote any one of $\{+,-,b\}$); such that $\mathcal{A}$ is any abelian category; has been our main tool of investigation into the geometry of varieties in context of both its classification and its role as the solutions to the moduli problems involving sheaves on schemes (see for instance, [1], [9], [18] and [19]), which is one of the most generalized setting of moduli of bundles on varieties (cf. [11] and [12]). In context of the latter, it was Shigeru Mukai (following Grothendieck and Verdier), who can be considered as one of the pioneers to have initiated the branch of derived algebraic geometry by developing a new functor -called Fourier-Mukai transform. Fourier-Mukai transform which was first introduced in [17] can be seen as the derived version of the usual Fourier transform where the latter could be considered as the transform from L^2 integrable functions to themselves, as a kind of pull back function to $\mathbb{R}^n \times \mathbb{R}^n$ multiplied by the kernel $\exp(2\pi\iota\,\langle x, y\rangle)$ and taking the direct image. The situation generalizes to the derived category and we obtain the transform from bounded derived category of an abelian variety to the bounded derived category of its dual, where the role of the exponential kernel is played by Poincar*é* line bundle and the direct image is derived. This was then further generalized in [5] from the case of abelian variety to any smooth projective variety. Thus it was along these lines, homological methods through derived categories and derived functors as initially pioneered by Grothendieck and his student Verdier (cf. [3] and [4]), helped bring these methods as important instruments into algebraic geometry and moduli theory.

2 Preliminary Notes

We are here interested in the ways $D^b(X)$ can help us characterize coherent sheaves on X up to quasi-isomorphic complexes. For this purpose we will use Fourier-Mukai technique and classical duality theorems and identities to work out some explicit calculations and establish the significance of their usefulness in the program of corresponding classification. Most of our work has its motivation in the classical sources of [14], [17], [19] on one hand and in comparatively later works such as [1], [6], [8], [9] and [10].

Let $D^b(X)$ and $D^b(Y)$ denote the bounded derived categories of coherent sheaves on X and Y respectively -where both X and Y are smooth complex projective varieties. We define Fourier-Mukai functor, as

$$\Phi^{\mathcal{K}}_{X\to Y} : D^b(X) \longrightarrow D^b(Y)$$

by letting

$$\Phi^{\mathcal{K}}_{X\to Y}(F) = Rp_*(q^*(F) \overset{L}{\otimes} \mathcal{K})$$

where p, q are projections from $X \times Y$ onto Y and X, F and $\mathcal{K}$ are complexes (sheaves if concentrated at degree zero) in $D^b(X)$, $D^b(X \times Y)$ respectively and $\overset{L}{\otimes}$ denoting the left derived tensor product between the corresponding derived categories. The object $\mathcal{K}$ in $D^b(X \times Y)$ is called kernel of the functor. Here we do not need to derive the pullback q^* since projections are flat. The same object $\mathcal{K}$ can be used to define the functor in the other direction, by reversing the role of the projections. But we keep the direction fixed as in our definition and denote the functor simply by $\Phi_{\mathcal{K}}$. So, we no longer need to specify the direction every time we mention the functor. However, our main interest lies in when this Fourier-Mukai functor becomes an equivalence. We then call it Fourier-Mukai transform; or FM-transform. Now if we let $\mathcal{K}_L = \mathcal{K}^\vee \otimes p^*\omega_Y[\dim(Y)]$ and $\mathcal{K}_R = \mathcal{K}^\vee \otimes q^*\omega_X[\dim(X)]$, where $\otimes$ is the usual tensor product as ω_X is a line bundle whenever X is a smooth variety, and $\mathcal{K}^\vee \simeq R\mathcal{H}om(\mathcal{K}, \mathcal{O}_X)$ the derived dual of $\mathcal{K}$ in $D^b(X)$ (i.e. the right derived functor of the usual local or sheaf Hom), then from [17] we have the following

Theorem 2.1. *(Mukai) Let $\Phi_{\mathcal{K}} : D^b(X) \longrightarrow D^b(Y)$ be the FM transform with FM-kernel $\mathcal{K}$, then*

$$\Phi_{\mathcal{K}_L} : D^b(Y) \longrightarrow D^b(X) \qquad \Phi_{\mathcal{K}_R} : D^b(Y) \longrightarrow D^b(X)$$

are respectively left and right adjoints of $\Phi_{\mathcal{K}}$.

Also from [9] and [10], we have the following two results.

Theorem 2.2. *(Orlov) Suppose $\Phi_{\mathcal{K}} : D^b(X) \longrightarrow D^b(Y)$ is a fully faithful Fourier-Mukai functor between smooth projective varieties. Then $\Phi_{\mathcal{K}}$ is an equivalence if and only if*

$$\dim(X) = \dim(Y) \quad \textit{and} \quad \mathcal{K} \otimes q^*\omega_X \simeq \mathcal{K} \otimes p^*\omega_Y$$

Theorem 2.3. *(Orlov) Let $\Phi : D^b(X) \longrightarrow D^b(Y)$ be an equivalence with X and Y both being smooth and projective, then ϕ is isomorphic to some FM-transform $\Phi_{\mathcal{K}}$ such that $\mathcal{K} \in D^b(X \times Y)$ which is unique up to isomorphism.*

However, as these derived categories of coherent sheaves of smooth projective varieties come equipped with Serre functor, so the condition on the existence of both adjoints can be weakened. In fact the result proved by A. Bondal and Van den Bergh states that for any smooth projective variety X, the bounded derived category is saturated i.e. every cohomological functor $D^b(X) \longrightarrow \mathrm{Vect}(k)$ is of finite type, from which one can see that we do not even have to assume the existence of any adjoint in Orlov's uniqueness statement. Since in that case for any X and Y, smooth projective varieties, every exact functor $F : D^b(X) \longrightarrow D^b(Y)$ admits a left and right adjoint, see [2] and [6] Prop 1.20.

3 Main Results

Based upon standard homological algebra of derived functors in algebraic geometry, we first derive one of our auxiliary result that would be needed later in proving Theorem 3.1.

Lemma 3.1. *Let $f : X \to Y$ be any morphism of smooth projective varieties. Then if Rf_* denotes the right derived functor of the corresponding pushforward, with $\mathcal{O}_\Delta^\vee =: R\mathcal{H}om(\mathcal{O}_\Delta, \mathcal{O}_{X\times X})$ denoting the derived dual of the structure sheaf $\mathcal{O}_\Delta =_{df} Ri_*(\mathcal{O}_X)$ of the diagonal embedding $i : X \to X \times X$, we have*

$$(Rf_*\mathcal{O}_X)^\vee \simeq Rf_*\omega_X[dim(X)] \otimes \omega_Y^*[-dim(Y)]$$

such that ω_X and ω_Y are canonical bundles of X and Y.

Proof. From Grothendieck-Verdier duality [6] appendix C.12, we have the isomorphism

$$Rf_*R\mathcal{H}om(\mathcal{E}, Lf^*\mathcal{F} \otimes \omega_X \otimes f^*\omega_Y^*[\dim(X) - \dim(Y)]) \simeq R\mathcal{H}om(Rf_*\mathcal{E}, \mathcal{F})$$

where Lf^* is the usual left derived pullback of f; $\mathcal{E} \in D^b(X), \mathcal{F} \in D^b(Y)$. If we let $\mathcal{E} = \mathcal{O}_X, \mathcal{F} = \mathcal{O}_Y$, $\dim(X) = m$, $\dim(Y) = n$ we obtain

$$Rf_*R\mathcal{H}om(\mathcal{O}_X, f^*\mathcal{O}_Y \otimes \omega_X \otimes f^*\omega_Y^*[m-n]) \simeq R\mathcal{H}om(Rf_*\mathcal{O}_X, \mathcal{O}_Y)$$

the left hand side of which yields following isomorphisms

$$\begin{aligned} Rf_*R\mathcal{H}om(\mathcal{O}_X, \omega_X \otimes f^*\omega_Y^*[m-n]) &\simeq Rf_*(\omega_X \otimes f^*\omega_Y^*[m-n]) \\ &\simeq Rf_*\omega_X[m] \otimes \omega_Y^*[-n] \end{aligned}$$

the last isomorphism is due to projection formula. □

Now with the help of above machinery, we want to work out the description of one of the most useful object from $D^b(X \times X)$; the derived dual of the diagonal structure sheaf $\mathcal{O}_\Delta$ of the diagonal embedding of X in $X \times X$, i.e. $\mathcal{O}_\Delta^\vee =: R\mathcal{H}om(\mathcal{O}_\Delta, \mathcal{O}_{X\times X})$. The description turns out to be highly non-trivial in $D^b(X \times X)$, the non-derived counterpart of which in $\mathcal{A} = \mathrm{Coh}(X \times X)$ is trivial; i.e. $\mathcal{O}_\Delta^* \simeq 0$ as $\mathcal{O}_\Delta$ is a torsion sheaf rendering the corresponding FM-transform trivial too. Thus $\Phi_{\mathcal{O}_\Delta^*}(\mathcal{F}^\bullet) \simeq 0$ in $D^b(X)$. Also, since $\mathcal{O}_\Delta^* \otimes q^*\omega[n]$

should give us quasi-inverse to the identity transform (from Theorem 2.2−2.3), this would yields quasi-inverse trivial too. The non-trivial description of $\mathcal{O}_\Delta^\vee$ is only stated in [6] 2.3 the proof of which is sketched that uses local computations with Koszul Complexes. We give the detail of our proof as follows

Theorem 3.2. *$\mathcal{O}_\Delta^\vee \simeq \mathcal{O}_\Delta[-m] \otimes p^*\omega_X^* \simeq \mathcal{O}_\Delta[-m] \otimes q^*\omega_X^*$ as an object in $D^b(X \times X)$, such that m=dim(X).*

Proof. Let $\mathcal{E} \in D^b(X)$ then if $\Phi_\mathcal{K} : D^b(X) \to D^b(X)$ is our FM-transform with $p : X \times X \to X$ and $q : X \times X \to X$ as projections onto the second and first factors respectively such that $\mathcal{K} = \mathcal{O}_\Delta[-\dim(X)] \otimes p^*\omega_X^*$ then if $\dim(X) = n$ and $i : X \to X \times X$ the usual diagonal embedding of X, we get

$$\begin{aligned}
\Phi_\mathcal{K}(\mathcal{E}) &= Rp_*(q^*(\mathcal{E}) \otimes \mathcal{O}_\Delta[-n] \otimes p^*\omega_X^*)\\
&= Rp_*(q^*(\mathcal{E}) \otimes Ri_*(\mathcal{O}_X)[-n] \otimes p^*\omega_X^*)\\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*((\mathcal{O}_X)[-n] \otimes Li^*p^*\omega_X^*))\\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*((\mathcal{O}_X)[-n] \otimes L(p \circ i)^*\omega_X^*))\\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*((\mathcal{O}_X)[-n] \otimes \omega_X^*))\\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*(\omega_X^*)[-n]))\\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*((\omega_X)[n])^\vee))\\
&\simeq Rp_*(Ri_*((Li^*q^*(\mathcal{E}) \otimes (\omega_X[n])^\vee)))\\
&\simeq Rp_*(Ri_*(L(q \circ i)^*(\mathcal{E}) \otimes (\omega_X[n])^\vee)))\\
&\simeq Rp_*(Ri_*(\mathcal{E} \otimes (\omega_X[n])^\vee)))\\
&\simeq R(p \circ i)_*(\mathcal{E} \otimes (\omega_X[n])^\vee)\\
&\simeq \mathcal{E} \otimes (\omega_X[n])^\vee\\
&\simeq \mathcal{E} \otimes \omega_X^*[-n] \qquad\qquad (1)
\end{aligned}$$

where Rp_*, Ri_*, Li^* are the right and left derived functors of the corresponding pushforwards and pullbacks of p and i respectively, $\overset{L}{\otimes}$ is the derived tensor product. Since ω_X is a line bundle, tensoring with it no longer requires to be derived, also since both projections, p and q are flat morphisms whenever X is smooth complex projective, thus their pullbacks are also not derived (see, [5], [9], [13] II and III). First and sixth isomorphisms above are due to projection formula; for second and ninth isomorphism, see [16] III.7. The rest is straight forward homological algebra of derived functors in algebraic geometry.

On the other hand if we let our kernel to be $\mathcal{O}_\Delta^\vee$, then we obtain

$$\begin{aligned}
\Phi_{\mathcal{O}_\Delta^\vee}(\mathcal{E}) &= Rp_*(q^*(\mathcal{E}) \overset{L}{\otimes} \mathcal{O}_\Delta^\vee) \\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*\omega_X[n] \otimes \omega^*_{X\times X}[-2n]) \\
&\simeq Rp_*(q^*(\mathcal{E}) \otimes Ri_*\omega_X \otimes p^*\omega_X^* \otimes q^*\omega_X^*[-n]) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*(\omega_X \otimes Li^*p^*\omega_X^*)) \otimes q^*\omega_X^*[-n]) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*(\omega_X \otimes L(p\circ i)^*\omega_X^*)) \otimes q^*\omega_X^*[-n]) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*(\omega_X \otimes \omega_X^*)) \otimes q^*\omega_X^*[-n]) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*(\mathcal{O}_X) \otimes q^*\omega_X^*[-n]) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*((\mathcal{O}_X) \otimes Li^*q^*\omega_X^*[-n])) \\
&\simeq Rp_*((q^*(\mathcal{E}) \otimes Ri_*(\omega_X^*[-n])) \\
&\simeq Rp_*Ri_*(Li^*q^*(\mathcal{E}) \otimes \omega_X^*[-n]) \\
&\simeq R(p\circ i)_*(\mathcal{E} \otimes \omega_X^*[-n]) \\
&\simeq \mathcal{E} \otimes \omega_X^*[-n] \qquad (2)
\end{aligned}$$

where first isomorphism is a consequence of Lemma 3.1; the rest is the same as above. This gives $\forall\, \mathcal{E} \in D^b(X)$

$$\Phi_{\mathcal{K}}(\mathcal{E}) \simeq \Phi_{\mathcal{O}_\Delta^\vee}(\mathcal{E})$$

Now if we can show that both $\Phi_{\mathcal{K}}$ and $\Phi_{\mathcal{O}_\Delta^\vee}$ are fully faithful then since from Theorem 2.1 both already admit left and right adjoints, thus from Theorem 2.3 above that guarantees the uniqueness of the corresponding kernels we would get the required isomorphism

$$\mathcal{O}_\Delta^\vee \simeq \mathcal{O}_\Delta[-\dim(X)] \otimes p^*\omega_X^*$$

However this follows from the following lemma and the fact that shifting by $[-n]$ is an autoequivalence. □

Lemma 3.3. *Let $F : D^b(X) \to D^b(X)$ be the functor determined by $F(\mathcal{E}) = \mathcal{E} \otimes \omega_X^*$. Then F is fully faithful.*

Proof. Since ω_X is a line bundle thus it has an inverse in Picard group of X which is ω_X^*, also double dual of any locally free sheaf is the sheaf itself (see [13] II.5 and II.6), and the fact that tensoring with line bundles are exact, we get the following $\forall \mathcal{E}, \mathcal{F} \in D^b(X)$

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(F(\mathcal{E}), F(\mathcal{F})) = &\mathrm{Hom}_{D^b(X)}(\mathcal{E}\otimes\omega_X^*, \mathcal{F}\otimes\omega_X^*) \\
&\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{E}\otimes\omega_X^*\otimes\omega_X, \mathcal{F}) \\
&\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}).
\end{aligned}$$

□

We have just proved the first isomorphism. The second follows immediately once the role of the projection is switched from p to q.

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Received: May 30, 2018; Published: June 26, 2018