

Identification Some Groups with Special Subgroups

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Abstract

A cover \mathcal{C} for a group G is called a \mathfrak{C}_n -cover whenever \mathcal{C} is an irredundant maximal core-free n -cover for G and in this case we say that G is a \mathfrak{C}_n -group. In this paper we prove that the only \mathfrak{C}_8 -group having at least three maximal subgroups with index 3 and four maximal subgroups with index 2 occurring as a subdirect products of three C_2 s and two primitive groups of degree 3, is (isomorphic to) $C_2 \times C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ for which $D = 1$.

Mathematics Subject Classification: 20F99

Keywords: covering groups by subgroups, maximal subgroups, maximal irredundant cover, core-free intersection

1 Introduction

Let G be a group. A set \mathcal{C} of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G . If the size of \mathcal{C} is n , we call \mathcal{C} an n -cover for the group G . A cover \mathcal{C} for a group G is called irredundant if no proper subset of \mathcal{C} is a cover for G . A cover \mathcal{C} for a group G is called core-free if the intersection $D = \bigcap_{M \in \mathcal{C}} M$ of \mathcal{C} is core-free in G , i.e. $D_G = \bigcap_{g \in G} g^{-1} D g$ is the trivial subgroup of G . A cover \mathcal{C} for a group G is called maximal if all the members of \mathcal{C} are maximal subgroups of G . A cover \mathcal{C} for a group G is called a \mathfrak{C}_n -cover whenever \mathcal{C} is an irredundant maximal core-free n -cover for G and in this case we say that G is a \mathfrak{C}_n -group. A finite group is called semisimple if it has no non-trivial normal abelian subgroups (see p. 86 of [20] for further information on such groups).

Also we use the usual notations ([20]); for example, C_n denotes the cyclic group of order n , $(C_n)^j$ is the direct product of j copies of C_n , the core of a subgroup H of G is denoted by H_G .

In [21], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

Theorem 1.1 (Scorza [21]) *Let $\{A_i : 1 \leq i \leq 3\}$ be an irredundant cover with core-free intersection D for a group G . Then $|D| = 1$ and $G \cong C_2 \times C_2$.*

In [18], Greco characterized all groups having an irredundant 4-cover with core-free intersection. Bryce et al.[16], characterized groups with maximal irredundant 5-cover with core-free intersection.

We characterized groups with maximal irredundant 6-cover with core-free intersection in [1], as below:

Theorem 1.2 *Let G be a group. Then G has a maximal irredundant 6-cover with core-free intersection D if and only if G satisfies one of the following properties.*

1. $|D| = 1$ and $G \cong C_5 \times C_5$;
2. $|D| = 1$ and $G \cong C_3 \times C_3 \times C_3$;
3. $|D| = 1$ and $G \cong Sym_3 \times Sym_3$;
4. $|D| = 1$ and $G \cong (C_3 \times C_3) \rtimes C_2$ with $Z(G) = 1$;
5. $|D| = 2$ and $G \cong (C_3)^3 \rtimes C_2$ with $Z(G) = 1$;
6. $|D| = 1$ and $G \cong C_2 \times C_2 \times Sym_3$ or $G \cong C_2 \times G_0$ where $G_0 = (C_3 \times C_3) \rtimes C_2$ with $Z(G_0) = 1$;
7. $|D| = 1$ and $G \cong C_5 \rtimes C_2$ or $G \cong C_5 \rtimes C_4$ and $Z(G) = 1$;
8. $|D| = 2$ and $G \cong (C_5 \times C_5) \rtimes C_2$ with $Z(G) = 1$;
9. $|D| = 4$ and $G \cong (C_5 \times C_5) \rtimes C_4$ with $Z(G) = 1$.

Abdollahi et al.[3], characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized p -groups with maximal irredundant 8-cover with core-free intersection in [2].

Theorem 1.3 (See [2]). *Let G be a \mathfrak{C}_8 -group. Then G is a p -group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Also we investigated covering groups by subgroups and semisimplity condition in [4], covering semisimple groups by subgroups in [7], C_8 -groups and nilpotency condition in [5], minimal normal subgroups and semisimplity condition in [8], characterization of 5-groups with a maximal irredundant 10-cover in [6], C_8 groups and subdirect product condition in [9] and subdirect product and covering groups by subgroups in [10]. Also we give some results on the number of C_8 -groups for some primitive subgroups in [11]. Also we investigated some results on C_8 -groups by index condition on maximal subgroups in [12].

Further problems of a similar nature, with slightly different aspects, have been studied by many people (see [13, 14, 15, 19, 22, 23]).

Let D denote the intersection of an arbitrary maximal irredundant 8-cover with core-free intersection. In this paper we prove that the only \mathfrak{C}_8 -group having at least three maximal subgroups with index 3 and four maximal subgroups with index 2 occuring as a subdirect products of three C_2 s and two primitive groups of degree 3, is (isomorphic to) $C_2 \times C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ for which $D = 1$.

2 Main Results

To obtain the following results using GAP; we had use several computers mostly for a long time.

Theorem 2.1 *The only \mathfrak{C}_8 -group having at least three maximal subgroups with index 3 and four maximal subgroups with index 2 occuring as a subdirect products of three C_2 s and two primitive groups of degree 3, is (isomorphic to) $C_2 \times C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ for which $D = 1$.*

Proof. We have used the following function written in GAP [17] to prove this theorem. The input of the function is a group G and the output are all irredundant 8-covers with core-free intersection of G three of whose members have index 3 and four of whose members have index 2, and if there is no such cover for G , the output is the empty list.

```
gfe3index34index2:=function(G,k) local
S,M,n,i,j,t,T,Q,R,F2,F3,C1,C2,C3,C4; n:=Size(G);
M:=MaximalSubgroups(G); F2:=Filtered(M,i->Index(G,i)=2);
C1:=Combinations(F2,4); F3:=Filtered(M,i->Index(G,i)=3);
C2:=Combinations(F3,3); C3:=Combinations(F2,5);
C4:=Combinations(F3,4); S=[]; for i in [1..Size(C1)] do for j in
[1..Size(C4)] do if Size(Union(Union(C1[i]),Union(C4[j])))=n then
Add(S,Union(C1[i],C4[j]));fi;od;od; for i in [1..Size(C3)] do for
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j in [1..Size(C2)] do if Size(Union(Union(C3[i]),Union(C2[j])))=n
then Add(S,Union(C2[j],C3[i]));fi;od;od; T:=[]; for i in
[1..Size(S)] do if Size(Core(G,Intersection(S[i])))=1 then
Add(T,S[i]); fi; od; R:=[]; for i in [1..Size(T)] do
Q:=Combinations(T[i],k-1); if (n in
List(Q,i->Size(Union(i))))=false then Add(R,T[i]); fi; od; return
R; end;

```

Acknowledgements. This research was supported by a grant from Payame Noor University.

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Received: October 28, 2017; Published: May 14, 2018