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On Reflexive Rings with Involution

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Abstract

Let R be a ring with involution *. Central quasi-*-IFP *-rings which generalize that of quasi-*-IFP *-rings are introduced. We introduce also the class of weakly *-reflexive *-rings which is a generalization of *-reflexive *-rings and investigate their properties. Moreover, we give the notion of central *-reflexive *-rings which generalizes weakly *-reflexive *-rings. We show that the class of central *-reduced *-rings extends naturally the class of *-reduced *-rings. Furthermore, some basic extensions for *-reflexive and central *-reflexive *-rings are given. Finally, for a quasi *-Armendariz *-ring R, it is proved that R is *-reflexive if and only if $R[x]$ is *-reflexive if and only if $R[x; x^{-1}]$ is *-reflexive.

Keywords: Quasi-*-IFP; Central quasi-*-IFP; Weakly quasi-*-IFP; *-reduced, Central *-reduced; *-reversible, Central *-reversible; *-reflexive; Central *-reflexive; Weakly *-reflexive *-rings

1 Introduction

Throughout this paper, all rings are associative with identity. A*-ring R will denote a ring with involution *. The right annihilator of a nonempty subset A of R is denoted by $r_R(A)$ and the right *-annihilator of A is denoted by $r_{*R}(A) = \{x \in R \mid Ax = Ax^* = 0\}$. If there is no ambiguity, we omit the subsuffix R. An involution * is called *proper* (resp., *semiproper*) if $aa^* = 0$ (resp., $aRa^* = 0$) implies $a = 0$, for every element $a \in R$. A proper involution

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is clearly semiproper. A *-*ideal* (*self-adjoint*) I of R is an ideal closed under involution. A self adjoint idempotent; $e^2 = e = e^*$, is called projection. A nonzero element a of a *-ring R is called *-*zero divisor* if $ab = 0 = a^*b$, for some nonzero element $b \in R$ and R is *-*domain* if it has no nonzero *-zero divisors, by [6]. A *-ring R is said to be *Abelian* (**-Abelian)* if every idempotent (projection) of R is centeral. A ring R is called *semicommutative* or has *(IFP)* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$ (equivalently $r(a)$ is an ideal of R for all $a \in R$) (see [11]). A *-ring R is said to have *-*IFP* if for all $a, b \in R$, $ab = 0$ implies $aRb^* = 0$ (equivalently $r(a)$ is a *-ideal of R for all $a \in R$) (see [4]). Following [10], a *-ring R is said to be *Baer *-ring* if the right annihilator of every nonempty subset of R is generated, as a right ideal, by a projection. In [5], a generalization of Baer *-ring is given which is consistent with the category of involution rings; that is *-Baer *-ring. A *-ring R is said to be **-Baer* if the *-right annihilator of every nonempty subset A of R is a principal *-biideal generated by a projection; that is $r^*(A) = eRe$. Recall from [5], an element a of R is said to be **-nilpotent* if $(aa^*)^n = 0$ and $a^m = 0$ for some positive integers n and m. A *-ring R is called *reduced* (**-reduced*) if it has no nonzero nilpotent (*-nilpotent) elements. From [12], recall a ring R is called *central reduced* if every nilpotent element of R is central. According to [7], a ring R is called *central semicommutative* if for all $a, b \in R$, $ab = 0$ implies arb is central. According to [3] and [9], a *-ring R is called **-reversible* (*reversible*) if for $a, b \in R$, $ab = 0 = ab^*(ab = 0)$ implies $ba = 0$, R has *quasi-*-IFP* if for all $a, b \in R$, $ab = ab^* = 0$ implies $aRb = 0$ and R is said to be **-reflexive* if $a, b \in R$, $aRb = 0 = aRb^*$ implies $bRa = 0$. From [2] a *-ring R is called *central *-reversible* (resp., *weakly quasi-*-IFP*) if for all $a, b \in R$, $ab = ab^* = 0$ implies ba belongs to the center of R (resp., arb is nilpotent for each $r \in R$). A ring R is called *weakly reflexive* if $aRb = 0$, implies $bra \in \mathrm{nil}(R)$; the set of all nilpotent elements of R, for all $r \in R$ (see [15]). The natural numbers and the integers will be denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over the ring R, while $T_n(R)$ ($T_{nE}(R)$) will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over R. $T_{2E}(R)$ is called the trivial extension of R and is always denoted by $T(R, R)$.

In this paper, we introduce *central quasi-*-IFP* *-rings which generalize the class of *quasi-*-IFP* *-rings. Moreover, we give nontrivial generalizations for the class of *-reflexive *-rings; namely, central and *weakly *-reflexive* *-rings, since, by definition, *-reflexive *-rings are *central *-reflexive* *-rings. We supply some examples to show that central *-reflexive *-rings need not be *- reflexive and weakly *-reflexive need not be *-reflexive. By the way, we show that the class of central *-reflexive *-rings lies strictly between that of *-reflexive *-rings and weakly *-reflexive *-rings. We show that the class of *central *-reduced* *-ring is a natural extension of the class of **-reduced* *-rings. Moreover, it is also shown that if R a commutative *-ring, then $T_{nE}(R)$ is weakly *-reflexive. Furthermore, it is shown that for a quasi *-Armendariz *-ring R, R is *-reflexive (central *-reflexive) if and only if $R[x]$ is *-reflexive (central *-reflexive) if and only if $R[x; x^{-1}]$ is *-reflexive (central *-reflexive). Finally, we show that the Dorroh extension of a central *-reflexive *-ring is also central *-reflexive and the classical quotient of a *-reflexive Ore *-ring is *-reflexive.

2 Central quasi-*-IFP *-rings

In this section, we introduce the class of central quasi-*-IFP *-rings which generalize quasi-*-IFP *-rings. We also investigate some properties of this new class.

Definition. A *-ring R is called *central quasi-*-IFP* if for $a, b \in R$, $ab = 0 = ab^*$ implies arb is central (or $arb \in C(R)$), for all $r \in R$. Consequently arb^* is also central.

Obviously, each quasi-*-IFP is central quasi-*-IFP. However, the converse is true when the ring is semiprime as shown in the next result.

Proposition 1. *Let R be a semiprime and central quasi-*-IFP *-ring then R is quasi-*-IFP.*

Proof. Let $a, b \in R$ with $ab = ab^* = 0$, then arb is central for all $r \in R$, and so a^2rb, arb^2 are also central. Now, $(aRbR)^2 = aRbR(aRb)R = (a^2Rb)RbRR = Ra(aRb)bRR = Rab(aRb)RR = 0$, then $aRbR = 0$ implies $aRb = 0$. Thus, R is quasi-*-IFP. □

Form [8, Proposition 3.20] and **Proposition 1**, we have the following corollary.

Corollary 1. *If R is a Baer and central quasi-*-IFP *-ring, then R is quasi-*-IFP.*

Clearly, each central IFP is central quasi-*-IFP. However, we see that the converse is true when the ring has also *-IFP.

Proposition 2. *Let R be a *-ring. If R is central quasi-*-IFP and has *-IFP, then R is central IFP.*

Proof. Obvious, since $ab = 0$, implies $aRb^* = 0$, by *-IFP property, and R is central IFP. □

Furthermore, One can easily see that the class of central quasi-*-IFP *-ring is closed under direct sums (using changeless involution) and under taking *-subrings.

Proposition 3. *The class of central quasi-*-IFP *-ring is closed under direct sums and under taking *-subrings.*

The following result is a direct consequence of **Proposition 3**.

Theorem 1. *R is central quasi-*-IFP if and only if R is *-Abelian and for any projection $e \in R$, eR and $(1-e)R$ are both central quasi-*-IFP.*

Proof. Since *-subrings of central quasi-*-IFP *-rings are central quasi-*-IFP, we prove only that R is *-Abelian. For any projection e, we have $e(1-e) = e(1-e)^* = 0$, and $(1-e)e = (1-e)e^* = 0$. Hence $er(1-e), (1-e)re \in C(R)$ for all $r \in R$, by hypothesis, which give $er = ere = re$. Conversely, Let $ab = ab^* = 0$. Then $eab = eab^* = 0$ and $(1-e)ab = (1-e)ab^* = 0$, by hypothesis. Hence $earb, (1-e)arb \in C(R)$ and $arb = (1-e)arb + earb \in C(R)$. □

However, the converse of **Theorem 1** is not true as in the following example.

Example 1. For the *-ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - d \equiv b \equiv c \equiv 0(mod\ 2), a, b, c, d \in \mathbb{Z} \right\},$$

with involution * defined as: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, the only projections are zero and the identity matrices. The matrix $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ satisfies $A^2 = AA^* = 0$ while ARA is not central.

Next, we show that a central quasi-*-IFP *-ring is weakly quasi-*-IFP.

Proposition 4. *Every central quasi-*-IFP *-ring is weakly quasi-*-IFP.*

Proof. Let $a, b \in R$ with $ab = ab^* = 0$, then arb is central for any $r \in R$, and so a^2rb, arb^2 are central. Now, $(arb)^2 = (arba)rb = (a^2rbr)b = ra^2rb^2 = ra(arb)b = rab(arb) = 0$ and arb is nilpotent. □

It is known that, if R is a commutative *-ring, then $T_{nE}(R)$ is not quasi-*-IFP for $n \geq 4$ and $T_{nE}(R)$ is weakly quasi-*-IFP, by [2]. The following example shows that the converse of **Proposition 4** is not always true.

Example 2. The *-ring $T_{4E}(\mathbb{Z}_4)$ over integers modulo 4 with involution * defined as $\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & a_{34} & a_{24} & a_{14} \\ 0 & a & a_{23} & a_{13} \\ 0 & 0 & a & a_{12} \\ 0 & 0 & 0 & a \end{pmatrix}$ is weakly quasi-*-IFP.

Further, the matrices $A = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ satisfy $AB = 0 = AB^*$, but for $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the matrix $ACB = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not central.

From [7, Lemma 2.8] we have the following result.

Proposition 5. *Let R be a commutative *-ring. Then $T_{2E}(R)$ and $T_{3E}(R)$ are central quasi-*-IFP.*

The next example demonstrates that the condition $T_{nE}(R), n = 2, 3$ in **Proposition 5**, cannot be weakened to the full matrix *-ring $\mathbb{M}_n(R)$, where $n > 1$.

Example 3. $\mathbb{Z}$ is central quasi-*-IFP *-ring with identical involution, while the *-ring $\mathbb{M}_2(\mathbb{Z})$ with adjoint involution * is not central quasi-*-IFP. Indeed, the matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ satisfy $AB = 0 = AB^*$, while $ACB = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}$ is not central.

3 Central *-reduced *-rings

In this section, we introduce and study a class of *-ring called central *-reduced *-rings which is a generalization of the class of *-reduced *-rings.

Definition. A *-ring R is called *central *-reduced* if every *-nilpotent element of R is central.

Clearly, a *-reduced *-ring is central *-reduced while the converse is not true as show by the next example.

Example 4. Let S be a commutative *-ring and $R = S[x]/ \prec x^2 \succ$. Then R is a commutative *-ring and so it is central *-reduced. For the polynomial $a = x + \prec x^2 \succ \neq 0$, we have $a^2 = 0$ and $aa^* = (x + \prec x^2 \succ)(x + \prec x^2 \succ)^* = 0$ and so R is not *-reduced.

Recall that a *-ring R is semiprime if $aRa = 0$ implies $a = 0$ for $a \in R$. In the next, we find a condition under which a central *-reduced *-ring is *-reduced.

Proposition 6. *Let R be a central *-reduced *-ring. Then R is *-reduced if one of the following conditions is satisfied.*

1. ** is semiproper.*
2. *R is semiprime.*

Proof. Assume $a \in R$ with $a^2 = 0 = aa^*$, then a is central. If the involution * is semiproper (the ring R is semiprime), then $aRa^* = 0$ ($aRa = 0$) and $a = 0$ follows. □

Proposition 7. *Let R be a semiprime central quasi *-IFP *-ring, then R is *-reduced.*

Proof. Let $a^2 = aa^* = 0$, then aRa is central and $araRara = ara^2raR = 0$, for every $r \in R$. Hence $aRa = 0$ which implies $a = 0$ and R is *-reduced. □

From [5, Proposition 2.9, Example 4.2, Proposition 4.6 and Theorem 4.8], [3, Proposition 2.5] and **Proposition 7**, we have the following corollaries.

Corollary 2. *A *-ring with proper involution is central *-reduced.*

Corollary 3. *Every *-Baer *-ring is central *-reduced.*

Corollary 4. *Every *-domain *-ring is central *-reduced.*

Corollary 5. *Let R be a semiprime a *-ring having quasi-*-IFP, then R is central *-reduced.*

Corollary 6. *Let R be a semiprime *-ring and central quasi-*-IFP, then R is central *-reduced.*

Note that each central reduced is central *-reduced and the converse is true for the rings in the previous Corollaries.

One can easily see that $\mathbb{M}_2(R)$ is not central *-reduced even if R is commutative, but $T(R, R)$ is central *-reduced as shown next.

Proposition 8. *Let R be a *-ring. Then R is commutative if and only if $T(R, R)$, with adjoint involution, is central *-reduced.*

Proof. If R is commutative, then $T(R,R)$ is commutative and so is central *-reduced. Conversely, for $a,b \in R$, since $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in T(R,R)$ is *-nilpotent, it commutes with $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R,R)$ and $ba = ab$ follows. □

From [12, Proposition 2.8, Proposition 2.10 and Propsition 2.17], [7, Prposition 2.10] and **Proposition 4**, we have the following immediate corollaries.

Corollary 7. *Every central reduced *-ring is weakly quasi-*-IFP.*

Corollary 8. *Every reduced *-ring is weakly quasi-*-IFP.*

Corollary 9. *A prime central IFP *-ring is central *-reduced.*

Corollary 10. *Every reduced *-ring is *-Abelian.*

4 *-Reflexive *-rings

In this section, we continue the study of *-reflexive *-rings ([3]) and show that the properties of quasi *-IFP and *-reflexive do not imply each other. More results are also studied.

Example 5. For a commutative ring R,

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d \in R \right\},$$

with involution * defined by $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$, has quasi-*-IFP ([3]). Moreover, $T_{3E}(R)$ is not *-reflexive, since $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, satisfy $ACB = 0 = ACB^*$, while $BCA = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, for any nonzero $C \in T_{3E}(R)$.

As a consequence from **Example 5**, $T_{nE}(R)$ is not *-reflexive for $n \geq 4$.

Noting that, the trivial extension of a semiprime *-ring need not be semiprime, by a proof similar to that of [13, Proposition 2.5], we have the following corollary.

Corollary 11. 1. *If the trivial extension $T(R, R)$ of *-ring R, with adjoint involution, is *-reflexive then so is R.*

2. *If R is semiprime *-ring, then*

   a. *For $a, b \in R$, $aRbRb = 0 = aRb^*Rb^*$ [resp., $aRaRb = aRaRb^* = 0$], if and only if $aRb = 0 = aRb^*$.*

   b. *The trivial extension $T(R, R)$, with adjoint involution, is *-reflexive.*

If R is reduced, by analogous proof to [15, Corollary 2.2.1] we have.

Corollary 12. *If R is a reduced *-ring, then $T(R, R)$ is *-reflexive *-ring.*

Using changeless involution, the class of *-reflexive *-ring is closed under direct sums.

Proposition 9. *The class of *-reflexive *-ring is closed under direct sums.*

The following is a consequence of **Proposition 9**.

Proposition 10. *For a central projection e of a *-ring R, eR and $(1-e)R$ are *-reflexive if and only if R is *-reflexive.*

Proof. Let $aRb = aRb^* = 0$ with $a, b \in R$. Then $eaRb = eaRb^* = 0$ and $(1-e)aRb = (1-e)aRb^* = 0$, so that $bRea = 0$ and $bR(1-e)a = 0$, by assumption. Hence, $bRa = bRea + [bR(1-e)a] = 0$ and R is *-reflexive. The converse is trivial by [3, Proposition 4.11]. □

5 Central *-reflexive *-rings

In this section, we introduce and study the class central *-reflexive*-rings, which is a generalization of *-reflexive *-rings. We start by giving the main definition.

Definition. A *-ring R is called *central *-reflexive* if for $a, b \in R$, $aRb = 0 = aRb^*$ implies bRa is central. Consequently b^*Ra is also central.

It is clear that each reflexive (resp., *-reflexive) is central reflexive (resp., central *-reflexive). The converse is true for semiprime *-rings as shown next.

Proposition 11. *A semiprime central *-reflexive (resp., central reflexive) *-ring is *-reflexive (resp., reflexive).*

Proof. If $arb = arb^* = 0$, then bra is central and consequently $braRbra = 0$, for all $r \in R$. By semiprimeness, we get $bra = 0$ and so R is *-reflexive. □

Proposition 12. *If R is a *-Baer and central *-reflexive *-ring, then R is *-reflexive.*

Proof. Since R is *-Baer then there exists a projection $e \in R$ such that $r_*(a) = eRe$ and $ae = 0$. If $aRb = 0 = aRb^*$, then $b = ebe = eb$ and $bRa = ebRa = bRae = 0$, since bRa is central, so R is *-reflexive. □

Since each Baer *-ring is *-Baer, we have the following corollary.

Corollary 13. *If R is a Baer and central *-reflexive *-ring, then R is *-reflexive.*

However, each central reflexive is central *-reflexive. The converse is true when the ring has *-IFP as shown in the next result.

Proposition 13. *Let R be a *-ring. If R is central *-reflexive and has *-IFP, then R is central reflexive.*

Proof. Clearly, since $aRb = 0$, implies $aRb^* = 0$, by *-IFP property, and so R is central reflexive. □

Clearly, a *-reflexive *-ring is central *-reflexive. The next result shows that $T_{3E}(R)$ in general is central *-reflexive and is not *-reflexive, by [2, Proposition 1].

Corollary 14. *Let R be a commutative *-ring, then the ring*

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

with involution defined as $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$ *is central *-reflexive *-ring.*

From [2, Example 1], we see that **Corollary 14** is not true for $n \geq 4$.

Moreover, the class of central *-reflexive *-rings is closed under direct sums (using changeless involution).

Proposition 14. *The class of central *-reflexive *-rings is closed under direct sums.*

The following result is a direct consequence of **Proposition 14**.

Corollary 15. *Let R be a *-ring. If eR and $(1 - e)R$ are central *-reflexive for some central projection e in R, then R is central *-reflexive.*

The question when a central *-reflexive *-ring is central *-reversible has a partial answer in the following.

Proposition 15. *A *-ring R is central *-reversible if and only if R is central *-reflexive and has quasi-*-IFP.*

Proof. The necessity is obvious. For sufficiency, let $ab = 0 = ab^*$ for some $a, b \in R$. Since R has quasi-*-IFP, then $aRb = 0 = aRb^*$ and bRa is central, because R is central *-reflexive. Hence ba is central. □

From [3, Proposition 2.5] and **Proposition 7** we have the following corollaries.

Corollary 16. *Let R be a semiprime *-ring having quasi-*-IFP, then R is central *-reflexive.*

Corollary 17. *Let R be a semiprime *-ring and central quasi-*-IFP, then R is central *-reflexive.*

6 Weakly *-reflexive *-rings

Here, we introduce weakly *-reflexive *-rings which investigate a weak form of a *-reflexive *-rings and generalize them.

Definition. A *-ring R is said to be *weakly *-reflexive* if for all $a, b \in R$, $aRb = 0 = aRb^*$ implies bra is nilpotent for all $r \in R$. Consequently b^*Ra is also nil.

Note that each commutative *-ring is weakly *-reflexive. Clearly, each weakly reflexive *-ring is weakly *-reflexive. The converse is true when the ring has *-IFP as shown in the following.

Proposition 16. *Let R be a *-ring. If R is weakly *-reflexive and has *-IFP, then R is weakly reflexive.*

Proof. Obvious, since $aRb = 0$, implies $aRb^* = 0$, by the *-IFP property, and R is weakly reflexive. □

Since every *-reflexive is weakly *-reflexive and, by [3, Corollary 4.4], every *-ring with semiproper involution is *-reflexive, we have the following.

Proposition 17. *Every *-ring with semproper involution is weakly *-reflexive.*

In the following, we see that a weakly *-reflexive *-ring is weakly *-reversible if it has quasi-*-IFP.

Proposition 18. *A *-ring R is a weakly *-reversible *-ring if R is weakly *-reflexive and has quasi-*-IFP.*

Proof. If $ab = 0 = ab^*$ then $aRb = 0 = aRb^*$, since R has quasi-*-IFP. Hence bRa is nil and ba is nilpotent. $\square$

Moreover, the class of weakly *-reflexive *-ring is closed under direct sums (using changeless involution) and undertaking *-subrings.

Proposition 19. *The class of weakly *-reflexive *-ring is closed under direct sums and undertaking *-subrings.*

Proposition 20. *If R a commutative *-ring, then $T_{nE}(R)$ is a weakly *-reflexive *-ring, with involution * defined by fixing the two diagonals considering the diagonal right / left lower as symmetric ones and interchange the symmetric elements about it; that is*

$$\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & 0 & a & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix}^* = \begin{pmatrix} a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{1(n-1)} \\ 0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{12} \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix}$$

Proof. Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij}) \in T_{nE}(R)$ with $ACB = ACB^* = 0$, where $1 \leq i \leq j \leq n$, then we have $acb = 0$, where a, b and c are the diagonal elements of A, B and C, respectively. Since R is weakly *-reflexive, there exists $k \in \mathbb{N}$ such that $(bca)^k = 0$. Hence $((BCA)^k)^n = 0$ and $T_{nE}(R)$ is weakly *-reflexive. $\square$

The next example shows that there exists a weakly *-reflexive *-ring having quasi *-IFP and is not *-reflexive.

Example 6. Let R be a commutative *-ring. Then the *-ring

$$T_{3E}(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},$$

has quasi-*-IFP and is weakly *-reflexive, by **Proposition 20**. For $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $AB = 0 = AB^*$ while $BRA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, so $T_{3E}(R)$ is not *-reflexive.

We note that if R is a commutative *-ring then the *-ring $T_{nE}(R)$ is not *-reflexive, by **Example 6** and is weakly *-reflexive, by **Proposition 20**. Moreover, it is clear that $T_{4E}(R)$ has not quasi-*-IFP and so $T_{nE}(R)$ has not quasi-*-IFP for $n \geq 4$.

The next example demonstrates that the condition $T_{nE}(R)$ in **Proposition 20**, cannot be weakened to the full matrix *-ring $\mathbb{M}_n(R)$, where $n > 1$.

Example 7. Let R be a weakly *-reflexive *-ring, then $\mathbb{M}_2(R)$, with adjoint involution, is not weakly *-reflexive. For $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $ACB = 0 = ACB^*$ and for $C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(R)$, we have $BCA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nil.

By a direct proof, the following indicate that the class of central *-reflexive *-rings lies properly between the classes of *-reflexive and weakly *-reflexive *-rings.

Theorem 2. *Let R be a *-ring and consider the following conditions.*

1. *R is *-reflexive.*
2. *R is central *-reflexive.*
3. *R is weakly *-reflexive.*
   Then $(1) \Longrightarrow (2) \Longrightarrow (3)$.

The converse implications of **Theorem 2** is not true by **Proposition 14** and **Example 6**.

Proposition 21. *If R is weakly quasi-*-IFP then R is weakly *-reflexive.*

Proof. Let $arb = 0 = arb^*$ for all $r \in R$, then there exists $m \in \mathbb{N}$ such that $(bra)^m = 0$ and R is weakly *-reflexive. □

From [5, Proposition 2.9, Example 4.2, Proposition 4.6 and Theorem 4.8] and **Proposition 21** we have the following corollaries.

Corollary 18. *Every *-ring with proper involution is weakly *-reflexive.*

Corollary 19. *Every *-Baer *-ring is weakly *-reflexive.*

Corollary 20. *Every *-domain *-ring is weakly *-reflexive.*

Corollary 21. *Every *-ring R having quasi-*-IFP is weakly *-reflexive.*

Corollary 22. *Every central quasi-*-IFP is weakly *-reflexive.*

7 Extensions of central quasi-*-IFP, central *-reduced and *-reflexive *-Rings

In this section, the properties of central quasi-*-IFP, central *-reduced and *-reflexive are shown to be extended from a *-ring to its localization, polynomial, Laurent polynomial, Dorroh extension and from Ore *-ring to its classical Quotient.

Let R be a *-ring and S be a multiplicatively closed subset of R consisting of nonzero central regular elements, then the localization of R to S; $S^{-1}R = \{u^{-1}a | u \in S, a \in R\}$, is a *-ring with involution $\diamond$ defined as:

$$(u^{-1}a)^{\diamond} = u^{-1*}a^* = u^{*-1}a^*.$$

Proposition 22. *A *-ring R is central quasi-*-IFP if and only if $S^{-1}R$ is central quasi-*-IFP.*

Proof. Let $\alpha\beta = 0 = \alpha\beta^{\diamond}$ with $\alpha = u^{-1}a$ and $\beta = v^{-1}b$ where $a, b \in R$ and $u, v \in S$. Hence $\alpha\beta = u^{-1}av^{-1}b = u^{-1}v^{-1}ab = (vu)^{-1}ab = 0$ and $\alpha\beta^{\diamond} = u^{-1}a(v^*)^{-1}b^* = u^{-1}(v^*)^{-1}ab^* = (v^*u)^{-1}ab^* = 0$, since S is contained in the center of R, and so $ab = 0 = ab^*$. By hypothesis acb is central for all $c \in R$ which implies $\alpha\gamma\beta = u^{-1}aw^{-1}cv^{-1}b = (vwu)^{-1}acb$ is central for every $w^{-1}c \in S^{-1}R$. The converse is clear. □

Proposition 23. *A *-ring R is central *-reduced if and only if $S^{-1}R$ is central *-reduced.*

Proof. Let $\alpha^n = 0 = (\alpha\alpha^{\diamond})^m$ with $\alpha = u^{-1}a$ where $a \in R$ and $u \in S$, then we have $\alpha^n = (u^{-1}a)^n = (u^{-1})^n a^n = 0$ and $(\alpha\alpha^{\diamond})^m = (u^{-1}a(u^*)^{-1}a^*)^m = (u^{-1}(u^*)^{-1})^m(aa^*)^m = ((u^*u)^{-1})^m(aa^*)^m = 0$ for some $n, m \in \mathbb{N}$. Since S is contained the center of R then $(a)^n = 0 = (aa^*)^m$. By hypothesis a is central, and consequently α is central. The converse is obvious. □

Proposition 24. *A *-ring R is *-reflexive if and only if $S^{-1}R$ is *-reflexive.*

Proof. Let $\alpha\gamma\beta = 0 = \alpha\gamma\beta^{\diamond}$ with $\alpha = u^{-1}a$, $\beta = v^{-1}b$ and $\gamma = w^{-1}c$ where $a, b, c \in R$ and $u, v, w \in S$. Since S is contained in the central of R, we have $\alpha\gamma\beta = u^{-1}aw^{-1}cv^{-1}b = (vwu)^{-1}acb = 0$ and $\alpha\gamma\beta^{\diamond} = u^{-1}aw^{-1}c(v^*)^{-1}b^* = (v^*wu)^{-1}acb^* = 0$ and so $acb = 0 = acb^*$. By hypothesis $bca = 0$ which implies $\beta\gamma\alpha = v^{-1}bw^{-1}cu^{-1}a = (uwv)^{-1}bca = 0$. The converse is clear. □

By a similar proof, we get analogous results for central *-reflexive and weakly *-reflexive *-rings.

Proposition 25. *A *-ring R is central *-reflexive if and only if $S^{-1}R$ is central *-reflexive.*

Proposition 26. *Let R be a *-ring, then R weakly *-reflexive if and only if $S^{-1}R$ is weakly *-reflexive.*

From **Propositions 23 and 24** we get the following corollaries.

Corollary 23. *If R is a *-reduced *-ring, then $S^{-1}R$ is central *-reduced.*

Corollary 24. *If $S^{-1}R$ is a *-reduced *-ring, then R is central *-reduced.*

Corollary 25. *If R is a reduced *-ring, then $S^{-1}R$ is central *-reduced.*

Corollary 26. *If $S^{-1}R$ is a reduced *-ring, then R is central *-reduced.*

Corollary 27. *If R is a *-reversible *-ring, then $S^{-1}R$ is weakly *-reflexive.*

Corollary 28. *If $S^{-1}R$ is a *-reversible *-ring, then R is weakly *-reflexive.*

Corollary 29. *If R is a reflexive *-ring, then $S^{-1}R$ is weakly *-reflexive.*

Corollary 30. *If $S^{-1}R$ is a reflexive *-ring, then R is weakly *-reflexive.*

Recall that the *-ring of Laurent polynomials in x, with coefficients in a *-ring R, consists of all formal sum $f(x) = \sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and $k, n \in \mathbb{N}$ and with involution * defined as $f^*(x) = \sum_{i=k}^{n} a_i^* x^i$. We denote this ring as usual by $R[x; x^{-1}]$.

Corollary 31. *Let R be a *-ring, then $R[x]$ is central quasi-*-IFP if and only if $R[x; x^{-1}]$ is central quasi-*-IFP.*

Proof. Let $S = \{1, x, x^2, \cdots\}$, then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. The necessity follows from **Proposition 22**. The sufficiency is clear. □

Corollary 32. *For a *-ring R, $R[x]$ is central *-reduced if and only if $R[x; x^{-1}]$ is central *-reduced.*

Proof. Similar to **Corollary 31** using **Proposition 23**. □

Corollary 33. *Let R be a *-ring, then $R[x]$ is *-reflexive if and only if $R[x; x^{-1}]$ is *-reflexive.*

Proof. Similar to **Corollary 31** using **Proposition 24**. □

Corollary 34. *Let R be a *-ring, then $R[x]$ is central *-reflexive if and only if $R[x; x^{-1}]$ is central *-reflexive.*

Proof. Similar to **Corollary 31** using **Proposition 25**. □

Corollary 35. *For a *-ring R, $R[x]$ is weakly *-reflexive if and only if $R[x;x^{-1}]$ is weakly *-reflexive.*

Proof. Similar to **Corollary 31** using **Proposition 26**. □

From **Corollary 31** and **Proposition 4** we get the same results as in [2]. Moreover, by **Corollary 32** we have the following Corollaries.

Corollary 36. *If $R[x]$ *-reduced, then $R[x;x^{-1}]$ is central *-reduced.*

Corollary 37. *If $R[x;x^{-1}]$ *-reduced, then $R[x]$ is central *-reduced.*

Recall, a *-ring R is **-Armendariz* if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j = 0$ for all i, j (Consequently $a_i b_j^* = 0$).

Theorem 3. *Let R be a *-Armendariz *-ring. Then the following statements are equivalent:*

1. *R is central quasi-*-IFP.*
2. *$R[x]$ is central quasi-*-IFP.*
3. *$R[x;x^{-1}]$ is central quasi-*-IFP.*

Proof. (1) $\Longrightarrow$ (2)**:** Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be such that $f(x)g(x) = 0 = f(x)g^*(x)$. By hypothesis, $a_i b_j = 0 = a_i b_j^*$ and $a_i r b_j \in C(R)$ for all i, j and $r \in R$. Hence $f(x)R[x]g(x)$ is central and $R[x]$ is central quasi-*-IFP.

(2) $\Longrightarrow$ (3)**:** Follows from **Corollary 31**.

(3) $\Longrightarrow$ (1)**:** Clear.

□

A *-ring R is *quasi-*-Armendariz* if whenever the polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0$, then $a_i R b_j = 0$ for all i, j (Consequently, $a_i R b_j^* = 0$).

Theorem 4. *Let R be a quasi-*-Armendariz *-ring, then the following statements are equivalent.*

1. *R is *-reflexive (central *-reflexive).*
2. *$R[x]$ is *-reflexive (central *-reflexive).*
3. *$R[x;x^{-1}]$ is *-reflexive (central *-reflexive).*

Proof. It suffices to show that (1) $\Longrightarrow$ (2). Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be such that $f(x)R[x]g(x) = 0 = f(x)R[x]g^*(x)$. Since R is quasi-*-Armendariz, we have $a_i R b_j = 0$ for all i, j. But R is *-reflexive (central *-reflexive), so $b_j R a_i = 0$ ($b_j R a_i$ is central) for all i, j. Consequently $g(x)R[x]f(x) = 0$ ($g(x)R[x]f(x)$ is central) and hence $R[x]$ is *-reflexive (central *-reflexive). □

The next corollaries follow from **Theorems 3 and 4**.

Corollary 38. *Let R be an Armendariz *-ring. Then the following statements are equivalent:*

1. *R is central quasi-*-IFP.*
2. *$R[x]$ is central quasi-*-IFP.*
3. *$R[x; x^{-1}]$ is central quasi-*-IFP.*

Corollary 39. *Let R be a quasi-Armendariz *-ring. Then the following statements equivalent.*

1. *R is *-reflexive (central *-reflexive).*
2. *$R[x]$ is *-reflexive (central *-reflexive).*
3. *$R[x; x^{-1}]$ is *-reflexive (central *-reflexive).*

The Dorroh extension $D(R, \mathbb{Z}) = \{(r, n) : r \in R, n \in \mathbb{Z}\}$ of a *-ring R is a ring with componentwise addition and multiplication given by $(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$. The involution of R can be extended naturally to $D(R, \mathbb{Z})$ as $(r, n)^* = (r^*, n)$ (see [1]).

Proposition 27. *A *-ring R is central *-reduced if and only if its Dorroh extension $D(R, \mathbb{Z})$ of R is central *-reduced.*

Proof. Let $(r, n) \in D(R, \mathbb{Z})$ with $(r, n)^m = 0$ and $((r, n)(r^*, n))^k = 0$ for some $m, k \in \mathbb{N}$. Hence $n^m = 0 = (n)^{2k}$, so that $n = 0$, $r^m = 0$ and $(rr^*)^k = 0$. By hypothesis r is central and so is (r, n), from which $D(R, \mathbb{Z})$ is central *-reduced. The converse is clear. □

For *-reflexive *-rings, a direct proof gives the following.

Proposition 28. *A *-ring R is *-reflexive if and only if its Dorroh extension $D(R, \mathbb{Z})$ is also *-reflexive.*

Similarly, we have the following.

Proposition 29. *A *-ring R is central *-reflexive if and only if its Dorroh extension $D(R, \mathbb{Z})$ of R is central *-reflexive.*

Recall that a ring R is called right Ore if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Left Ore is defined similarly and R is *Ore ring* if it is both right and left Ore. For * rings, right Ore implies left Ore and vice versa. It is a known fact that R is Ore if and only if its classical quotient ring Q of R exists and for *-rings, * can be extended to Q by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see[14, Lamme 4]).

Theorem 5. *Let R be an Ore *-ring and Q be its classical quotient *-ring of R. If R is *-reflexive, then Q is *-reflexive.*

Proof. The proof is similar to that of [15, Proposition 3.8]. □

From[15, Proposition 3.8] and **Theorem 5**, we have the following corollaries.

Corollary 40. *If R is reflexive *-ring, then Q is *-reflexive (central *-reflexive, weakly *-reflexive).*

Corollary 41. *If R is *-reflexive *-ring, then Q is central *-reflexive (weakly *-reflexive).*

Conclusion

Finally, we can sate following implications in the class of rings with involution.

$$
\begin{array}{ccccccccc}
\Leftarrow & weak.\ qu. -^* -IFP & & & & & & & \\
\Downarrow & \Uparrow & & & & & & & \\
\Downarrow & cent.\ qu. -^* -IFP & \Longrightarrow & ^* - Abelian & & & & weak.\ refl & \Longrightarrow \\
\Downarrow & \Uparrow & & \Uparrow & & & & \Uparrow & \Downarrow \\
\Downarrow & cent.\ redu. & \Longleftarrow & redu. & \Longrightarrow & refl. & \Longrightarrow & cent.\ refl. & \Downarrow \\
\Downarrow & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \Downarrow \\
\Downarrow & cent.\ ^* - redu. & \Longleftarrow & ^* - redu. & \Longrightarrow & ^* - refl. & \Longrightarrow & cent.\ ^* - refl. & \Downarrow \\
\Downarrow & & & & & & & \Downarrow & \Downarrow \\
\Downarrow & \Longrightarrow & \Longrightarrow & \Longrightarrow & \Longrightarrow & \Longrightarrow & \Longrightarrow & weak.\ ^* - refl. & \Longleftarrow
\end{array}
$$

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