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About Symmetric and Standard Polynomials in Algebras A and $A \otimes E$

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Abstract

In this paper we will show how to relate symmetric and standard polynomials as polynomial identities in algebras A and $A \otimes E$. Here E is Grassmann algebra.

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1 Introduction

Let V be an vector space over K of countable infinite dimension with basis $\{e_1, e_2, ...\}$. The Grassmann algebra E of V is the associative algebra with K-basis consisting of 1 and all products of the form $e_{i_1}e_{i_2}...e_{i_m}$ with $i_1 < i_2... < i_m$, $m \geq 1$ and with multiplication induced by $e_i^2 = 0$ and $e_ie_j + e_je_i = 0$. When $\operatorname{char} K = p = 2$, then obviously E is commutative and hence are not very "interesting" from the PI point of view. Therefore, we restrict our attention the case $p \neq 2$.

We denote by $M_n(E)$ the algebra of $n \times n$ matrices over E. The algebra $M_{a,b}(E)$ is the subalgebra of $M_{a+b}(E)$ that consists of the block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in M_a(E_0)$, $B \in M_{a\times b}(E_1)$, $C \in M_{b\times a}(E_1)$ and $D \in M_b(E_0)$.

If two algebras A and B satisfy the same polynomial identities we say that A is PI-equivalent to B and we denote by $A \sim B$. An important consequence of the Kemer's structure theory is the Tensor Product Theorem (TPT).

Theorem 1 *(TPT). Let* $\mathrm{char} K = 0$. *Then we have* $M_{1,1}(E) \sim E \otimes E$, $M_{a,b}(E) \otimes E \sim M_{a+b}(E)$ *and* $M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)$.

In [4] it has been proved that Kemer's tensor product theorem can not be transposed to fields of positive characteristic. However we have the following multilinear version. Let $P(I)$ be the set of multilinear polynomials in the T -ideal I:

Theorem 2 *([6], theorem 5) Let* $\mathrm{char} K = p \neq 2$. *Then we have* $P(T(M_{1,1}(E))) = P(T(E \otimes E))$, $P(T(M_{a,b}(E) \otimes E)) = P(T(M_{a+b}(E)))$ *and* $P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E)))$.

We denote by

$$w_k(x_1, ..., x_k) = \sum_{\sigma \in S_k} x_{\sigma(1)} ... x_{\sigma(k)}$$

and

$$s_k(x_1, ..., x_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) x_{\sigma(1)} ... x_{\sigma(k)}$$

such as polynomials symmetric and standard, respectively, of degree k where $\epsilon(\sigma)$ is the sign of σ.

2 Relating w_n and s_n

Theorem 3 *Let A be an algebra.*

(a) If the polynomial w_n is an identity for algebra $A \otimes E$ then s_n is an identity for algebra A;

(b) If the polynomial s_n is an identity for algebra $A \otimes E$ then w_n is an identity for algebra A.

Proof. Let $a_1, a_2, \ldots, a_n \in A$ and $e_1, e_2, \ldots, e_n$ elements of the basis de E.

(a) Suppose that s_m is polynomial identity for $A \otimes E$. Then

$$0 = s_m(a_1 \otimes e_1, \ldots, a_m \otimes e_m) = w_m(a_1 \ldots, a_m) \otimes e_1 \ldots e_m$$

as $e_1 \ldots e_m \neq 0$, we have to $w_m(a_1 \ldots, a_m) = 0$ and therefore w_m is a identity polynomial for A.

(b) Suppose that w_m is polynomial identity for $A \otimes E$. Analogously, we get that

$$0 = w_m(a_1 \otimes e_1, \ldots, a_m \otimes e_m) = s_m(a_1, \ldots, a_m) \otimes e_1 \ldots e_m$$

as $e_1 \ldots e_m \neq 0$, we have to $s_m(a_1 \ldots, a_m) = 0$ and therefore s_m is a identity polynomial for A.

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3 An application

How w_p is an polynomial identity for algebra E, using [3] Theorem 3.1, we obtain that w_{np} is a identity polynomial for $M_n(E)$.

In [9] was proved that

Theorem 4 *The polinomial s_m is a identity for $M_{n,n}(E)$ if and only if $n(2p-1) \leq m \leq 2np$.*

Theorem 5 *If the polynomial w_m is a identity for A and $m = np + r$ with $0 \leq r \leq p-1$ then w_{np} is a identity polynomial for A*

Using the above results and the multilienar equivalence in between the algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$ we will show the following result as application of our theorem

Theorem 6 *Let p be a prime number such that $p > n-1$. Then $w_m \in T(M_{2n}E)$ if and only is $m \geq 2np$*

Proof. If $m \geq 2np$ follows from [3] Theorem 3.1 that w_m is a identity polynomial from $M_{2n}(E)$.

Suppose that w_{2np-1} is a identity polynomial from $M_{2n}(E)$. Using the Theorem 5 and the fact that $2np - 1 = (2n-1)p + (p-1)$ we obtain that $w_{(2n-1)p}$ is a identity polynomial from $M_{2n}(E)$. Being $M_{2n}(E)$ and $M_{n,n}(E) \otimes E$ multi-linearly equivalent, we obtain that, $w_{(2n-1)p}$ is a identity polynomial from $M_{n,n}(E) \otimes E$. Using the Theorem 3 obtain that $s_{(2n-1)p}$ is a identity polynomial from $M_{n,n}(E)$ and how $(2n-1)p$ is odd we obtain that $s_{(2n-1)p-1}$ is a identity polynomial from $M_{n,n}(E)$ which contradicts Theorem 4 since $m = (2n-1)p - 1 < 2np - n = [(2n-1)p - 1] + [p - (n-1)]$.

Therefore if w_m is a identity polynomial from $M_{2n}(E)$ then $m \geq 2np$.

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