MI-Injective and MI-Flat Modules

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Abstract

We introduce MI-injective modules and MI-flat modules. The properties and characterizations of these two classes of modules are provided. We also characterize rings such that all min-injective modules are min-flat in term of MI-flat modules.

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1 Introduction

Min-injective modules were first introduced by Harada [3] and then studied by Nicholson and Yousif in [5]. In [4], Mao introduced min-flat modules and studied min-coherent rings in term of min-injective modules and min-flat modules. In this paper, we would like introduce two new classes of modules based from min-injective modules, namely MI-injective modules and MI-flat modules. The properties of these two classes of modules are provided. We also characterize MI-injective modules as kernels of some precovers and MI-flat modules as cokernels of some preenvelopes. As a main result, we prove that

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every MI-injective module is a direct sum of a reduced MI-injective module and an injective module. We also characterize rings such that all min-injective modules are min-flat in term of MI-flat modules.

Throughout the paper, \( R \) is an associative ring with identity and all modules are unitary. For an \( R \)-module \( M \), the character module \( M^+ := \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})^+ \).

For two left \( R \)-modules \( M \) and \( N \), we use \( \text{Hom}(M, N) \) to stand for \( \text{Hom}_R(M, N) \). Similarly, we have notations \( M \otimes N \), \( \text{Ext}^k(M, N) \) and \( \text{Tor}_k(M, N) \) for \( k \geq 1 \).

Let \( L \) be a class of \( R \)-modules. \( L \) is said to be closed under extensions, if for any exact sequence \( 0 \to X \to Y \to Z \to 0 \) with \( X, Z \in L \), it holds that \( Y \in F \) too.

Let \( C \) be a class of \( R \)-modules and \( M \) is a \( R \)-module. Following [2], we say that a homomorphism \( \phi : M \to C \) is a \( C \)-preenvelope if \( C \in C \) and the abelian group homomorphism \( \text{Hom}(\phi, C') : \text{Hom}(C, C') \to \text{Hom}(M, C') \) is surjective for each \( C' \in C \). A \( C \)-preenvelope \( \phi : M \to C \) is said to be a \( C \)-envelope if every endomorphism \( g : C \to C \) such that \( g\phi = \phi \) is an isomorphism. Dually we have the definitions of a \( C \)-precover and a \( C \)-cover. \( C \)-envelopes (\( C \)-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let \( R \) be a ring. A left \( R \)-module \( M \) is said to be finitely generated (finitely presented, resp.) if there is an exact sequence \( P_0 \to M \to 0 \) (\( P_1 \to P_0 \to M \to 0 \), resp.), where each \( P_i \) is finitely generated projective left \( R \)-modules. In particular, a simple module is always finitely generated.

Recall that a ring \( R \) is called left coherent if every finitely generated left ideal is finitely presented. A ring \( R \) is called left min-coherent if every simple left ideal of \( R \) is finitely presented.

## 2 MI-injective and MI-flat modules

We firstly recall the following definition, see for instance [5].

**Definition 2.1** Let \( R \) be a ring. A left \( R \)-module \( M \) is called min-injective if \( \text{Ext}^1(R/I, M) = 0 \) or equivalently, the sequence \( \text{Hom}(R, M) \to \text{Hom}(I, M) \to 0 \) is exact for any simple left ideal \( I \) of \( R \).

In the following, we introduce two new classes of modules.

**Definition 2.2** (1) A left \( R \)-module \( M \) is called MI-injective if \( \text{Ext}^1(G, M) = 0 \) for any min-injective left \( R \)-module \( G \).

(2) A right \( R \)-module \( N \) is said to be MI-flat if \( \text{Tor}_1(N, G) = 0 \) for any min-injective left \( R \)-module \( G \).

Immediately from the standard isomorphism: \( \text{Ext}^1(N, M^+) \cong \text{Tor}_1(M, N)^+ \) for any left \( R \)-module \( N \) and right \( R \)-module \( M \), we have the following results.
**Proposition 2.3** A right $R$-module $M$ is MI-flat if and only if $M^+$ is MI-injective.

The following result gives some properties of MI-injective modules and MI-flat modules. The proof is also easy, so we omit it.

**Proposition 2.4** (1) The class of MI-injective modules is closed under direct products, direct summands and extensions.

(2) The class of MI-flat modules is closed under direct sums, direct summands and extensions.

Recall that, the min-injective dimension of $M$, denoted by $\text{mid}(M)$, is defined to be the smallest nonnegative integer $n$ such that $\text{Ext}^{n+1}(R/I, M) = 0$ for every simple left ideal $I$ (if no such $n$ exists, set $\text{mid}(M) = \infty$). For instance, an $R$-module $M$ has the min-injective dimension 0 if and only if $M$ itself is min-injective.

It is easy to see that injective modules are MI-injective and that flat modules are MI-flat. The following result consider the inverse part in the case of coherent rings.

**Proposition 2.5** We have the following result for a left coherent ring $R$:

(1) A left $R$-module $M$ is injective if and only if $M$ is MI-injective and $\text{mid}(M) \leq 1$.

(2) A right $R$-module $N$ is flat if and only if $N$ is MI-flat and $\text{fd}(M) \leq 1$.

**Proof.** (1) Suppose that $M$ is an MI-injective left $R$-module. Consider an exact sequence $0 \to M \to E \to L \to 0$ with $E$ injective. Then there exists an exact sequence

$$0 = \text{Ext}^1(R/I, E) \to \text{Ext}^1(R/I, L) \to \text{Ext}^2(R/I, M)$$

for any simple left ideal $I$. Since $\text{mid}(M) \leq 1$, we have that $\text{Ext}^2(R/I, M) = 0$. Hence $\text{Ext}^1(R/I, L) = 0$, that is, $L$ is min-injective. It follows that $\text{Ext}^1(L, M) = 0$ since $M$ is MI-injective. So the exact sequence $0 \to M \to E \to L \to 0$ is split by the definition. Thus $M$ is isomorphic to a direct summand of $E$, and hence injective.

(2) Suppose $N$ is any MI-flat right $R$-module. Then $N^+$ is MI-injective by Proposition 2.3. Note that $\text{Ext}^{n+1}(R/I, N^+) \cong \text{Tor}_{n+1}(N, R/I)^+$ for any simple left ideal $I$, since $R$ is coherent. So we have that $\text{mid}(N^+) \leq 1$ since $\text{fd}(N) \leq 1$ by the hypothesis. Then $N^+$ is injective by (1), which implies that $N$ is flat, as desired. \qed

**Proposition 2.6** The following are equivalent for a left $R$-module $M$.

(1) $M$ is MI-injective.

(2) For every exact sequence $0 \to M \to E \to L \to 0$ with $E$ min-injective, $E \to L$ is a min-injective precover of $L$. 

(3) $M$ is a kernel of a min-injective precover $f : A \rightarrow B$ with $A$ injective.

(4) $M$ is injective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where $C$ is min-injective.

**Proof.** (1) $\Rightarrow$ (2) Note that there exists an exact sequence $\text{Hom}(G, E) \rightarrow \text{Hom}(G, L) \rightarrow \text{Ext}^1(G, M)$ for any min-injective left $R$-module $G$ and that $\text{Ext}^1(G, M) = 0$ as $M$ is $MI$-injective, so $\text{Hom}(G, E) \rightarrow \text{Hom}(G, L)$ is an epimorphism. It follows that $E \rightarrow L$ is a min-injective precover of $L$ by the definition.

(2) $\Rightarrow$ (3) It is trivial since for any left $R$-module $M$, there exists a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow E/M \rightarrow 0$, where $E$ is the injective envelope of $M$.

(3) $\Rightarrow$ (1). Suppose $M$ is the kernel of a min-injective precover $f : A \rightarrow B$ with $A$ injective. Then there exists an exact sequence $0 \rightarrow M \rightarrow A \xrightarrow{\pi} \text{im}(f) \rightarrow 0$. For any any min-injective left $R$-module $N$, we get the induced exact sequence

$$
\text{Hom}(N, A) \xrightarrow{\text{Hom}(N, \pi)} \text{Hom}(N, \text{im}(f)) \xrightarrow{\text{Ext}^1(N, M)} \text{Ext}^1(N, A) = 0.
$$

It is easy to verify that $\pi : A \rightarrow \text{im}(f)$ is also a min-injective precover. So $\text{Hom}(N, \pi)$ is surjective by the definition of precovers. Then $\text{Ext}^1(N, M) = 0$ and $M$ is $MI$-injective by the definition.

(1) $\Rightarrow$ (4) Clearly, there exists an exact sequence

$$0 \rightarrow \text{Hom}(C, M) \rightarrow \text{Hom}(B, M) \rightarrow \text{Hom}(A, M) \rightarrow \text{Ext}^1(C, M).$$

Note that $C$ is min-injective and that $M$ is $MI$-injective by the assumption, so we have that $\text{Ext}^1(C, M) = 0$. Hence (4) follows.

(4) $\Rightarrow$ (1). Take any a min-injective left $R$-module $N$ and consider a short exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with $P$ projective. Then there is an induced exact sequence

$$\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow \text{Ext}^1(N, M) \rightarrow \text{Ext}^1(P, M) = 0.$$

But $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M)$ is surjective by the assumption of (4). Therefore $\text{Ext}^1(N, M) = 0$ and $M$ is $MI$-injective, as desired.

The following result characterizes kernels of min-injective covers. Recall that a left $R$-module $M$ is called reduced if $M$ has no nonzero injective submodules.

**Proposition 2.7** Suppose $R$ is a left coherent ring. Then the following are equivalent for a left $R$-module $M$:

(1) $M$ is a reduced $MI$-injective left $R$-module.

(2) $M$ is a kernel of a min-injective cover $f : A \rightarrow B$ with $A$ injective.

**Proof.** (1) $\Rightarrow$ (2) Note that the natural map $\pi : E \rightarrow E/M$ is a min-injective precover by Proposition 2.6(3), where $E$ is the injective envelope of $M$. Since $R$ is left coherent, the module $E/M$ has a min-injective cover [4]. Moreover,
$E$ has no nonzero direct summand $K$ contained in $M$ since $M$ is reduced. So we can get that $\pi : E \to E/M$ is indeed a min-injective cover by [6, Corollary 1.2.8], and then (2) follows.

(2) $\Rightarrow$ (1). Suppose $M$ is the kernel of a min-injective cover $\alpha : A \to B$ with $A$ injective. Then $M$ is MI-injective by Proposition 2.6. Now suppose $K$ is an injective submodule of $M$. Then $K$ is also an injective submodule of $A$ and hence a direct summand of $A$. Let $A = K \oplus L$ for some $L$. Assume that $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$. In fact, let $a = (k + l) \in A$, then $\alpha(ip)(a) = \alpha(ip)(k + l) = \alpha(l) = \alpha(k + l) = \alpha(a)$, since $\alpha(K) = 0$. Hence $ip$ is an isomorphism since $\alpha$ is a cover. It follows that $i$ is epic. Thus $A = L$ and then $K = 0$. So $M$ is reduced.

\textbf{Theorem 2.8} Let $R$ be a left coherent ring. Then a left $R$-module $M$ is MI-injective if and only if $M$ is a direct sum of a reduced MI-injective left $R$-module and an injective left $R$-module.

\textbf{Proof.} The if part is obvious.

Only if part. Let $M$ be an MI-injective left $R$-module and consider the exact sequence $0 \to M \to E \to E/M \to 0$, where $E$ is the injective envelope of $M$. We know that $E \to E/M$ is a min-injective precover of $E/M$ by Proposition 2.6. Since $R$ is left coherent, $E/M$ also has a min-injective cover $L \to E/M$. So we get the following commutative diagram with exact rows for some $K$:

\[
\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{f} & L & \xrightarrow{\phi} & E/M & \xrightarrow{\gamma} & 0 \\
| & | & \phi & & | & | & | & | & \\
0 & \to & M & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E/M & \xrightarrow{\gamma} & 0 \\
| & \sigma & | & | & | & | & | & | & \\
0 & \to & K & \xrightarrow{f} & L & \xrightarrow{\beta} & E/M & \xrightarrow{\gamma} & 0.
\end{array}
\]

Note that $\beta \gamma$ is an isomorphism since $L \to E/M$ is a cover. Then $E \simeq L \oplus \ker(\beta)$. It follows that $L$ and $\ker(\beta)$ is injective. Therefore $K$ is a reduced MI-injective module by Proposition 2.7. Since $\sigma \phi$ is also an isomorphism by the Five Lemma, we have that $\sigma$ is epic and $\phi$ is monic and that $M = \ker(\sigma) \oplus \im(\phi)$, where $\im(\phi) \simeq K$ since $\phi$ is monic. In addition, we have the following commutative diagram:
Hence \( \ker(\sigma) \cong \ker(\beta) \) is injective and we have \( M = \ker(\sigma) \oplus \text{im}(\phi) \), where \( \text{im}(\phi) \cong K \) is a reduced MI-injective left \( R \)-module and \( \ker(\sigma) \) is injective. \( \square \)

**Proposition 2.9** Suppose \( R \) is a left coherent ring.

(1) If \( L \) is a cokernel of a min-flat preenvelope \( f : K \to F \) of a right \( R \)-module \( K \) with \( F \) being MI-flat, then \( L \) is MI-flat.

(2) If \( M \) is a finitely presented MI-flat right \( R \)-module, then \( M \) is a cokernel of a flat preenvelope.

**Proof.** (1) Consider the exact sequence \( 0 \to \text{im}(f) \to F \to L \to 0 \) obtained from the assumption. It is easy to check that \( i : \text{im}(f) \to F \) is also a min-flat preenvelope. Let \( N \) be any min-injective left \( R \)-module, then \( N^+ \) is min-flat by [4, Lemma 3.2]. Thus we get an exact sequence \( \text{Hom}(F, N^+) \to \text{Hom}(\text{im}(f), N^+) \to 0 \) by the definition of preenvelope. Equivalently, we have the exact sequence \( (F \otimes N)^+ \to (\text{im}(f) \otimes N)^+ \to 0 \). This implies that the sequence \( 0 \to \text{im}(f) \otimes N \to F \otimes N \) is exact. On the other hand, since \( F \) is MI-flat, we have \( \text{Tor}_1(F, N) = 0 \). Thus, from the induced long exact sequence

\[ 0 = \text{Tor}_1(F, N) \to \text{Tor}_1(L, N) \to \text{im}(f) \otimes N \to F \otimes N, \]

we get that \( \text{Tor}_1(L, N) = 0 \). So \( L \) is MI-flat, as desired.

(2) Since \( M \) is a finitely presented right \( R \)-module, we have an exact sequence \( 0 \to K \to P \to M \to 0 \) with \( P \) projective and both \( P \) and \( K \) finitely generated. It is enough to show that \( K \to P \) is a flat preenvelope. In fact, suppose \( F \) is any flat right \( R \)-module, then \( F^+ \) is injective. Obviously, it is also min-injective. Hence \( \text{Tor}_1(M, F^+) = 0 \), and so we have the following commutative diagram with the first row exact:

\[
\begin{array}{ccc}
0 & \to & K \otimes F^+ \\
\downarrow \tau_{K,F} & & \downarrow \tau_{P,F} \\
\Hom(K, F)^+ & \xrightarrow{\theta} & \Hom(P, F)^+ \\
\end{array}
\]

Note that, by [1, Lemma 2], \( \tau_{K,F} \) is an epimorphism since \( K \) is finitely generated and \( \tau_{P,F} \) is an isomorphism since \( P \) is finitely presented. Since \( \theta \tau_{K,F} = \tau_{P,F} \alpha \), we obtain that \( \tau_{K,F} \) is also a monomorphism and hence an
isomorphism. Thus $\theta$ is a monomorphism. It follows that the homomorphism $\text{Hom}(P, F) \to \text{Hom}(K, F)$ is epic. So $K \to P$ is a flat preenvelope and $M$ is a cokernel of a flat preenvelope.

We call that $R$ is said to be a left $MIF$ ring if every min-injective left $R$-module is min-flat.

An exact sequence $0 \to A \to B \to C \to 0$ is said to be min-pure exact, if for any simple left ideal $I$, the induced sequence $0 \to \text{Hom}(R/I, A) \to \text{Hom}(R/I, B) \to \text{Hom}(R/I, C) \to 0$ is exact, or equivalently, the induced sequence $0 \to A \otimes R/I \to B \otimes R/I \to C \otimes R/I \to 0$ is exact (by [1, Lemma 2], since $R/I$ is finitely presented).

$M$ is called a min-pure-injective left $R$-module, if the functor $\text{Hom}(\cdot, M)$ preserves the exactness of all min-pure exact sequences.

**Theorem 2.10** The following are equivalent for a ring $R$.

1. $R$ is a left $MIF$ ring.
2. Every min-pure-injective left $R$-module is MI-injective.
3. Every right $R$-module is MI-flat.
4. Every finitely presented right $R$-module is MI-flat.

**Proof.** (1) $\Rightarrow$ (2). Let $M$ be an arbitrary min-pure-injective left $R$-module. Take any min-injective left $R$-module $N$ and consider an exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective. The sequence is in fact min-pure from the definition, since there is an induced exact sequence $0 \to \text{Hom}(R/I, N) \to \text{Hom}(R/I, E) \to \text{Hom}(R/I, L) \to \text{Ext}^1(R/I, N) = 0$. On the other hand, there exists an exact sequence $0 \to K \to P \to N \to 0$ with $P$ projective. Note that the exact sequence is also min-pure exact sequence, since $N$ is also min-flat by the assumption and we have an induced exact sequence $0 = \text{Tor}_1(N, R/I) \to K \otimes R/I \to P \otimes R/I \to N \otimes R/I \to 0$. Applying the functor $\text{Hom}(M, \cdot)$, we obtain an induced long exact sequence $0 \to \text{Hom}(P, K) \to \text{Hom}(P, M) \to \text{Hom}(K, M) \to \text{Ext}^1(N, M) \to \text{Ext}^1(P, M) = 0$. As $M$ is min-pure-injective, the sequence $\text{Hom}(P, M) \to \text{Hom}(K, M) \to 0$ is exact. It follows that $\text{Ext}^1(N, M) = 0$ and that $M$ is MI-injective.

(2) $\Rightarrow$ (3). Take any right $R$-module $M$, then $M^+$ is pure injective. It is obviously min-pure injective. So it is also $MI$-injective by the assumption. Then $M$ is $MI$-flat by Proposition 2.3.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Suppose $E$ is any min-injective left $R$-module and let $M$ be any finitely presented right $R$-module. Then $M$ is $MI$-flat by the assumption. Hence $\text{Tor}_1(M, E) = 0$. It follows that $E$ is flat for the arbitrariness of $M$. Obviously it is also min-flat. $\square$
References


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