About Symmetric and Standard Polynomials in Algebras $A$ and $A \otimes E$

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Abstract
In this paper we will show how to relate symmetric and standard polynomials as polynomial identities in algebras $A$ and $A \otimes E$. Here $E$ is Grassmann algebra.

Mathematics Subject Classification: 16R10, 16R20, 16R40, 15A75

Keywords: PI Theory, Standard polynomial, Symmetric polynomial

1 Introduction
Let $V$ be a vector space over $K$ of countable infinite dimension with basis \{${e_1, e_2, ...}$\}. The Grassmann algebra $E$ of $V$ is the associative algebra with $K$-basis consisting of 1 and all products of the form $e_{i_1}e_{i_2}...e_{i_m}$ with $i_1 < i_2 ... < i_m$, $m \geq 1$ and with multiplication induced by $e_i^2 = 0$ and $e_ie_j + e_je_i = 0$. When $\text{char } K = p = 2$, then obviously $E$ is commutative and hence are not very
"interesting" from the PI point of view. Therefore, we restrict our attention the case \( p \neq 2 \).

We denote by \( M_n(E) \) the algebra of \( n \times n \) matrices over \( E \). The algebra \( M_{a,b}(E) \) is the subalgebra of \( M_{a+b}(E) \) that consists of the block matrices \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( A \in M_a(E_0) \), \( B \in M_a \times b(E_1) \), \( C \in M_b \times a(E_1) \) and \( D \in M_b(E_0) \).

If two algebras \( A \) and \( B \) satisfy the same polynomial identities we say that \( A \) is PI-equivalent to \( B \) and we denote by \( A \sim B \). An important consequence of the Kemer’s structure theory is the Tensor Product Theorem (TPT).

**Theorem 1 (TPT).** Let \( \text{char}K = 0 \). Then we have \( M_{1,1}(E) \sim E \otimes E, M_{a,b}(E) \otimes E \sim M_{a+b}(E) \) and \( M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E) \).

In [4] it has been proved that Kemer’s tensor product theorem can not be transposed to fields of positive characteristic. However we have the following multilinear version. Let \( P(I) \) be the set of multilinear polynomials in the \( T \)-ideal \( I \):

**Theorem 2 ([6], theorem 5) Let \( \text{char}K = p \neq 2 \). Then we have \( P(T(M_{1,1}(E))) = P(T(E \otimes E)), P(T(M_{a,b}(E) \otimes E)) = P(T(M_{a+b}(E))) \) and \( P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E))). \)

We denote by

\[
w_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)}
\]

and

\[
s_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)}
\]

such as polynomials symmetric and standard, respectively, of degree \( k \) where \( \epsilon(\sigma) \) is the sign of \( \sigma \).

## 2 Relating \( w_n \) and \( s_n \)

**Theorem 3** Let \( A \) be an algebra.

(a) If the polynomial \( w_n \) is an identity for algebra \( A \otimes E \) then \( s_n \) is an identity for algebra \( A \);

(b) If the polynomial \( s_n \) is an identity for algebra \( A \otimes E \) then \( w_n \) is an identity for algebra \( A \).

**Proof.** Let \( a_1, a_2, \ldots, a_n \in A \) and \( e_1, e_2, \ldots, e_n \) elements of the basis de \( E \).
About symmetric and standard polynomials in algebras $A$ and $A \otimes E$

(a) Suppose that $s_m$ is polynomial identity for $A \otimes E$. Then

$$0 = s_m(a_1 \otimes e_1, \ldots, a_m \otimes e_m) = w_m(a_1, \ldots, a_m) \otimes e_1 \ldots e_m$$

as $e_1 \ldots e_m \neq 0$, we have to $w_m(a_1, \ldots, a_m) = 0$ and therefore $w_m$ is a identity polynomial for $A$.

(b) Suppose that $w_m$ is polynomial identity for $A \otimes E$. Analogously, we get

$$0 = w_m(a_1 \otimes e_1, \ldots, a_m \otimes e_m) = s_m(a_1, \ldots, a_m) \otimes e_1 \ldots e_m$$

as $e_1 \ldots e_m \neq 0$, we have to $s_m(a_1, \ldots, a_m) = 0$ and therefore $s_m$ is a identity polynomial for $A$.

3 An application

How $w_p$ is an polynomial identity for algebra $E$, using [3] Theorem 3.1, we obtain that $w_{np}$ is a identity polynomial for $M_n(E)$.

In [9] was proved that

**Theorem 4** The polinomial $s_m$ is a identity for $M_{n,n}(E)$ if and only if $n(2p - 1) \leq m \leq 2np$.

**Theorem 5** If the polynomial $w_m$ is a identity for $A$ and $m = np + r$ with $0 \leq r \leq p - 1$ then $w_{np}$ is a identity polynomial for $A$

Using the above results and the multilienar equivalence in between the algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$ we will show the following result as application of our theorem

**Theorem 6** Let $p$ be a prime number such that $p > n - 1$. Then $w_m \in T(M_{2n}, E)$ if and only is $m \geq 2np$

**Proof.** If $m \geq 2np$ follows from [3] Theorem 3.1 that $w_m$ is a identity polynomial from $M_{2n}(E)$.

Suppose that $w_{2np - 1}$ is a identity polynomial from $M_{2n}(E)$. Using the Theorem 5 and the fact that $2np - 1 = (2n - 1)p + (p - 1)$ we obtain that $w_{(2n-1)p}$ is a identity polynomial from $M_{2n}(E)$. Being $M_{2n}(E)$ and $M_{n,n}(E) \otimes E$ multi-linearly equivalent, we obtain that, $w_{(2n-1)p}$ is a identity polynomial.
from $M_{n,n}(E) \otimes E$. Using the Theorem 3 obtain that $s_{(2n-1)p}$ is a identity polynomial from $M_{n,n}(E)$ and how $(2n - 1)p$ is odd we obtain that $s_{(2n-1)p-1}$ is a identity polynomial from $M_{n,n}(E)$ which contradicts Theorem 4 since
\[ m = (2n - 1)p - 1 < 2np - n = [(2n - 1)p - 1] + [p - (n - 1)]. \]
Therefore if $w_m$ is a identity polynomial from $M_{2n}(E)$ then $m \geq 2np$.

Acknowledgements. Sérgio M. Alves is partially supported by PROPP/UESC. Proj. 00220.1300.1732.

References


Received: November 8, 2017; Published: January 14, 2018