On Reflexive Rings with Involution

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Abstract

Let \( R \) be a ring with involution \(*\). Central quasi-*-IFP *-rings which generalize that of quasi-*-IFP *-rings are introduced. We introduce also the class of weakly *-reflexive *-rings which is a generalization of *-reflexive *-rings and investigate their properties. Moreover, we give the notion of central *-reflexive *-rings which generalizes weakly *-reflexive *-rings. We show that the class of central *-reduced *-rings extends naturally the class of *-reduced *-rings. Furthermore, some basic extensions for *-reflexive and central *-reflexive *-rings are given. Finally, for a quasi *-Armendariz *-ring \( R \), it is proved that \( R \) is *-reflexive if and only if \( R[x] \) is *-reflexive if and only if \( R[x; x^{-1}] \) is *-reflexive.

Keywords: Quasi-*-IFP; Central quasi-*-IFP; Weakly quasi-*-IFP; *-reduced, Central *-reduced; *-reversible, Central *-reversible; *-reflexive; Central *-reflexive; Weakly *-reflexive *-rings

1 Introduction

Throughout this paper, all rings are associative with identity. A*-ring \( R \) will denote a ring with involution \(*\). The right annihilator of a nonempty subset \( A \) of \( R \) is denoted by \( r_{R}(A) \) and the right *-annihilator of \( A \) is denoted by \( r_{*R}(A) = \{ x \in R \mid Ax = Ax^{*} = 0 \} \). If there is no ambiguity, we omit the subsuffix \( R \). An involution \(*\) is called proper (resp., semiproper) if \( aa^{*} = 0 \) (resp., \( aRa^{*} = 0 \)) implies \( a = 0 \), for every element \( a \in R \). A proper involution

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is clearly semiproper. A *-ideal (self-adjoint) $I$ of $R$ is an ideal closed under involution. A self-adjoint idempotent; $e^2 = e = e^*$, is called projection. A nonzero element $a$ of a *-ring $R$ is called *-zero divisor if $ab = 0 = a^*b$, for some nonzero element $b \in R$ and $R$ is *-domain if it has no nonzero *-zero divisors, by [6]. A *-ring $R$ is said to be Abelian (*-Abelian) if every idempotent (projection) of $R$ is central. A ring $R$ is called semicommutative or has (IFP) if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$ (equivalently $r(a)$ is an ideal of $R$ for all $a \in R$) (see [11]). A *-ring $R$ is said to have *-IFP if for all $a, b \in R$, $ab = 0$ implies $aRb^* = 0$ (equivalently $r(a)$ is a *-ideal of $R$ for all $a \in R$) (see [4]). Following [10], a *-ring $R$ is said to be Baer *-ring if the right annihilator of every nonempty subset of $R$ is generated, as a right ideal, by a projection. In [5], a generalization of Baer *-ring is given which is consistent with the category of involution rings; that is *-Baer *-ring. A *-ring $R$ is said to be *-Baer if the *-right annihilator of every nonempty subset $A$ of $R$ is a principal *-bideal generated by a projection; that is $r^*(A) = eRe$. Recall from [5], an element $a$ of $R$ is said to be *-nilpotent if $(aa^*)^n = 0$ and $a^m = 0$ for some positive integers $n$ and $m$. A *-ring $R$ is called reduced (*-reduced) if it has no nonzero nilpotent (*-nilpotent) elements. From [12], recall a ring $R$ is called central reduced if every nilpotent element of $R$ is central. According to [7], a ring $R$ is called central semicommutative if for all $a, b \in R$, $ab = 0$ implies $arb$ is central. According to [3] and [9], a *-ring $R$ is called *-reversible (reversible) if for all $a, b \in R$, $ab = 0 = ab^*(ab = 0)$ implies $ba = 0$, $R$ has quasi-*-IFP if for all $a, b \in R$, $ab = ab^* = 0$ implies $aRb = 0$ and $R$ is said to be *-reflexive if for all $a, b \in R$, $aRb = 0 = aRb^*$ implies $bRa = 0$. From [2] a *-ring $R$ is called central *-reversible (resp., weakly quasi-*-IFP) if for all $a, b \in R$, $ab = ab^* = 0$ implies $ba$ belongs to the center of $R$ (resp., $arb$ is nilpotent for each $r \in R$). A ring $R$ is called weakly reflexive if $aRb = 0$, implies $bra \in \text{nil}(R)$; the set of all nilpotent elements of $R$, for all $r \in R$ (see [15]). The natural numbers and the integers will be denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. $M_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over the ring $R$, while $T_n(R)$ $(T_{2E}(R))$ will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over $R$. $T_{2E}(R)$ is called the trivial extension of $R$ and is always denoted by $T(R, R)$.

In this paper, we introduce central quasi-*-IFP *-rings which generalize the class of quasi-*-IFP *-rings. Moreover, we give nontrivial generalizations for the class of *-reflexive *-rings; namely, central and weakly *-reflexive *-rings, since, by definition, *-reflexive *-rings are central *-reflexive *-rings. We supply some examples to show that central *-reflexive *-rings need not be *-reflexive and weakly *-reflexive need not be *-reflexive. By the way, we show that the class of central *-reflexive *-rings lies strictly between that of *-reflexive *-rings and weakly *-reflexive *-rings. We show that the class of central *-reduced *-ring is a natural extension of the class of *-reduced *-rings. Moreover, it is also
shown that if $R$ a commutative *-ring, then $T_{nE}(R)$ is weakly *-reflexive. Furthermore, it is shown that for a quasi *-Armendariz *-ring $R$, $R$ is *-reflexive (central *-reflexive) if and only if $R[x]$ is *-reflexive (central *-reflexive) if and only if $R[x; x^{-1}]$ is *-reflexive (central *-reflexive). Finally, we show that the Dorroh extension of a central *-reflexive *-ring is also central *-reflexive and the classical quotient of a *-reflexive Ore *-ring is *-reflexive.

2 Central quasi-*-IFP *-rings

In this section, we introduce the class of central quasi-*-IFP *-rings which generalize quasi-*-IFP *-rings. We also investigate some properties of this new class.

Definition. A *-ring $R$ is called central quasi-*-IFP if for $a, b \in R$, $ab = 0 = ab^*$ implies $arb$ is central (or $arb \in C(R)$), for all $r \in R$. Consequently $arb^*$ is also central.

Obviously, each quasi-*-IFP is central quasi-*-IFP. However, the converse is true when the ring is semiprime as shown in the next result.

Proposition 1. Let $R$ be a semiprime and central quasi-*-IFP *-ring then $R$ is quasi-*-IFP.

Proof. Let $a, b \in R$ with $ab = ab^* = 0$, then $arb$ is central for all $r \in R$, and so $a^2rb$, $arb^2$ are also central. Now, $(aRb)^2 = aRb(aRb)R = (a^2Rb)RbRR = Ra(aRb)bRR = Rab(aRb)RR = 0$, then $aRbR = 0$ implies $aRb = 0$. Thus, $R$ is quasi-*-IFP.

Form [8, Proposition 3.20] and Proposition 1, we have the following corollary.

Corollary 1. If $R$ is a Baer and central quasi-*-IFP *-ring, then $R$ is quasi-*-IFP.

Clearly, each central IFP is central quasi-*-IFP. However, we see that the converse is true when the ring has also *-IFP.

Proposition 2. Let $R$ be a *-ring. If $R$ is central quasi-*-IFP and has *-IFP, then $R$ is central IFP.

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by *-IFP property, and $R$ is central IFP.
Furthermore, one can easily see that the class of central quasi-*-IFP *-ring is closed under direct sums (using changeless involution) and under taking *-subrings.

**Proposition 3.** The class of central quasi-*-IFP *-ring is closed under direct sums and under taking *-subrings.

The following result is a direct consequence of **Proposition 3**.

**Theorem 1.** \( R \) is central quasi-*-IFP if and only if \( R \) is *-Abelian and for any projection \( e \in R \), \( eR \) and \( (1 - e)R \) are both central quasi-*-IFP.

**Proof.** Since *-subrings of central quasi-*-IFP *-rings are central quasi-*-IFP, we prove only that \( R \) is *-Abelian. For any projection \( e \), we have \( e(1 - e) = e(1 - e)^* = 0 \), and \( (1 - e)e = (1 - e)e^* = 0 \). Hence \( er(1 - e), (1 - e)re \in C(R) \) for all \( r \in R \), by hypothesis, which give \( er = ere = re \). Conversely, Let \( ab = ab^* = 0 \). Then \( eab = eab^* = 0 \) and \( (1 - e)ab = (1 - e)ab^* = 0 \), by hypothesis. Hence \( earb, (1 - e)arb \in C(R) \) and \( arb = (1 - e)arb + earb \in C(R) \).

However, the converse of **Theorem 1** is not true as in the following example.

**Example 1.** For the *-ring

\[
R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - d \equiv b \equiv c \equiv 0 \text{(mod } 2), a, b, c, d \in \mathbb{Z} \right\},
\]

with involution * defined as: \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \), the only projections are zero and the identity matrices. The matrix \( A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \) satisfies \( A^2 = AA^* = 0 \) while \( ARA \) is not central.

Next, we show that a central quasi-*-IFP *-ring is weakly quasi-*-IFP.

**Proposition 4.** Every central quasi-*-IFP *-ring is weakly quasi-*-IFP.

**Proof.** Let \( a, b \in R \) with \( ab = ab^* = 0 \), then \( arb \) is central for any \( r \in R \), and so \( a^2rb, arb^2 \) are central. Now, \( (arb)^2 = (arba)rb = (a^2rbr)b = ra^2rb^2 = ra(ARB)b = rab(ARB) = 0 \) and \( arb \) is nilpotent.

It is known that, if \( R \) is a commutative *-ring, then \( T_{nE}(R) \) is not quasi-*-IFP for \( n \geq 4 \) and \( T_{nE}(R) \) is weakly quasi-*-IFP, by [2]. The following example shows that the converse of **Proposition 4** is not always true.
Example 2. The $*$-ring $T_{4E}(\mathbb{Z}_4)$ over integers modulo 4 with involution $*$ defined as
\[
\begin{pmatrix}
  a & a_{12} & a_{13} & a_{14} \\
  0 & a & a_{23} & a_{24} \\
  0 & 0 & a & a_{34} \\
  0 & 0 & 0 & a
\end{pmatrix}^* =
\begin{pmatrix}
  a & a_{34} & a_{24} & a_{14} \\
  0 & a & a_{23} & a_{13} \\
  0 & 0 & a & a_{12} \\
  0 & 0 & 0 & a
\end{pmatrix}
\] is weakly quasi-$*$-IFP.

Further, the matrices $A = \begin{pmatrix} 2 & 2 & 1 & 2 \\
                    0 & 2 & 1 & 0 \\
                    0 & 0 & 2 & 0 \\
                    0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 & 2 \\
                    0 & 2 & 1 & 0 \\
                    0 & 0 & 2 & 0 \\
                    0 & 0 & 0 & 2 \end{pmatrix}$ satisfy $AB = 0 = AB^*$, but for $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\
                    0 & 1 & 1 & 1 \\
                    0 & 0 & 1 & 1 \\
                    0 & 0 & 0 & 1 \end{pmatrix}$, the matrix $ACB = \begin{pmatrix} 0 & 0 & 2 & 2 \\
                     0 & 0 & 0 & 2 \\
                     0 & 0 & 0 & 0 \\
                     0 & 0 & 0 & 0 \end{pmatrix}$ is not central.

From [7, Lemma 2.8] we have the following result.

Proposition 5. Let $R$ be a commutative $*$-ring. Then $T_{2E}(R)$ and $T_{3E}(R)$ are central quasi-$*$-IFP.

The next example demonstrates that the condition $T_{nE}(R), n = 2,3$ in Proposition 5, cannot be weakened to the full matrix $*$-ring $\mathbb{M}_n(R)$, where $n > 1$.

Example 3. $\mathbb{Z}$ is central quasi-$*$-IFP $*$-ring with identical involution, while the $*$-ring $\mathbb{M}_2(\mathbb{Z})$ with adjoint involution $*$ is not central quasi-$*$-IFP. Indeed, the matrices $A = \begin{pmatrix} 0 & 0 \\
                    0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\
                    0 & 0 \end{pmatrix}$ satisfy $AB = 0 = AB^*$, while $ACB = \begin{pmatrix} 0 & 0 & 0 \\
                       0 & 0 & 0 \\
                       0 & 0 & -c \end{pmatrix}$ is not central.

3 Central $*$-reduced $*$-rings

In this section, we introduce and study a class of $*$-ring called central $*$-reduced $*$-rings which is a generalization of the class of $*$-reduced $*$-rings.

Definition. A $*$-ring $R$ is called central $*$-reduced if every $*$-nilpotent element of $R$ is central.

Clearly, a $*$-reduced $*$-ring is central $*$-reduced while the converse is not true as show by the next example.
Example 4. Let $S$ be a commutative $*$-ring and $R = S[x]/\langle x^2 \rangle$. Then $R$ is a commutative $*$-ring and so it is central $*$-reduced. For the polynomial $a = x + \langle x^2 \rangle \neq 0$, we have $a^2 = 0$ and $aa^* = (x + \langle x^2 \rangle)(x + \langle x^2 \rangle)^* = 0$ and so $R$ is not $*$-reduced.

Recall that a $*$-ring $R$ is semiprime if $aRa = 0$ implies $a = 0$ for $a \in R$. In the next, we find a condition under which a central $*$-reduced $*$-ring is $*$-reduced.

**Proposition 6.** Let $R$ be a central $*$-reduced $*$-ring. Then $R$ is $*$-reduced if one of the following conditions is satisfied.

1. $*$ is semiproper.
2. $R$ is semiprime.

**Proof.** Assume $a \in R$ with $a^2 = 0 = aa^*$, then $a$ is central. If the involution $*$ is semiproper (the ring $R$ is semiprime), then $aRa^* = 0$ ($aRa = 0$) and $a = 0$ follows.

**Proposition 7.** Let $R$ be a semiprime central quasi $*$-IFP $*$-ring, then $R$ is $*$-reduced.

**Proof.** Let $a^2 = aa^* = 0$, then $aRa$ is central and $araRa = ara^2raR = 0$, for every $r \in R$. Hence $aRa = 0$ which implies $a = 0$ and $R$ is $*$-reduced.

From [5, Proposition 2.9, Example 4.2, Proposition 4.6 and Theorem 4.8], [3, Proposition 2.5] and **Proposition 7**, we have the following corollaries.

**Corollary 2.** A $*$-ring with proper involution is central $*$-reduced.

**Corollary 3.** Every $*$-Baer $*$-ring is central $*$-reduced.

**Corollary 4.** Every $*$-domain $*$-ring is central $*$-reduced.

**Corollary 5.** Let $R$ be a semiprime $*$-ring having quasi-$*$-IFP, then $R$ is central $*$-reduced.

**Corollary 6.** Let $R$ be a semiprime $*$-ring and central quasi-$*$-IFP, then $R$ is central $*$-reduced.

Note that each central reduced is central $*$-reduced and the converse is true for the rings in the previous Corollaries.

One can easily see that $M_2(R)$ is not central $*$-reduced even if $R$ is commutative, but $T(R, R)$ is central $*$-reduced as shown next.

**Proposition 8.** Let $R$ be a $*$-ring. Then $R$ is commutative if and only if $T(R, R)$, with adjoint involution, is central $*$-reduced.
Proof. If $R$ is commutative, then $T(R, R)$ is commutative and so is central *-reduced. Conversely, for $a, b \in R$, since \( \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in T(R, R) \) is *-nilpotent, it commutes with \( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R, R) \) and $ba = ab$ follows.

From [12, Proposition 2.8, Proposition 2.10 and Proposition 2.17], [7, Proposition 2.10] and Proposition 4, we have the following immediate corollaries.

**Corollary 7.** Every central reduced *-ring is weakly quasi-*-IFP.

**Corollary 8.** Every reduced *-ring is weakly quasi-*-IFP.

**Corollary 9.** A prime central IFP *-ring is central *-reduced.

**Corollary 10.** Every reduced *-ring is *-Abelian.

### 4 *-Reflexive *-rings

In this section, we continue the study of *-reflexive *-rings ([3]) and show that the properties of quasi *-IFP and *-reflexive do not imply each other. More results are also studied.

**Example 5.** For a commutative ring $R$,

\[
T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},
\]

with involution * defined by \( \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \), has quasi-*-IFP ([3]). Moreover, $T_{3E}(R)$ is not *-reflexive, since $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, satisfy $ACB = 0 = ACB^*$, while $BCA = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\neq 0$, for any nonzero $C \in T_{3E}(R)$.

As a consequence from Example 5, $T_{nE}(R)$ is not *-reflexive for $n \geq 4$.

Noting that, the trivial extension of a semiprime *-ring need not be semiprime, by a proof similar to that of [13, Proposition 2.5], we have the following corollary.
Corollary 11. 1. If the trivial extension $T(R, R)$ of $*$-ring $R$, with adjoint involution, is $*$-reflexive then so is $R$.

2. If $R$ is semiprime $*$-ring, then

   a. For $a, b \in R$, $aRbRb = 0 = aRb^* Rb^*$ [resp., $aRaRb = aRaRb^* = 0$], if and only if $aRb = 0 = aRb^*$.

   b. The trivial extension $T(R, R)$, with adjoint involution, is $*$-reflexive.

If $R$ is reduced, by analogous proof to [15, Corollary 2.2.1] we have.

Corollary 12. If $R$ is a reduced $*$-ring, then $T(R, R)$ is $*$-reflexive $*$-ring.

Using changeless involution, the class of $*$-reflexive $*$-ring is closed under direct sums.

Proposition 9. The class of $*$-reflexive $*$-ring is closed under direct sums.

The following is a consequence of Proposition 9.

Proposition 10. For a central projection $e$ of a $*$-ring $R$, $eR$ and $(1 - e)R$ are $*$-reflexive if and only if $R$ is $*$-reflexive.

Proof. Let $arb = aRb^* = 0$ with $a, b \in R$. Then $eaRb = eaRb^* = 0$ and $(1 - e)aRb = (1 - e)aRb^* = 0$, so that $bRea = 0$ and $bR(1 - e)a = 0$, by assumption. Hence, $bRa = bRea + [bR(1 - e)a] = 0$ and $R$ is $*$-reflexive. The converse is trivial by [3, Proposition 4.11].

5 Central $*$-reflexive $*$-rings

In this section, we introduce and study the class central $*$-reflexive $*$-rings, which is a generalization of $*$-reflexive $*$-rings. We start by giving the main definition.

Definition. A $*$-ring $R$ is called central $*$-reflexive if for $a, b \in R$, $aRb = 0 = aRb^*$ implies $bRa$ is central. Consequently $b^*Ra$ is also central.

It is clear that each reflexive (resp., $*$-reflexive) is central reflexive (resp., central $*$-reflexive). The converse is true for semiprime $*$-rings as shown next.

Proposition 11. A semiprime central $*$-reflexive (resp., central reflexive) $*$-ring is $*$-reflexive (resp., reflexive).

Proof. If $arb = arb^* = 0$, then $bra$ is central and consequently $braRbra = 0$, for all $r \in R$. By semiprimeness, we get $bra = 0$ and so $R$ is $*$-reflexive.
Proposition 12. If $R$ is a $*$-Baer and central $*$-reflexive $*$-ring, then $R$ is $*$-reflexive.

Proof. Since $R$ is $*$-Baer then there exists a projection $e \in R$ such that $r_*(a) = eRe$ and $ae = 0$. If $aRb = 0 = aRb^*$, then $b = ebe = eb$ and $bRa = ebRa = bRae = 0$, since $bRa$ is central, so $R$ is $*$-reflexive.

Since each Baer $*$-ring is $*$-Baer, we have the following corollary.

Corollary 13. If $R$ is a Baer and central $*$-reflexive $*$-ring, then $R$ is $*$-reflexive.

However, each central reflexive is central $*$-reflexive. The converse is true when the ring has $*$-IFP as shown in the next result.

Proposition 13. Let $R$ be a $*$-ring. If $R$ is central $*$-reflexive and has $*$-IFP, then $R$ is central reflexive.

Proof. Clearly, since $aRb = 0$, implies $aRb^* = 0$, by $*$-IFP property, and so $R$ is central reflexive.

Clearly, a $*$-reflexive $*$-ring is central $*$-reflexive. The next result shows that $T_{3E}(R)$ in general is central $*$-reflexive and is not $*$-reflexive, by [2, Proposition 1].

Corollary 14. Let $R$ be a commutative $*$-ring, then the ring

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

with involution defined as \( \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* \) is central $*$-reflexive $*$-ring.

From [2, Example 1], we see that Corollary 14 is not true for $n \geq 4$.

Moreover, the class of central $*$-reflexive $*$-rings is closed under direct sums (using changeless involution).

Proposition 14. The class of central $*$-reflexive $*$-rings is closed under direct sums.

The following result is a direct consequence of Proposition 14.

Corollary 15. Let $R$ be a $*$-ring. If $eR$ and $(1 - e)R$ are central $*$-reflexive for some central projection $e$ in $R$, then $R$ is central $*$-reflexive.
The question when a central $\ast$-reflexive $\ast$-ring is central $\ast$-reversible has a partial answer in the following.

**Proposition 15.** A $\ast$-ring $R$ is central $\ast$-reversible if and only if $R$ is central $\ast$-reflexive and has quasi-$\ast$-IFP.

**Proof.** The necessity is obvious. For sufficiency, let $ab = 0 = ab^*$ for some $a, b \in R$. Since $R$ has quasi-$\ast$-IFP, then $aRb = 0 = aRb^*$ and $bRa$ is central, because $R$ is central $\ast$-reflexive. Hence $ba$ is central. $\Box$

From [3, Proposition 2.5] and **Proposition 7** we have the following corollaries.

**Corollary 16.** Let $R$ be a semiprime $\ast$-ring having quasi-$\ast$-IFP, then $R$ is central $\ast$-reflexive.

**Corollary 17.** Let $R$ be a semiprime $\ast$-ring and central quasi-$\ast$-IFP, then $R$ is central $\ast$-reflexive.

## 6 Weakly $\ast$-reflexive $\ast$-rings

Here, we introduce weakly $\ast$-reflexive $\ast$-rings which investigate a weak form of a $\ast$-reflexive $\ast$-rings and generalize them.

**Definition.** A $\ast$-ring $R$ is said to be weakly $\ast$-reflexive if for all $a, b \in R$, $aRb = 0 = aRb^*$ implies $bra$ is nilpotent for all $r \in R$. Consequently $b^*Ra$ is also nil.

Note that each commutative $\ast$-ring is weakly $\ast$-reflexive. Clearly, each weakly reflexive $\ast$-ring is weakly $\ast$-reflexive. The converse is true when the ring has $\ast$-IFP as shown in the following.

**Proposition 16.** Let $R$ be a $\ast$-ring. If $R$ is weakly $\ast$-reflexive and has $\ast$-IFP, then $R$ is weakly reflexive.

**Proof.** Obvious, since $aRb = 0$, implies $aRb^* = 0$, by the $\ast$-IFP property, and $R$ is weakly reflexive. $\Box$

Since every $\ast$-reflexive is weakly $\ast$-reflexive and, by [3, Corollary 4.4], every $\ast$-ring with semiproper involution is $\ast$-reflexive, we have the following.

**Proposition 17.** Every $\ast$-ring with semiproper involution is weakly $\ast$-reflexive.

In the following, we see that a weakly $\ast$-reflexive $\ast$-ring is weakly $\ast$-reversible if it has quasi-$\ast$-IFP.
Proposition 18. A *-ring $R$ is a weakly *-reversible *-ring if $R$ is weakly *-reflexive and has quasi *-IFP.

Proof. If $ab = 0 = ab^*$ then $aRb = 0 = aRb^*$, since $R$ has quasi *-IFP. Hence $bRa$ is nil and $ba$ is nilpotent.

Moreover, the class of weakly *-reflexive *-ring is closed under direct sums (using changeless involution) and undertaking *-subrings.

Proposition 19. The class of weakly *-reflexive *-ring is closed under direct sums and undertaking *-subrings.

Proposition 20. If $R$ a commutative *-ring, then $T_nE(R)$ is a weakly *-reflexive *-ring, with involution * defined by fixing the two diagonals considering the diagonal right / left lower as symmetric ones and interchange the symmetric elements about it; that is

$$
\begin{pmatrix}
a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\
0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\
0 & 0 & a & \cdots & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & a_{(n-1)n} \\
0 & 0 & 0 & \cdots & \cdots & a
\end{pmatrix}^* =
\begin{pmatrix}
a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\
0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{1(n-1)} \\
0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & a_{12} \\
0 & 0 & 0 & \cdots & \cdots & a
\end{pmatrix}
$$

Proof. Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij}) \in T_nE(R)$ with $ACB = ACB^* = 0$, where $1 \leq i \leq j \leq n$, then we have $acb = 0$, where $a, b$ and $c$ are the diagonal elements of $A, B$ and $C$, respectively. Since $R$ is weakly *-reflexive, there exists $k \in \mathbb{N}$ such that $(bca)^k = 0$. Hence $((BCA)^k)^n = 0$ and $T_nE(R)$ is weakly *-reflexive.

The next example shows that there exists a weakly *-reflexive *-ring having quasi *-IFP and is not *-reflexive.

Example 6. Let $R$ be a commutative *-ring. Then the *-ring

$$
T_3E(R) = \left\{ \begin{pmatrix} a & b & c \\
0 & a & d \\
0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},
$$

has quasi *-IFP and is weakly *-reflexive, by Proposition 20. For $A =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
$ and $B =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},$ we have $AB = 0 = AB^*$ while $BRA =
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \neq 0$, so $T_3E(R)$ is not *-reflexive.
We note that if $R$ is a commutative *-ring then the *-ring $T_{nE}(R)$ is not *-reflexive, by Example 6 and is weakly *-reflexive, by Proposition 20. Moreover, it is clear that $T_{4E}(R)$ has not quasi-*-IFP and so $T_{nE}(R)$ has not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_{nE}(R)$ in Proposition 20 cannot be weakened to the full matrix *-ring $M_n(R)$, where $n > 1$.

Example 7. Let $R$ be a weakly *-reflexive *-ring, then $M_2(R)$, with adjoint involution, is not weakly *-reflexive. For $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $ACB = 0 = ACB^*$ and for $C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M_2(R)$, we have $BCA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nil.

By a direct proof, the following indicate that the class of central *-reflexive *-rings lies properly between the classes of *-reflexive and weakly *-reflexive *-rings.

Theorem 2. Let $R$ be a *-ring and consider the following conditions.

1. $R$ is *-reflexive.
2. $R$ is central *-reflexive.
3. $R$ is weakly *-reflexive.

Then (1) $\implies$ (2) $\implies$ (3).

The converse implications of Theorem 2 is not true by Proposition 14 and Example 6.

Proposition 21. If $R$ is weakly quasi-*-IFP then $R$ is weakly *-reflexive.

Proof. Let $arb = 0 = arb^*$ for all $r \in R$, then there exists $m \in \mathbb{N}$ such that $(bra)^m = 0$ and $R$ is weakly *-reflexive. $\square$

From [5, Proposition 2.9, Example 4.2, Proposition 4.6 and Theorem 4.8 ] and Proposition 21 we have the following corollaries.

Corollary 18. Every *-ring with proper involution is weakly *-reflexive.

Corollary 19. Every *-Baer *-ring is weakly *-reflexive.

Corollary 20. Every *-domain *-ring is weakly *-reflexive.

Corollary 21. Every *-ring $R$ having quasi-*-IFP is weakly *-reflexive.

Corollary 22. Every central quasi-*-IFP is weakly *-reflexive.
7 Extensions of central quasi-*-IFP, central *-reduced and *-reflexive *-Rings

In this section, the properties of central quasi-*-IFP, central *-reduced and *-reflexive are shown to be extended from a *-ring to its localization, polynomial, Laurent polynomial, Dorroh extension and from Ore *-ring to its classical *-reflexive are shown to be extended from a *-ring to its localization, polynomial, Laurent polynomial, Dorroh extension and from Ore *-ring to its classical Quotient.

Let \( R \) be a *-ring and \( S \) be a multiplicatively closed subset of \( R \) consisting of nonzero central regular elements, then the localization of \( R \) to \( S \); \( S^{-1}R = \{u^{-1}a | u \in S, a \in R\} \), is a *-ring with involution \( \circ \) defined as:

\[
(u^{-1}a)^{\circ} = u^{-1}a^{*} = u^{-1}a^{*}.
\]

**Proposition 22.** A *-ring \( R \) is central quasi-*-IFP if and only if \( S^{-1}R \) is central quasi-*-IFP.

**Proof.** Let \( \alpha \beta = 0 = \alpha \beta^{\circ} \) with \( \alpha = u^{-1}a \) and \( \beta = v^{-1}b \) where \( a, b \in R \) and \( u, v \in S \). Hence \( \alpha \beta = u^{-1}av^{-1}b = u^{-1}v^{-1}ab = (vu)^{-1}ab = 0 \) and \( \alpha \beta^{\circ} = u^{-1}a(v^{*})^{-1}b^{*} = u^{-1}(v^{*})^{-1}ab^{*} = (v^{*}u)^{-1}ab^{*} = 0 \), since \( S \) is contained in the center of \( R \), and so \( ab = 0 = ab^{*} \). By hypothesis \( abc \) is central for all \( c \in R \) which implies \( \alpha \gamma \beta = u^{-1}aw^{-1}cv^{-1}b = (vwu)^{-1}acb \) is central for every \( w^{-1}c \in S^{-1}R \). The converse is clear. \( \square \)

**Proposition 23.** A *-ring \( R \) is central *-reduced if and only if \( S^{-1}R \) is central *-reduced.

**Proof.** Let \( \alpha^{n} = 0 = (\alpha$\alpha^{\circ}$)^{m} \) with \( \alpha = u^{-1}a \) and \( u \in S \), then we have \( \alpha^{n} = (u^{-1}a)^{n} = (u^{-1})^{n}a^{n} = 0 \) and \( (\alpha \alpha^{\circ})^{m} = (u^{-1}a^{*})^{-1}a^{*})^{m} = (u^{-1}(u^{*})^{-1})^{m}(aa^{*})^{m} = (u^{*}u)^{-1}m(aa^{*})^{m} = 0 \) for some \( n, m \in \mathbb{N} \). Since \( S \) is contained the center of \( R \) then \( (a)$^{n} = 0 = (aa^{*})^{m} \). By hypothesis \( a \) is central, and consequently \( \alpha \) is central. The converse is obvious. \( \square \)

**Proposition 24.** A *-ring \( R \) is *-reflexive if and only if \( S^{-1}R \) is *-reflexive.

**Proof.** Let \( \alpha \gamma \beta = 0 = \alpha \gamma \beta^{\circ} \) with \( \alpha = u^{-1}a \), \( \beta = v^{-1}b \) and \( \gamma = w^{-1}c \) where \( a, b, c \in R \) and \( u, v, w \in S \). Since \( S \) is contained in the central of \( R \), we have \( \alpha \gamma \beta = u^{-1}aw^{-1}cv^{-1}b = (vwu)^{-1}acb = 0 \) and \( \alpha \gamma \beta^{\circ} = u^{-1}aw^{-1}c(v^{*})^{-1}b^{*} = (v^{*}w)^{-1}acb^{*} = 0 \), so \( acb = 0 = acb^{*} \). By hypothesis \( bca = 0 \) which implies \( \beta \gamma \alpha = v^{-1}bw^{-1}cu^{-1}a = (uvw)^{-1}bca = 0 \). The converse is clear. \( \square \)

By a similar proof, we get analogous results for central *-reflexive and weakly *-reflexive *-rings.

**Proposition 25.** A *-ring \( R \) is central *-reflexive if and only if \( S^{-1}R \) is central *-reflexive.
Proposition 26. Let $R$ be a $*$-ring, then $R$ weakly $*$-reflexive if and only if $S^{-1}R$ is weakly $*$-reflexive.

From Propositions 23 and 24 we get the following corollaries.

Corollary 23. If $R$ is a $*$-reduced $*$-ring, then $S^{-1}R$ is central $*$-reduced.

Corollary 24. If $S^{-1}R$ is a $*$-reduced $*$-ring, then $R$ is central $*$-reduced.

Corollary 25. If $R$ is a reduced $*$-ring, then $S^{-1}R$ is central $*$-reduced.

Corollary 26. If $S^{-1}R$ is a reduced $*$-ring, then $R$ is central $*$-reduced.

Corollary 27. If $R$ is a $*$-reversible $*$-ring, then $S^{-1}R$ is weakly $*$-reflexive.

Corollary 28. If $S^{-1}R$ is a $*$-reversible $*$-ring, then $R$ is weakly $*$-reflexive.

Corollary 29. If $R$ is a reflexive $*$-ring, then $S^{-1}R$ is weakly $*$-reflexive.

Corollary 30. If $S^{-1}R$ is a reflexive $*$-ring, then $R$ is weakly $*$-reflexive.

Recall that the $*$-ring of Laurent polynomials in $x$, with coefficients in a $*$-ring $R$, consists of all formal sum $f(x) = \sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and $k, n \in \mathbb{N}$ and with involution $*$ defined as $f^*(x) = \sum_{i=k}^{n} a_i^* x^i$. We denote this ring as usual by $R[x; x^{-1}]$.

Corollary 31. Let $R$ be a $*$-ring, then $R[x]$ is central quasi-$*$-IFP if and only if $R[x; x^{-1}]$ is central quasi-$*$-IFP.

Proof. Let $S = \{1, x, x^2, \cdots \}$, then $S$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. The necessity follows from Proposition 22. The sufficiency is clear.

Corollary 32. For a $*$-ring $R$, $R[x]$ is central $*$-reduced if and only if $R[x; x^{-1}]$ is central $*$-reduced.

Proof. Similar to Corollary 31 using Proposition 23.

Corollary 33. Let $R$ be a $*$-ring, then $R[x]$ is $*$-reflexive if and only if $R[x; x^{-1}]$ is $*$-reflexive.

Proof. Similar to Corollary 31 using Proposition 24.

Corollary 34. Let $R$ be a $*$-ring, then $R[x]$ is central $*$-reflexive if and only if $R[x; x^{-1}]$ is central $*$-reflexive.

Proof. Similar to Corollary 31 using Proposition 25.
Corollary 35. For a *-ring $R$, $R[x]$ is weakly *-reflexive if and only if $R[x; x^{-1}]$ is weakly *-reflexive.


From Corollary 31 and Proposition 4 we get the same results as in [2]. Moreover, by Corollary 32 we have the following Corollaries.

Corollary 36. If $R[x]$ *-reduced, then $R[x; x^{-1}]$ is central *-reduced.

Corollary 37. If $R[x; x^{-1}]$ *-reduced, then $R[x]$ is central *-reduced.

Recall, a *-ring $R$ is *-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j = 0$ for all $i, j$ (Consequently, $a_i b_j^* = 0$).

Theorem 3. Let $R$ be a *-Armendariz *-ring. Then the following statements are equivalent:

1. $R$ is central quasi-*-IFP.
2. $R[x]$ is central quasi-*-IFP.
3. $R[x; x^{-1}]$ is central quasi-*-IFP.

Proof. (1) $\implies$ (2): Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be such that $f(x)g(x) = f(x)g^*(x) = 0$. By hypothesis, $a_i b_j = 0 = a_i b_j^*$ and $a_i \tau b_j \in C(R)$ for all $i, j$ and $\tau \in R$. Hence $f(x)R[x]g(x)$ is central and $R[x]$ is central quasi-*-IFP.

(2) $\implies$ (3): Follows from Corollary 31.

(3) $\implies$ (1): Clear. □

A *-ring $R$ is quasi-*-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0$, then $a_i R b_j = 0$ for all $i, j$ (Consequently, $a_i R b_j^* = 0$).

Theorem 4. Let $R$ be a quasi-*-Armendariz *-ring, then the following statements are equivalent.

1. $R$ is *-reflexive (central *-reflexive).
2. $R[x]$ is *-reflexive (central *-reflexive).
3. $R[x; x^{-1}]$ is *-reflexive (central *-reflexive).
Proof. It suffices to show that (1) \(\implies\) (2). Let \(f(x) = \sum_{i=0}^{m} a_i x^i\), \(g(x) = \sum_{j=0}^{n} b_j x^j\) be such that \(f(x)R[x]g(x) = 0 = f(x)R[x]g^*(x)\). Since \(R\) is quasi-*-Armendariz, we have \(a_i R b_j = 0\) for all \(i, j\). But \(R\) is *-reflexive (central *-reflexive), so \(b_j R a_i = 0\) (\(b_j R a_i\) is central) for all \(i, j\). Consequently \(g(x)R[x]f(x) = 0\) (\(g(x)R[x]f(x)\) is central) and hence \(R[x]\) is *-reflexive (central *-reflexive).

The next corollaries follow from Theorems 3 and 4.

Corollary 38. Let \(R\) be an Armendariz *-ring. Then the following statements are equivalent:

1. \(R\) is central quasi-*-IFP.
2. \(R[x]\) is central quasi-*-IFP.
3. \(R[x; x^{-1}]\) is central quasi-*-IFP.

Corollary 39. Let \(R\) be a quasi-Armendariz *-ring. Then the following statements are equivalent.

1. \(R\) is *-reflexive (central *-reflexive).
2. \(R[x]\) is *-reflexive (central *-reflexive).
3. \(R[x; x^{-1}]\) is *-reflexive (central *-reflexive).

The Dorroh extension \(D(R, Z) = \{(r, n) : r \in R, n \in Z\}\) of a *-ring \(R\) is a ring with componentwise addition and multiplication given by \((r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)\). The involution of \(R\) can be extended naturally to \(D(R, Z)\) as \((r, n)^* = (r^*, n)\) (see [1]).

Proposition 27. A *-ring \(R\) is central *-reduced if and only if its Dorroh extension \(D(R, Z)\) of \(R\) is central *-reduced.

Proof. Let \((r, n) \in D(R, Z)\) with \((r, n)^m = 0\) and \(((r, n)(r^*, n))^k = 0\) for some \(m, k \in \mathbb{N}\). Hence \(n^m = 0 = (n)^{2k}\), so that \(n = 0\), \(r^m = 0\) and \((rr^*)^k = 0\). By hypothesis \(r\) is central and so is \((r, n)\), from which \(D(R, Z)\) is central *-reduced. The converse is clear.

For *-reflexive *-rings, a direct proof gives the following.

Proposition 28. A *-ring \(R\) is *-reflexive if and only if its Dorroh extension \(D(R, Z)\) is also *-reflexive.

Similarly, we have the following.
Proposition 29. A *-ring $R$ is central *-reflexive if and only if its Dorroh extension $D(R, Z)$ of $R$ is central *-reflexive.

Recall that a ring $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. Left Ore is defined similarly and $R$ is Ore ring if it is both right and left Ore. For * rings, right Ore implies left Ore and vice versa. It is a known fact that $R$ is Ore if and only if its classical quotient ring $Q$ of $R$ exists and for *-rings, * can be extended to $Q$ by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see [14, Lamme 4]).

Theorem 5. Let $R$ be an Ore *-ring and $Q$ be its classical quotient *-ring of $R$. If $R$ is *-reflexive, then $Q$ is *-reflexive.

Proof. The proof is similar to that of [15, Proposition 3.8].

From [15, Proposition 3.8] and Theorem 5, we have the following corollaries.

Corollary 40. If $R$ is reflexive *-ring, then $Q$ is *-reflexive (central *-reflexive, weakly *-reflexive).

Corollary 41. If $R$ is *-reflexive *-ring, then $Q$ is central *-reflexive (weakly *-reflexive).

Conclusion

Finally, we can state following implications in the class of rings with involution.

\[ \iff \text{weak. qu. - } \ast - \text{IFP} \]
\[ \downarrow \uparrow \]
\[ \downarrow \text{cent. qu. - } \ast - \text{IFP} \iff \ast - \text{Abelian} \]
\[ \uparrow \downarrow \ast - \text{Abelian} \iff \text{weak. refl} \iff \]
\[ \downarrow \uparrow \]
\[ \downarrow \text{cent. redu.} \iff \text{redu.} \iff \text{refl.} \iff \text{cent. refl.} \iff \]
\[ \uparrow \downarrow \]
\[ \downarrow \text{cent. * - redu.} \iff \ast - \text{redu.} \iff \ast - \text{refl.} \iff \text{cent. * - refl.} \iff \]
\[ \downarrow \uparrow \]
\[ \downarrow \text{weak. } \ast - \text{refl.} \iff \]

References


https://doi.org/10.1556/012.2016.53.2.1338


https://doi.org/10.5666/kmj.2011.51.3.283


https://doi.org/10.1016/s0022-4049(03)00109-9


https://doi.org/10.1080/00927872.2011.554474

https://doi.org/10.1016/0021-8693(69)90053-2

https://doi.org/10.2478/s12175-013-0106-5

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