Some Generalizations for *-Armendariz *-Rings

Usama A. Aburawash and Bsmaa M. ELgamudi

Department of Mathematics and Computer Science
Faculty of Science, Alexandria University, Alexandria, Egypt

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Abstract

In this paper we introduce some classes of *-rings which generalize that of *-Armendariz *-rings and investigate their properties. We introduce the concepts of central *-Armendariz, weak *-Armendariz, *-weak *-Armendariz and quasi *-Armendariz. Moreover, sufficient conditions are given for central and quasi *-rings to be *-Armendariz. We give also sufficient conditions for central *-Armendariz, weak *-Armendariz, *-weak *-Armendariz and quasi *-Armendariz *-rings to be central Armendariz, weak Armendariz and quasi Armendariz, respectively. Furthermore, We show that the classes of weak Armendariz and weak *-Armendariz *-rings lie strictly between the classes of Armendariz and *-weak *-Armendariz *-rings. Also, we discuss the relation between weak *-Armendariz and *-IFP *-rings. Finally, we show that the properties of central, weak and quasi are extended to its polynomial *-ring \( R[x] \), Laurent polynomial *-ring \( R[x, x^{-1}] \), localization \( S^{-1}R \) of \( R \) to \( S \), from Ore *-ring to its classical Quotient \( Q \), upper triangular matrices with equal diagonal elements \( T_{nE}(R) \) over a commutative *-ring and the *-corner *-ring \( eRe \).

Keywords: *-Armendariz, reduced, central *-Armendariz, weak *-Armendariz, *-weak *-Armendariz, quasi *-Armendariz, *-Baer, IFP, *-IFP, quasi-*-IFP, *-domain, *-Abelian

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Faculty of Science, Benghazi University, Benghazi, Libya
1 Introduction

By a ring we always mean an associative ring with identity. A ring $R$ is said to be $\ast$-ring if on $R$ there is defined an involution $\ast$. $\ast$-rings are objects of the category of rings with involution with morphisms also preserving involution. Therefore the consistent way of investigating $\ast$-rings is to study them within this category, as done in a series of papers (for instance [5], [3] and [6]). The purpose of this note is to study some classes of $\ast$-rings which generalize that of $\ast$-Armendariz $\ast$-rings within its category.

Throughout this paper, the natural numbers, the integer numbers and the integers modulo $n$ will be denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively, $\mathcal{C}(R)$ denotes the center of a $\ast$-ring $R$, $\text{nil}(R)$ ($\ast$-$\text{nil}(R)$) will denote the set of all nilpotent ($\ast$-nilpotent) elements of $R$ and $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over the ring $R$, while $T_n(R)$ ($T_{nE}(R)$) will denote the $n \times n$ upper triangular matrix ring (with equal diagonal elements) over $R$. Furthermore, for a commutative ring $R$, the involution $\diamond$ defined on $T_{nE}(R)$ for $n > 2$ is given by replacing each entry by its involutive image and fixing the two diagonals considering the diagonal right upper / left lower as symmetric ones and interchanging the symmetric elements about it. For $n = 2$ (trivial extension $T(R,R)$, the involution $\diamond$ is the adjoint involution.

The right annihilator of a nonempty set $A$ of $R$ is denoted by $r_R(A)$ and the $\ast$-right annihilator of $A$ is denoted by $r_*(A) = \{ x \in R \mid Ax = Ax^* = 0 \}$. If there is no ambiguity, we write $r(A)$ and $r_*(A)$ for $r_R(A)$ and $r_*(A)$, respectively. Left and $\ast$-left annihilators ($l_R(A)$ and $l_*(A)$, respectively) are defined similarly. A self adjoint idempotent element $e$ (i.e., $e^* = e = e^2$) is called projection. A $\ast$-ring $R$ is said to be Abelian ($\ast$-Abelian) if every idempotent (projection) of $R$ is central. Recall from [5], an element $a$ of $R$ is said to be $\ast$-nilpotent if $(aa^*)^n = 0 = a^m$, for some positive integers $n$ and $m$. Obviously, a $\ast$-nilpotent element is nilpotent, but the converse is not true [5, Example 2.3]. A $\ast$-ring $R$ is called reduced ($\ast$-reduced) if it has no nonzero nilpotent ($\ast$-nilpotent) elements. An involution $\ast$ is called proper (resp., semiproper) if $aa^* = 0$ (resp., $aRa^* = 0$) implies $a = 0$, for every element $a \in R$. A proper involution is clearly semiproper. A $\ast$-ring $R$ is said to have IFP ($\ast$-IFP) if for all $a, b \in R, ab = 0$ implies $aRb = 0$ ($aRb^* = 0$) ([13], [4]). $R$ is reversible if $ab = 0$ implies $ba = 0$ ([8]). Recall from [3], a $\ast$-ring $R$ is said to have quasi-$\ast$-IFP if for all $a, b \in R, ab = ab^* = 0$ implies $aRb = 0$. Following [11], a $\ast$-ring $R$ is said to be Baer if the right annihilator of every nonempty subset of $R$ is generated, as a right ideal, by a projection. In [5], a generalization of Baer $\ast$-ring is given which is consistent with the category of involution rings; that is $\ast$-Baer $\ast$-ring. A $\ast$-ring $R$ is said to be $\ast$-Baer if the $\ast$-right ($\ast$-left) annihilator of every nonempty subset $A$ of $R$ is a principal $\ast$-biideal generated by a projection; that is $r_*(A) = eRe$. 

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From [15], recall a ring $R$ is Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_{i}x^{i},\ g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_{i}b_{j} = 0$ for each $i, j$. However, reduced rings are Armendariz ([7, Lemma1]). By [2], a $*$-ring $R$ is called $*$-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_{i}x^{i},\ g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)g(x) = f(x)g'(x) = 0$, then $a_{i}b_{j} = 0$ for all $i, j$ (consequently $a_{i}b^{*}_{j} = 0$). According to [17], a ring $R$ is called weak Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_{i}x^{i},\ g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_{i}b_{j} \in \text{nil}(R)$ for each $i, j$. Clearly, Armendariz rings are weak Armendariz while the converse is not true. A ring $R$ is called central Armendariz if for any $f(x) = \sum_{i=0}^{m} a_{i}x^{i},\ g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x],\ f(x)g(x) = 0$ implies that $a_{i}b_{j} \in C(R)$ ([9]). Clearly, Armendariz rings are central Armendariz. From [16], a ring $R$ is called quasi-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_{i}x^{i},\ g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_{i}Rb_{j} = 0$ for each $i, j$. Clearly, Armendariz rings are quasi-Armendariz. Moreover, several examples and counterexamples are included which answers questions that occur naturally in the process of this paper.

## 2 Central $*$-Armendariz $*$-rings

In this section, central $*$-Armendariz $*$-rings are introduced as a generalization of $*$-Armendariz $*$-rings. If $R$ is a $*$-ring, then the involution $*$ can naturally be extended to $R[x]$ as:

$$(f(x))^* = (\sum_{i=0}^{m} a_{i}x^{i})^* = \sum_{i=0}^{m} a_{i}^{*} x^{i} \text{ for all } f(x) \in R[x].$$

**Definition.** A $*$-ring $R$ is called central $*$-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_{i}x^{i}$ and $g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)g(x) = f(x)g^{*}(x) = 0$, then $a_{i}b_{j} \in C(R)$ for all $i, j$ (consequently $a_{i}b^{*}_{j} \in C(R)$).

Clearly, each $*$-Armendariz $*$-ring is central $*$-Armendariz, but the converse is not true as shown by the following example:

**Example 1.** If $n$ is non-square-free number (that is divides at least perfect square), then the $\circ$-ring $\mathbb{T}(\mathbb{Z}_{n}, \mathbb{Z}_{n})$, is commutative and so central $\circ$-Armendariz. Moreover, $\mathbb{T}(\mathbb{Z}_{8}, \mathbb{Z}_{8})$ is not $\circ$-Armendariz, since the polynomial $f(x) = \left(\begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array}\right) + \left(\begin{array}{cc} 4 & 1 \\ 0 & 4 \end{array}\right) x \in \mathbb{T}(\mathbb{Z}_{8}, \mathbb{Z}_{8})$, satisfies $(f(x))^{2} = f(x)f^{*}(x) = 0$, while

$$\left(\begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array}\right) \left(\begin{array}{cc} 4 & -1 \\ 0 & 4 \end{array}\right) = \left(\begin{array}{cc} 0 & -4 \\ 0 & 0 \end{array}\right) \neq 0.$$

The question when a central $*$-Armendariz $*$-ring is $*$-Armendariz has a partial answer in next proposition.
Proposition 1. If $R$ is a *-Baer and central *-Armendariz *-ring, then $R$ is *-Armendariz.

Proof. Let $R$ be a *-Baer *-ring and $f(x)g(x) = f(x)g^*(x) = 0$ with $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$. Then we have the following equations:

\[
\begin{align*}
  a_0b_0 &= 0 & a_0b_0^* &= 0 & \text{(1)} \\
  a_0b_1 + a_1b_0 &= 0 & a_0b_1^* + a_1b_0^* &= 0 & \text{(2)} \\
  a_0b_2 + a_1b_1 + a_2b_0 &= 0 & a_0b_2^* + a_1b_1^* + a_2b_0^* &= 0 & \text{(3)} \\
  & \vdots & & \vdots & \text{(4)} \\
  a_0b_m + \ldots + a_nb_0 &= 0 & a_0b_m^* + \ldots + a_nb_0^* &= 0. & \text{(5)}
\end{align*}
\]

By hypothesis, there exist a projection $e_i \in R$ such that $L_i(a_i^*) = e_iRe_i$ for all $i$. We have $e_0a_0^* = e_0a_0 = 0$ and $b_0 = e_0b_0e_0 = b_0e_0$, since $b_0 \in L_i(a_i^*) = e_iRe_i$. Multiplying equation (2) by $e_0$ from left yields $0 = e_0a_0b_1 + e_0a_1b_0 = a_1b_0$ and $0 = e_0a_0b_1^* + e_0a_1b_0^* = a_1b_0^*$, since $R$ is central *-Armendariz. Hence $a_0b_1 = 0, a_0b_1^* = 0$ and so $b_1 = b_1e_0$. Multiplying equation (3) by $e_0$ from left yields $0 = e_0a_0b_2 + e_0a_1b_1 + e_0a_2b_0$ and $e_0a_0b_2^* + e_0a_1b_1^* + e_0a_2b_0^*$. Hence equation (3) is reduced to:

\[
\begin{align*}
  a_1b_1 + a_2b_0 &= 0 & a_1b_1^* + a_2b_0^* &= 0 & \text{(6)}
\end{align*}
\]

Similarly, multiplying equation (6) by $e_1$ from left, we get $a_2b_0 = 0 = a_2b_0^*$ and so $a_1b_1 = 0 = a_1b_1^*$. Continuing this process, we get $a_i b_j = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ and $R$ is *-Armendariz.

Since each Baer is *-Baer, the condition of *-Baer in the previous proposition can be replaced by Baer. The next example shows that the condition of *-Baer is essential.

Example 2. By Example 1, the *-ring $T(Z_8, Z_8)$ is central *-Armendariz and is not *-Armendariz. Moreover, $T(Z_8, Z_8)$ is not *-Baer, since $r_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Z_8 \\ 0 & 0 \end{pmatrix}$ cannot generated be a projection, since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z_8 & Z_8 \\ Z_8 & Z_8 \end{pmatrix} = \begin{pmatrix} Z_8 & 0 \\ 0 & Z_8 \end{pmatrix} \neq \begin{pmatrix} 0 & Z_8 \\ Z_8 & 0 \end{pmatrix}$

Each central Armendariz *-ring is clearly central *-Armendariz and the converse is true with the following condition.

Proposition 2. If $R$ is central *-Armendariz and $R[x]$ has *-IFP, then $R$ is central Armendariz.
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Proof. Obvious, since \( f(x)g(x) = 0 \), implies \( f(x)R[x]g^*(x) = 0 \), by *-IFP property, and \( R \) is central Armendariz.

One can easily show that the class of central *-Armendariz *-rings is closed under finite direct sums (with changeless involution) and under taking *-subrings.

**Proposition 3.** The class of central *-Armendariz *-rings is closed under finite direct sums and under taking *-subrings.

Since each reduced *-ring is *-Armendariz [2, Proposition 1], we have the following corollary.

**Corollary 1.** Each reduced *-ring is central *-Armendariz.

The converse of the previous corollary is not true by **Example 1**, since \( T(Z_8, Z_8) \) is not reduced because the nonzero matrix \( A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \) satisfies \( A^2 = 0 \).

From [2, Proposition 4 and Corollary 2], if \( R \) is a commutative reduced *-ring, then the \( \circ \)-rings \( T_{3E}(R) \) and \( T(R, R) \) are \( \circ \)-Armendariz and so they are central \( \circ \)-Armendariz. we note that the reduced condition is not essential for \( T(R, R) \) (**Example 1**).

The full matrix \( M_n(R) \) over a *-ring \( R \) with transpose involution is not central *-Armendariz, for \( n \geq 3 \), according to the following examples:

**Example 3.** The *-ring \( M_3(R) \) is not central *-Armendariz, since the polynomials \( f(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( x, g(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), \( x \in M_3(R)[x] \), satisfy \( f(x)g(x) = f(x)g^*(x) = 0 \), while \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) \( \notin \mathcal{C}(M_3(R)) \).

**Example 4.** The *-ring \( M_4(R) \) is not central *-Armendariz, since the polynomials \( f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)
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\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] \( x \in M_4(R)[x] \), satisfy \( f(x)g(x) = f(x)g^*(x) = 0 \), while

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \notin \mathcal{C}(M_4(R)).
\]

The next example gives a *-Abelian *-ring which is not central *-Armendariz.

**Example 5.** \( T_2(\mathbb{Z}_4) \), with adjoint involution * defined by:

\[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}^* =
\begin{pmatrix}
c & -b \\
0 & a
\end{pmatrix}
\] is not central *-Armendariz, since the polynomials

\[
f(x) = \begin{pmatrix}
2 & 2 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} x, \quad g(x) = \begin{pmatrix}
2 & 2 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\]

satisfy \( f(x)g(x) = f(x)g^*(x) = 0 \), while

\[
\begin{pmatrix}
2 & 2 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix} \notin \mathcal{C}(T_2(\mathbb{Z}_4)).
\]

Moreover, the only projections are

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

which are central, so \( T_2(\mathbb{Z}_4) \) is *-Abelian.

We now give necessary and sufficient conditions for a *-Abelian *-ring \( R \) to be central *-Armendariz.

**Proposition 4.** For a *-Abelian *-ring \( R \) the following statements are equivalent:

1. \( R \) is central *-Armendariz.

2. \( eR \) and \( (1 - e)R \) are central *-Armendariz for every projection \( e \) of \( R \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious by **Proposition 3.**

(2) \( \Rightarrow \) (1). Let \( f(x)g(x) = f(x)g^*(x) = 0 \) with

\[
f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x],
\]

then \( ef(x)g(x) = ef(x)eg(x) = ef(x)g^*(x) = ef(x)eg^*(x) = 0 \) and

\[
(1 - e)f(x)g(x) = (1 - e)f(x)(1 - e)g(x) = (1 - e)f(x)g^*(x) = (1 - e)f(x)(1 - e)g^*(x) = 0,
\]

since \( e \) is central. By assumption, we have \( ea_ib_j \) is central in \( eR \) and \( (1 - e)a_ib_j \) is central in \( (1 - e)R \) for all \( 0 \leq i \leq m, 0 \leq j \leq n \). Hence \( a_ib_j = ea_ib_j + (1 - e)a_ib_j \) is central in \( R \) and \( R \) is central *-Armendariz.

\[\square\]

Summarizing the results of this section, we have:

\[
\begin{array}{c}
\text{Reduced} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\text{Armendariz} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\text{central} - \text{Armendariz} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\text{Abelian} \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{*-Armendariz} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\text{central} \text{-*-Armendariz} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\text{*-Abelian}
\end{array}
\]
3 Extensions of central *-Armendariz *-rings

In this section, the property of central *-Armendariz is shown to be extended from the *-ring to its polynomial, localization and Laurent polynomial *-rings.

**Theorem 1.** A *-ring $R$ is central *-Armendariz if and only if $R[x]$ is central *-Armendariz.

*Proof.* Let $R$ be a central *-Armendariz *-ring and $f(y)g(y) = f(y)g^*(y) = 0$ with $f(y) = \sum_{i=0}^{m} f_i y^i, g(y) = \sum_{j=0}^{n} g_j y^j \in R[x][y]$ with $f_i = a_i + a_i x + \cdots + a_{im} x^m, g_j = b_{j0} + b_{j1} x + \cdots + b_{jn} x^n \in R[x]$. Let $t = \deg f_0 + \deg f_1 + \cdots + \deg f_m + \deg g_0 + \deg g_1 + \cdots + \deg g_n$ where the degree is as polynomials in $x$ and the degree of the zero polynomials is taken to be zero. Then $f(x^t) = \sum_{i=0}^{m} f_i x^{ti}, g(x^t) = \sum_{j=0}^{n} g_j x^{tj} \in R[x]$ and the set of coefficients of the $f_i$'s (resp., $g_j$'s) equals the set of coefficients of the $f(x^t)$ (resp., $g(x^t)$). Since $f(y)g(y) = f(y)g^*(y) = 0$ and $x$ commutes with elements of $R$, $f(x^t)g(x^t) = f(x^t)g^*(x^t) = 0$. Since $R$ is central *-Armendariz, $a_{is} b_{jr} \in \mathcal{C}(R)$, where $0 \leq s \leq m, 0 \leq r \leq n$ and $\mathcal{C}(R)$ is closed under addition. Thus $f_i g_j \in \mathcal{C}(R[x])$.

The sufficient condition is clear by **Proposition 3.**

Let $R$ be a *-ring and $S$ be a multiplicatively closed subset of $R$ consisting of nonzero central regular elements, then the localization of $R$ to $S$ is the *-ring $S^{-1}R = \{u^{-1} a | u \in S, a \in R\}$, with involution * defined as:

$$ (u^{-1}a)^* = u^{-1}a^* $$

**Proposition 5.** A *-ring $R$ is central *-Armendariz if and only if $S^{-1}R$ is central *-Armendariz.

*Proof.* By **Proposition 3**, it suffices to prove the necessary condition. Let $R$ be a central *-Armendariz *-ring and $F(x)G(x) = F(x)G^*(x) = 0$ with $F(x) = \sum_{i=0}^{m} \alpha_i x^i, G(x) = \sum_{j=0}^{n} \beta_j x^j \in S^{-1}R[x]$, where $\alpha_i = u^{-1}a_i, \beta_j = v^{-1}b_j$, and $a_i, b_j \in R, u, v \in S$. Hence

$$ F(x)G(x) = (u^{-1}a_0 + u^{-1}a_1 x + \cdots + u^{-1}a_m x^m)(v^{-1}b_0 + v^{-1}b_1 x + \cdots + v^{-1}b_n x^n) $$

$$ = u^{-1}v^{-1}a_0 b_0 + u^{-1}v^{-1}(a_0 b_1 + a_1 b_0)x + \cdots + u^{-1}v^{-1}(a_0 b_n + \cdots + a_m b_0)x^{m+n} $$

$$ = (vu)^{-1}(a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots + (a_0 b_n + \cdots + a_m b_0)x^{m+n}) $$

$$ = (vu)^{-1}f(x)g(x) = 0, $$

$$ F(x)G^*(x) = (u^{-1}a_0 + u^{-1}a_1 x + \cdots + u^{-1}a_m x^m)(v^* - 1 b_0^* + v^* - 1 b_1^* x + \cdots + v^* - 1 b_n^* x^n) $$

$$ = u^{-1}v^* - 1 a_0 b_0^* + u^{-1}v^* - 1 (a_0 b_1^* + a_1 b_0^*)x + \cdots + u^{-1}v^* - 1 (a_0 b_n^* + \cdots + a_m b_0^*)x^{m+n} $$

$$ = (v^* u)^{-1}(a_0 b_0^* + (a_0 b_1^* + a_1 b_0^*)x + \cdots + (a_0 b_n^* + \cdots + a_m b_0^*)x^{m+n}) $$

$$ = (v^* u)^{-1}f(x)g^*(x) = 0, $$

since $S$ is contained in the center of $R$, so $f(x)g(x) = f(x)g^*(x) = 0$. By hypothesis $a_i b_j \in \mathcal{C}(R)$ for all $i, j$ which implies $a_i \beta_j = (uv)^{-1}a_i b_j = (uv)^{-1}r_k a_i b_j = \delta_k a_i \beta_j$ for all $\delta_k = w^{-1}r_k$ with $w \in S, r \in R$. Therefore $S^{-1}R$ is central *-Armendariz. □
From Proposition 5, the following results are straightforward.

**Corollary 2.** If $R$ is a *-Armendariz *-ring, then $S^{-1}R$ is central *-Armendariz.

**Corollary 3.** If $S^{-1}R$ is a *-Armendariz *-ring, then $R$ is central *-Armendariz.

The *-ring of Laurent polynomials in $x$, with coefficients in a *-ring $R$, consists of all formal sums $f(x) = \sum_{i=k}^{m} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and $k, m$ are (possibly negative) integers and with involution $*$ defined as $f^*(x) = \sum_{i=k}^{m} a_i^* x^i$. We denote this ring as usual by $R[x,x^{-1}].$

**Corollary 4.** For a *-ring $R$, $R[x]$ is central *-Armendariz if and only if $R[x,x^{-1}]$ is central *-Armendariz. 

**Proof.** The sufficient condition is obvious by Proposition 3. Clearly $S = \{1,x,x^2,\cdots\}$ is a multiplicatively closed subset of $R[x]$. Since $R[x,x^{-1}] = S^{-1}R[x]$, it follows that $R[x,x^{-1}]$ is central *-Armendariz by Proposition 5.

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**4 weak *-Armendariz *-rings**

In this section, we give another generalization for *-Armendariz *-rings.

**Definition.** A *-ring $R$ is said to be weak *-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^{i}$ and $g(x) = \sum_{j=0}^{n} b_j x^{j} \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_i b_j \in \text{nil}(R)$ for all $i, j$ (consequently $a_i^* b_j \in \text{nil}(R)$).

Each weak Armendariz *-ring is clearly weak *-Armendariz and the converse is true with the following condition.

**Proposition 6.** If $R$ is weak *-Armendariz and $R[x]$ has *-IFP, then $R$ is weak Armendariz.

**Proof.** Let $f(x)g(x) = 0$ for some $f(x), g(x) \in R[x]$. By the *-IFP property $f(x)R[x]g^*(x) = 0$, hence $a_i b_j \in \text{nil}(R)$ and $R$ is weak Armendariz.

One can easily show that the class of weak *-Armendariz *-rings is closed under finite subdirect sums (with changeless involution) and under taking *-subrings.

**Proposition 7.** Let $R$ be a finite subdirect sum of weak *-Armendariz *-rings. Then $R$ is weak *-Armendariz.
Proof. Let $I_k(k = 1, 2, \cdots, l)$ be $*$-ideals of $R$ such that each $R/I_k$ is weak $*$-Armendariz and $\cap_{k=1}^l I_k = 0$. Suppose that two polynomials $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$. Then there exists $p_k \in N$ such that $(\bar{a}_i \bar{b}_j)^p_k = 0$ in $R/I_k$. Thus $(a_ib_j)^p_k \in I_k$. Set $p = p_1 p_2 \cdots p_l$. Then $(a_ib_j)^p \in I_k$ for any $k$. Which implies that $(a_ib_j)^p = 0$. Thus $R$ is weak $*$-Armendariz. □

**Proposition 8.** The class of weak $*$-Armendariz $*$-rings is closed under taking $*$-subrings.

**Proposition 9.** A commutative $*$-ring $R$ is weak $*$-Armendariz if and only if the $\circ$-ring $\mathbb{T}_{nE}(R)$, with adjoint involution $\circ$, is weak $\circ$-Armendariz.

**Proof.** By **Proposition 8**, it suffices to prove the necessary condition. 
Let $R$ be a weak $*$-Armendariz $*$-ring and $f(x)g(x) = f(x)g^*(x) = 0$ with $f(x) = \sum_{i=0}^m A_i x^i, g(x) = \sum_{j=0}^n B_j x^j \in \mathbb{T}_{nE}(R)[x]$, where 

$$A_i = \begin{pmatrix} a_i & a_{12i} & a_{13i} & \cdots & a_{1ni} \\ 0 & a_i & a_{23i} & \cdots & a_{2ni} \\ 0 & 0 & a_i & \cdots & a_{3ni} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_i \end{pmatrix}, \quad B_j = \begin{pmatrix} b_j & b_{12j} & b_{13j} & \cdots & b_{1nj} \\ 0 & b_j & b_{23j} & \cdots & b_{2nj} \\ 0 & 0 & b_j & \cdots & b_{3nj} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_j \end{pmatrix} \in \mathbb{T}_{nE}(R).$$

Hence $(\sum_{i=0}^m a_i x^i)(\sum_{j=0}^n b_j x^j) = 0 = (\sum_{i=0}^m a_i x^i)(\sum_{j=0}^n b_j x^j)$. Since $R$ is weak $*$-Armendariz, there exists $k \in \mathbb{N}$ such that $(a_ib_j)^k = 0$ for all $i, j$

and $(A_iB_j)^k = \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Therefore $(A_iB_j)^{kn} = 0$ and $\mathbb{T}_{nE}(R)$ is weak $\circ$-Armendariz. [denotes an element of $R$]. □

In case of trivial extension $\mathbb{T}(R, R)$ with adjoint involution $\circ$ given by 

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^\circ = \begin{pmatrix} a^* & -b^* \\ 0 & a^* \end{pmatrix},$$

we have the following result.

**Proposition 10.** A commutative $*$-ring $R$ is weak $*$-Armendariz if and only if the $\circ$-ring $\mathbb{T}(R, R)$ is weak $\circ$-Armendariz.

Each $*$-Armendariz $*$-ring is clearly weak $*$-Armendariz, but the converse is not true by the following example.

**Example 6.** Consider the $\circ$-ring $\mathbb{T}_{4E}(R)$, with adjoint involution $\circ$. $\mathbb{T}_{4E}(R)$ is weak $\circ$-Armendariz by **Proposition 9**. Moreover, $\mathbb{T}_{4E}(R)$ is not $\circ$-Armendariz,
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since the polynomials $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} x, g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$ satisfy $f(x)g(x) = f(x)g^*(x) = 0,$ while $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \neq 0.$

Since each reduced *-ring is *-Armendariz [2, Proposition 1], we have the following corollary.

**Corollary 5.** Each reduced *-ring is weak *-Armendariz.

The converse of the previous corollary is not true since, by Proposition 10, the ∗-ring $T(R, R)$ is weak ∗-Armendariz and $T(Z_8, Z_8)$ is not reduced.

A necessary and sufficient conditions for a *-Abelian *-ring $R$ to be weak *-Armendariz is now given.

**Proposition 11.** For a *-Abelian *-ring $R$ the following statements are equivalent:

1. $R$ is weak *-Armendariz.
2. $eR$ and $(1 - e)R$ are weak *-Armendariz.

**Proof.** (1) ⇒ (2) is obvious by Proposition 8.

(2) ⇒ (3). Let $f(x)g(x) = f(x)g^*(x) = 0$ with $f(x) = \sum_{i=0}^{m} q_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x], then ef(x)g(x) = ef(x)eg(x) = ef(x)g^*(x) = ef(x)eg^*(x) = 0$ and $(1 - e)f(x)g(x) = (1 - e)f(x)(1 - e)g(x) = (1 - e)f(x)g^*(x) = (1 - e)f(x)(1 - e)g^*(x) = 0,$ since $e$ is central. By assumption, there exists $p$ and $q$ such that $(ea_ib_j)^p = 0$ and $(1 - e)a_ib_j)^q = 0$ for all $0 \leq i \leq m, 0 \leq j \leq n.$ Let $k = \max\{p, q\}.$ Then $e(a_ib_j)^k = 0$ and $(1 - e)(a_ib_j)^k = 0.$ Hence $(a_ib_j)^k = 0$ and $R$ is weak *-Armendariz.

5 Extensions of weak *-Armendariz *-rings

Note first that if a *-ring $R$ has IFP, then $R[x]$ may not have IFP (see, [13, Example 2]). For weak *-Armendariz we have the following results.

**Proposition 12.** If $R$ has *-IFP, then $R[x]$ is weak *-Armendariz.
Proof. Clear from [4] and [17, Theorem 3.8], since $R$ has *-IFP (has also IFP), then $R[x]$ is weak Armendariz and so weak *-Armendariz.

Recall that a ring $R$ is called right Ore if given $a,b \in R$ with $b$ regular there exist $a_1,b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. Left Ore is defined similarly and $R$ is Ore ring if it is both right and left Ore. For *-rings, right Ore implies left Ore and vice versa. It is a known fact that $R$ is Ore if and only if its classical quotient ring $Q$ of $R$ exists and for *-rings, * can be extended to $Q$ by $(a^{-1}b)^* = b^*(a^*)^{-1}$ (see[14, Lamme 4]).

Proposition 13. Let $R$ be an Ore *-ring and $Q$ be its classical quotient *-ring. If $R$ has *-IFP, then $Q$ is weak *-Armendariz.

Proof. Clear from [4] and [17, Proposition 3.10], since $R$ has *-IFP (has also IFP), then $Q$ is weak Armendariz and so weak *-Armendariz.

From [2, Theorem 1, Proposition 7 and Corollary 5], we have the following:

Corollary 6. Let $R$ be a *-Armendariz *-ring. Then $R[x], S^{-1}R, R[x, x^{-1}]$ are all weak *-Armendariz.

Corollary 7. Let $S^{-1}R$ be a *-Armendariz *-ring, then $R$ is weak *-Armendariz.

6 *-Weak *-Armendariz *-rings

In this section, we introduction further generalization for *-Armendariz; that is *-weak *-Armendariz *-rings. This class is a proper subclass of the class of weak *-Armendariz *-rings.

Definition. A *-ring $R$ is said to be *-weak *-Armendariz if whenever the polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = f(x)g^*(x) = 0$, then $a_ib_j \in *-nil(R)$ for all $i, j$ (consequently $a_i b_j^* \in *-nil(R)$).

Each *-weak *-Armendariz *-ring is obviously weak *-Armendariz, while there is no clear connection between *-weak *-Armendariz and weak Armendariz. However, *-weak *-Armendariz $R$ is weak Armendariz if $R[x]$ has *-IFP.

Proposition 14. If $R$ is *-weak *-Armendariz and $R[x]$ has *-IFP, then $R$ is weak Armendariz.

Proof. Obvious, since $f(x)g(x) = 0$, implies $f(x)R[x]g^*(x) = 0$, by *-IFP property, and $R$ is weak Armendariz.

One can easily show that the class of *-weak *-Armendariz *-rings is closed under finite subdirect sums (with changeless involution) and under taking *-subrings.
Proposition 15. Let \( R \) be a finite subdirect sum of \(*\)-weak \(*\)-Armendariz \(*\)-rings. Then \( R \) is \(*\)-weak \(*\)-Armendariz.

Proof. The proof is similar to that of Proposition 7. \( \square \)

Proposition 16. The class of \(*\)-weak \(*\)-Armendariz \(*\)-rings is closed under taking \(*\)-subrings.

By a similar proof to that of Proposition 9 and using Proposition 16, we have analogous results for \(*\)-weak \(*\)-Armendariz \(*\)-ring.

Proposition 17. A commutative \(*\)-ring \( R \) is \(*\)-weak \(*\)-Armendariz if and only if the \( \circ\)-ring \( \Theta_n E(R) \), with adjoint involution \( \circ \), is \( \circ\)-weak \( \circ\)-Armendariz.

Proposition 18. A commutative \(*\)-ring \( R \) is \(*\)-weak \(*\)-Armendariz if and only if the \( \circ\)-ring \( \Theta(R,R) \), with adjoint involution \( \circ \), is \( \circ\)-weak \( \circ\)-Armendariz.

Since each reduced \( \circ\)-ring is \( \circ\)-Armendariz [2, Proposition 1], we have:

Corollary 8. Each reduced \( \circ\)-ring is \(*\)-weak \(*\)-Armendariz.

The converse of the previous corollary is not true since, by Proposition 18, the \( \circ\)-ring \( \Theta(R,R) \) is \( \circ\)-weak \( \circ\)-Armendariz and \( \Theta(Z_8,Z_8) \) is not reduced.

From [2, Proposition 4 and Example 5], if \( R \) is a commutative reduced \( \circ\)-ring, then the \( \circ\)-ring \( \Theta_3 E(R) \) is \( \circ\)-Armendariz while \( \Theta_n E(R) \) is not \( \circ\)-Armendariz when \( n \geq 4 \). Meanwhile, by Propositions 9 and 17, we have the following results.

Corollary 9. If \( R \) is a commutative \(*\)-weak \(*\)-Armendariz \(*\)-ring, then the \( \circ\)-ring \( \Theta_n E(R) \) is weak \( \circ\)-Armendariz.

Corollary 10. If \( R \) is a commutative reduced \( \circ\)-ring, then the \( \circ\)-ring \( \Theta_n E(R) \) is \( \circ\)-weak \( \circ\)-Armendariz.

Examples 3 and 4 declare that the full matrix \(*\)-ring \( M_n(R) \), with transpose involution is not \(*\)-weak \(*\)-Armendariz, for \( n \geq 3 \), since

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\notin \ast-nil(M_3(R)) \text{ and } \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\notin \ast-nil(M_4(R)).
\]

Clearly, each \( \circ\)-Armendariz \(*\)-ring is \(*\)-weak \(*\)-Armendariz, but the converse is not true by Example 6, since the \( \circ\)-ring \( \Theta_{1E}(R) \) is not \( \circ\)-Armendariz and by Proposition 17, \( \Theta_{1E}(R) \) is \( \circ\)-weak \( \circ\)-Armendariz.

By a proof similar to that of Proposition 11, necessary and sufficient conditions for a \(*\)-Abelian \(*\)-ring to be \(*\)-weak \(*\)-Armendariz is now given.
Proposition 19. For a $\ast$-Abelian $\ast$-ring $R$ the following statements are equivalent:

1. $R$ is $\ast$-weak $\ast$-Armendariz.
2. $eR$ and $(1 - e)R$ are $\ast$-weak $\ast$-Armendariz.

From [2, Proposition 4], Proposition 11 and Proposition 19, we have:

Corollary 11. Let $R$ be a $\ast$-Abelian $\ast$-ring. Consider the following conditions:

1. $R$ is $\ast$-Armendariz.
2. $eR$ and $(1 - e)R$ are $\ast$-Armendariz.
3. $eR$ and $(1 - e)R$ are $\ast$-weak $\ast$-Armendariz.
4. $R$ is $\ast$-weak $\ast$-Armendariz.
5. $R$ is weak $\ast$-Armendariz.
6. $eR$ and $(1 - e)R$ are weak $\ast$-Armendariz.

Then $1 \iff 2 \Rightarrow 3 \iff 4 \Rightarrow 5 \iff 6$

7 Weak $\ast$-Armendariz $\ast$-rings and $\ast$-rings With $\ast$-IFP

In this section, the relation between weak $\ast$-Armendariz $\ast$-rings and $\ast$-rings having $\ast$-IFP is studied.

From [4] and [17, Corollary 3.4], the following results are straightforward.

Corollary 12. Every $\ast$-ring with IFP is weak $\ast$-Armendariz.

Corollary 13. Every $\ast$-ring with $\ast$-IFP is weak $\ast$-Armendariz.

By [12, Lemma 1.4], each reversible $\ast$-ring has IFP. Hence, from Corollary 12, it follows that all reversible $\ast$-ring is weak $\ast$-Armendariz. However, the converse is not true by the next example.

Example 7. By Proposition 9, the $\diamond$-ring $T_{4E}(R)$ is weak $\diamond$-Armendariz. Moreover, $T_{4E}(R)$ has not IFP, since the matrices

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

satisfy $AB = 0$, while $ACB =$
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\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & a_{23} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\neq 0.
\]

Now we see that there exists a weak Armendariz *-ring which has not *-IFP.

**Example 8.** By [17, Proposition 2.2], the ring \( T_2(R) \) is weak Armendariz. Moreover, \( T_2(R) \) with adjoint involution * has not *-IFP, since the matrices

\[
A = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 \\
0 & 1 \\
\end{pmatrix},
\text{satisfy } AB = 0 \text{ while } ACB^* = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}\begin{pmatrix}
a & b \\
0 & c \\
\end{pmatrix}\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
a & 0 \\
0 & 0 \\
\end{pmatrix} \neq 0.
\]

The summarize of the results of the previous three sections are as follows:

\[
\begin{array}{c}
\text{reversible } \Rightarrow \text{IFP} \\
\text{Armendariz } \Rightarrow \text{weak - Armendariz} \\
\text{weak - Armendariz } \Rightarrow \text{weak - * - Armendariz} \\
\text{weak - * - Armendariz } \Rightarrow \text{weak - * - Armendariz} \\
\end{array}
\]

8 Quasi-*-Armendariz *-rings

Here, we give another generalization for *-Armendariz *-rings; that is quasi-*-Armendariz.

**Definition.** A *-ring \( R \) is called quasi-*-Armendariz if whenever the polynomials \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \) satisfy \( f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0 \), then \( a_i Rb_j = 0 \) for all \( i,j \) (consequently \( a_i Rb_j^* = 0 \)).

By a proof similar to [10, Lemma 2.1], we get immediately.

**Lemma 1.** Let \( f(x), g(x) \) be two elements of \( R[x] \). Then \( f(x)Rg(x) = f(x)Rg^*(x) = 0 \) if and only if \( f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0 \).

By **Lemma 1**, *-Armendariz *-rings are quasi-*-Armendariz, but the converse is not true by the next example:

**Example 9.** Let \( R \) be a quasi-Armendariz *-ring and \( S \) be a subring of \( M_n(R) \) such that \( e_i S e_j \subseteq S \) for all \( i, j \in \{1, 2, ..., n\} \), then \( S \) is quasi-Armendariz [10, Theorem 3.12] and so quasi-*-Armendariz. Moreover, by **Example 3**, \( M_3(R) \) is not *-Armendariz.
The question when a quasi *-Armendariz *-ring is *-Armendariz has a partial answer.

**Proposition 20.** Let $R$ be a quasi-*-Armendariz *-ring and $R[x]$ has quasi-*-IFP, then $R$ is *-Armendariz.

*Proof.* Since $f(x)g(x) = f(x)g^*(x) = 0$, implies $f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0$, by quasi *-IFP property, hence $R$ is *-Armendariz. □

Each quasi Armendariz *-ring is clearly quasi *-Armendariz and the converse is true with the following condition.

**Proposition 21.** If $R$ is quasi *-Armendariz and $R[x]$ has *-IFP, then $R$ is quasi Armendariz.

*Proof.* Obvious, since $f(x)R[x]g(x) = 0$, implies $f(x)R[x]g^*(x) = 0$, by *-IFP property, and $R$ is quasi Armendariz. □

Since each *-ring having *-IFP has quasi-*-IFP, the condition of quasi-*-IFP in **Proposition 20** can be replaced by *-IFP.

One can easily show that the class of quasi-*-Armendariz *-rings is closed under finite subdirect sums (with changeless involution), with proof similar to that of **Proposition 7**, and under taking *-subrings.

**Proposition 22.** Let $R$ be a finite subdirect sum of quasi-*-Armendariz *-rings. Then $R$ is quasi-*-Armendariz.

**Proposition 23.** The class of quasi-*-Armendariz *-rings is closed under taking *-subrings.

Since a semiprime ring is quasi-Armendariz [10, Corollary 3.8], we have:

**Corollary 14.** A semiprime-*-ring is quasi-*-Armendariz.

Recall from [1], a *-ring $R$ is *-domain if it has no nonzero *-zero divisor elements and from [5], each domain (Bear) *-ring is *-Baer, each *-Baer (reduced) *-ring is *-reduced and each *-reduced *-ring is semiprime, the following results are straightforward.

**Corollary 15.** A reduced *-ring is quasi-*-Armendariz.

**Corollary 16.** A Baer *-ring is quasi-*-Armendariz.

**Corollary 17.** A domain *-ring is quasi-*-Armendariz.
Example 9 declare that the converse of the previous corollaries is not true, since the *-ring $M_2(R)$ is quasi-*-Armendariz and $M_2(Z_2)$ is not *-reduced where the nonzero matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies $A^2 = AA^* = 0$.

From Proposition 23, if the ⋄-rings $T_{nE}(R)$ and $T(R,R)$ are quasi-⋄-Armendariz, then $R$ is quasi-*-Armendariz and the converse is true for $n \leq 3$ if $R$ is a commutative reduced *-ring, from [2, Proposition 4, Corollary 2]; that is:

Corollary 18. Let $R$ be a commutative reduced *-ring, then the ⋄-ring $T_{3E}(R)$ is quasi-⋄-Armendariz.

Corollary 19. Let $R$ be a commutative reduced *-ring, then the ⋄-ring $T(R,R)$ is quasi-⋄-Armendariz.

Now, we show that the property of quasi-*-Armendariz is restricted from the full matrix ring to its underlying ring.

Proposition 24. If $M_n(R)$ is a quasi-*-Armendariz *-ring for some $n \geq 1$, with the transpose involution *, then $R$ is also quasi-*- Armendariz.

Proof. Let $M_n(R)$ be a quasi-*-Armendariz *-ring for some $n \geq 1$. Since $R \cong e_{11}M_n(R)e_{11}$, as *-rings, then $R$ is quasi-*-Armendariz, by Proposition 25. □

Summarizing the results of this section, we have:

\[
\begin{array}{c}
domain \\
\downarrow \\
* - domain \\
\downarrow \\
Baer * - ring \quad \implies \quad * - Baer * - ring \\
\downarrow \\
reduced \quad \implies \quad * - reduced \\
\downarrow \\
\downarrow \\
semiprime \quad \implies \quad prime \\
\downarrow \\
Armendariz \quad \implies \quad quasi - Armendariz \\
\downarrow \\
* - Armendariz \quad \implies \quad quasi - * - Armendariz \\
\end{array}
\]

9 Extensions of quasi *-Armendariz *-rings

Finally, the property of quasi *-Armendariz is shown to be extended from the *-ring to its polynomial and *-corner *-rings.

By a similar proof to Theorem 1 and using Proposition 23, we get analogous result for quasi-*-Armendariz *-rings.
Theorem 2. A *-ring $R$ is quasi *-Armendariz if and only if $R[x]$ is quasi *-Armendariz.

Proposition 25. A *-ring $R$ is quasi-*-Armendariz if and only if $eRe$ for every projection $e$ of $R$ is quasi-*-Armendariz.

Proof. By Proposition 23, it suffices to prove the necessary condition. Let $R$ be a quasi *-Armendariz *-ring and $f(x)eRe[x]g(x) = f(x)eRe[x]g^*(x) = 0$ with $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in eRe[x]$. Since $f(x)e = f(x)$ and $eg(x) = g(x)$, we obtain $f(x)R[x]g(x) = f(x)R[x]g^*(x) = 0$. By hypothesis $a_iRb_j = 0$ for each $i, j$ which implies $a_i eReb_j = a_i Rb_j = 0$. Therefore $eRe$ is quasi-*-Armendariz.

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