Prime Ideals in $B$-Algebras

Elsi Fitria$^1$, Sri Gemawati, and Kartini

Department of Mathematics
University of Riau, Pekanbaru 28293, Indonesia

$^1$Corresponding author

Copyright © 2017 Elsi Fitria, Sri Gemawati, and Kartini. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we introduce the definition of ideal in $B$-algebra and some relate properties. Also, we introduce the definition of prime ideal in $B$-algebra and we obtain some of its properties.

Keywords: $B$-algebras, $B$-subalgebras, ideal, prime ideal

1 Introduction

In 1996, Y. Imai and K. Iseki introduced a new algebraic structure called $BCK$-algebra. In the same year, K. Iseki introduced the new idea be called $BCI$-algebra, which is generalization from $BCK$-algebra. In 2002, J. Neggers and H. S. Kim [9] constructed a new algebraic structure, they took some properties from $BCI$ and $BCK$-algebra be called $B$-algebra. A non-empty set $X$ with a binary operation $*$ and a constant $0$ satisfying some axioms will construct an algebraic structure be called $B$-algebra.

The concepts of $B$-algebra have been discussed, e.g., a note on normal subalgebras in $B$-algebras by A. Walendziak in 2005, Direct Product of $B$-algebras by Lingcong and Endam in 2016, and Lagrange’s Theorem for $B$-algebras by JS. Bantug in 2017. Earlier, in 2010, N. O. Al-Shehrie [1] applied the notion of left-right derivation in $BCI$-algebra to $B$-algebra and obtained some related properties. Then, in 2012, the new definition of prime ideal was introduced by R.A. Borzooei and O. Zahiri in this paper entitled "Prime Ideals in $BCI$ and $BCK$-algebras". They found a new definition of prime ideal in $BCI$-algebra and some of its properties. Furthermore, we
apply the concept of prime ideals in $BCI$-algebra to $B$-algebra and investigate some its properties.

## 2 Preliminaries

We begin with some definitions and some theorems of $B$-algebra and $BCI$-algebra.

**Definition 2.1** [9] A $B$-algebra is a non-empty set $X$ with a constant 0 as identity element and a binary operation $\ast$ satisfying the following axioms:

- $(B1)$ $x \ast x = 0$,
- $(B2)$ $x \ast 0 = x$,
- $(B3)$ $(x \ast y) \ast z = x \ast (z \ast (0 \ast y))$,

for all $x, y, z \in X$.

**Example 2.2** [9] Let $X = \{0, 1, 2\}$ be a set with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

As known $0 \in X$ and $1, 2 \in X$, it can be seen from the Table 1 above that $2 \ast 0 = 2, 2 \ast 2 = 0$ and $(1 \ast 2) \ast 0 = 1 \ast (0 \ast (0 \ast 2)) = 2$ that satisfying the all axioms $B$-algebra. Then, $(X; \ast, 0)$ is a $B$-algebra.

**Definition 2.3** [2] A nonempty subset $S$ of $B$-algebra $X$ is called a subalgebra ($B$-subalgebra) of $X$ if $0 \in S$ and $a \ast b \in S$, for all $a, b \in S$.

**Definition 2.4** [9] A $B$-algebra $(X; \ast, 0)$ is said to be commutative $B$-algebra if $a \ast (0 \ast b) = b \ast (0 \ast a)$, for any $a, b \in X$.

**Example 2.5** Let $X = \{0, 1, 2\}$ be a set with the table on Example 2.2. Then $(X; \ast, 0)$ is a commutative $B$-algebra (see [9]).

**Definition 2.6** [5] A $BCI$-algebra is an algebra $(X; \ast, 0)$ satisfying the following axioms:
Prime ideals in B-algebras

(BCI1) \(((x \ast y) \ast (x \ast z) \ast (z \ast y)) = 0,
(BCI2) \((x \ast (x \ast y)) \ast y = 0,
(BCI3) \(x \ast 0 = x,
(BCI4) \(x \ast y = 0 \text{ and } y \ast x = 0 \implies x = y,

for all \(x, y, z \in X.

BCI-algebra \(X\) called BCK-algebra if satisfying \(0 \ast x = 0, \text{ for all } x \in X\). A non-empty subset \(S\) of BCI-algebra \((X; \ast, 0)\) is called a subalgebra of \(X\) if \(x \ast y \in S, \text{ for any } x, y \in S\). The set \(P=\{x \in X \mid 0 \ast (0 \ast x) = x\}\) is called a \(P\)-semisimple part of BCI-algebra \(X\) and \(X\) is called a \(P\)-semisimple BCI-algebra if \(P = X\) (see [3]).

**Definition 2.7** [3] Let \(I\) be a nonempty subset of BCI-algebra \(X\) containing 0. \(I\) is called an ideal of \(X\) if \(y \ast x \in I\) and \(x \in I\) imply \(y \in I\), for any \(x, y \in X\).

**Definition 2.8** [3] A proper ideal \(I\) of BCI-algebra \(X\) is called prime if \(A \cap B \subseteq I\) implies \(A \subseteq I\) or \(B \subseteq I\), for all ideals \(A\) and \(B\) of \(X\).

**Theorem 2.9** [6] Every a commutative B-algebra is a BCI-algebra.

The converse of this theorem may not true in general.

**Theorem 2.10** [6] Every commutative B-algebra is a \(P\)-semisimple BCI-algebra.

The converse of this theorem is true in general.

3 Main Result

R.A. Borzooei and O. Zahiri have been discussed a new definition of prime ideal in BCI and BCK-algebra. By the similar way, we obtain the definition and some theorems of ideal and prime ideal in B-algebra. Some other similar properties from BCI-algebra as a base of this definition.

**Definition 3.1** A non-empty subset \(S\) of B-algebra \(X\) is called ideal of \(X\) if

(i) \(0 \in S, \text{ and}\)

(ii) \(a \in S \text{ and } b \ast a \in S, \implies b \in S, \text{ for any } a, b \in X.\)
Table 2: Cayley table for \((X; *, 0)\)

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
* & 0 & 1 & a & b & c & d \\
\hline
0 & 0 & a & 1 & b & c & d \\
1 & 1 & 0 & a & c & d & b \\
a & a & 1 & 0 & d & b & c \\
b & b & c & d & 0 & a & 1 \\
c & c & d & b & 1 & 0 & a \\
d & d & b & c & a & 1 & 0 \\
\hline
\end{tabular}
\end{center}

Clearly, \(\{0\}\) is an ideal of \(B\)-algebra \(X\). An ideal \(S\) called proper ideal if \(S \neq X\) and is called closed ideal if \(a \ast b \in S\), for all \(a, b \in S\). The least ideal of \(X\) containing \(I\), the generated ideal of \(X\) by \(I\) and is denoted by \(\langle I \rangle\).

**Example 3.2** Let \(X = \{0, 1, a, b, c, d\}\) is \(B\)-algebra with Cayley table as follows:

Let \(S = \{0, 1, a\}\), so \(0 \in S\). From the Table 2, we have \(1 \in S\) and \(a \ast 1 = 1 \in S\), so \(a \in S\). Hence, \(S\) is an ideal of \(X\) and since \(1 \ast a = a \in S\), then \(S\) a \(B\)-subalgebra.

In the next example, we will be checked, are every ideals in \(B\)-algebra is \(B\)-subalgebra?.

**Example 3.3** Let \(X = (Z; -, 0)\) with "-" subtraction operation of integers \(Z\). Then, it is easy to prove that \(X\) is \(B\)-algebra. Let \(I = Z^+ \cup \{0\}\) is a subset of \(X\) with \(Z^+\) is positive integers, so \(I\) is an ideal of \(X\) and \(I\) is not a \(B\)-subalgebra of \(X\).

So, every ideal in \(B\)-algebra is not always \(B\)-subalgebra. From the definition of \(B\)-subalgebra that is closed of binary operation \(\ast\) and the closed ideal in \(B\)-algebra, we obtain that every \(B\)-subalgebra is closed ideal in \(B\)-algebra.

**Definition 3.4** Let \(X\) is a \(B\)-algebra. A proper ideal \(M\) of \(X\) is called a maximal ideal of \(X\) if \(\langle M \cup \{x\} \rangle = X\), for any \(x \in X\), where \(\langle M \cup \{x\} \rangle\) is an ideal generated by \(M \cup \{x\}\). \(M\) is an maximal ideal of \(X\) if and only if \(M \subseteq A \subseteq X\) implies that \(M = A\) or \(A = X\), for any ideal \(A\) of \(X\).

**Example 3.5** Let \(X = \{0, a, b, c\}\) be a \(B\)-algebra with Cayley table as follows:
Prime ideals in $B$-algebras

Table 3: Cayley table for $(X; *, 0)$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

We obtain the set of all proper ideal of $X$ is $\{\{0\}, \{0, a\}, \{0, b\}\}$. Let $A = \langle a \rangle = \{0, a\}$ and $B = \langle b \rangle = \{0, b\}$, then $A$ and $B$ are maximal ideals of $X$, because there are no other ideal of $X$ that containing $A$ and $B$ rather than itself.

Let $A$ is an ideal of $B$-algebra $X$. Then relation $\theta$ is defined $(x, y) \in \theta \iff x * y, y * x \in A$ is a congruence relation on $X$, denoted $A_x$ for $[x]=\{y \in X \mid (x, y) \in \theta \}$. So, $A_0$ is a closed ideal of $B$-algebra $X$. Let $X/A = \{A_x \mid x \in X\}$, then $(X/A; *, A_0)$ is a $B$-algebra with $A_x * A_y = A_{(x*y)}$, for all $x, y \in X$ (see [3]).

**Lemma 3.6** Let $A$ and $B$ ideals of $B$-algebra $X$ such that $A \subseteq B$. Denote $B/A = \{A_x \mid x \in B\}$. Then

(i) $x \in B$ if and only if $A_x \in B/A$, for any $x \in X$,

(ii) $B/A = \{A_x \mid x \in B\}$ is an ideal from $X/A$.

**Definition 3.7** A proper ideal $S$ of $B$-algebra $X$ called irreducible ideal of $X$ if $A \cap B = S$ implies $A = S$ or $B = S$, for any ideals $A$ and $B$ of $X$.

**Definition 3.8** A proper ideal $S$ of $B$-algebra $X$ called prime ideal of $X$ if $A \cap B \subseteq S$ implies $A \subseteq S$ or $B \subseteq S$, for all ideals $A$ and $B$ of $X$.

**Theorem 3.9** Let $S$ be an ideal of $B$-algebra $X$. Then $S$ is a prime ideal of $X$ if and only if $\langle x \rangle \cap \langle y \rangle \subseteq S$ implies $x \in S$ or $y \in S$, for any $x, y \in X$.

**Proof** Let $S$ be a prime ideal of $X$. From Definition 3.8 if $A$ and $B$ are two ideals of $X$ and $A \cap B \subseteq S$ implies $A \subseteq S$ or $B \subseteq S$, its mean if $A = \langle x \rangle$ and $B = \langle y \rangle$ then $\langle x \rangle \subseteq S$. So that, for any $x \in A$ then $x \in S$ or $\langle y \rangle \subseteq S$ and for any $y \in B$ then $y \in S$. Hence, we obtain if $S$ be a prime ideal of $X$ then $\langle x \rangle \cap \langle y \rangle \subseteq S$ implies $x \in S$ or $y \in S$, for any $x, y \in X$. Conversely, Let $S$ be an ideal of $X$ and $\langle x \rangle \cap \langle y \rangle \subseteq S$ implies $x \in S$ or $y \in S$, for any $x, y \in X$. Let $A$ and $B$ be two ideals of $X$ so that $A \cap B \subseteq S$. We assuming $A \not\subseteq S$, there exists $x \in A$ and $x \not\in S$. Because for any $y \in B$ then $\langle x \rangle \cap \langle y \rangle \subseteq A \cap B \subseteq S$ and $x \not\in S$ then $y \in S$, so that $B \subseteq S$. Therefore, $S$ be a prime ideal of $X$. 
Definition 3.10 A non-empty subset $F$ of $X$ is called a finite $\cap$-structure, if $(\langle x \cap \langle y \rangle \rangle \cap F \neq \emptyset$, for all $x, y \in F$ and $X$ is called a finite $\cap$-structure if $X \setminus \{0\}$ is a finite $\cap$-structure.

Let $X$ and $Y$ are $B$-algebra. The mapping $f : X \to Y$ is called homomorphism of $B$-algebra or $B$-homomorphism if $f(x \cdot y) = f(x) \cdot f(y)$, for all $x, y \in X$. A $B$-homomorphism called $B$-monomorphism if $f$ is one-to-one and $B$-epimorphism if $f$ is onto. A $B$-homomorphism $f : X \to Y$ called $B$-isomorphism if $f$ is one-to-one and onto (bijection), and labeled by $X \cong Y$. If $f : X \to Y$ $B$-isomorphism so $f^{(-1)} : Y \to X$ also $B$-isomorphism (see [7]).

Theorem 3.11 Let $X$ and $Y$ are $B$-algebra and $f : X \to Y$ $B$-epimorphism. Then

(i) An ideal $A$ of $X$ is prime if and only if $F = X - A$ is a finite $\cap$-structure.

(ii) Let $A$ a closed ideal of $X$ and $B$ an ideal of $X$ containing $A$. If $B$ is a prime ideal of $X$ then $B/A$ is a prime ideal of $X/A$.

Proof

(i) Let $A$ be a prime ideal of $X$ and $F = X - A$, for any $x, y \in F$. We assume $(\langle x \cap \langle y \rangle \rangle \cap F = \emptyset$, so $\langle x \rangle \cap \langle y \rangle \subseteq A$. Because $A$ be a prime ideal of $X$, then $x \in A$ or $y \in A$, while $x, y \in F$. This is impossible, hence $\langle x \rangle \cap \langle y \rangle \cap F \neq \emptyset$, so that $F = X - A$ is a finite $\cap$-structure. Conversely, if $F = X - A$ is a finite $\cap$-structure and $x, y \in X$, so that $\langle x \rangle \cap \langle y \rangle \subseteq A$. If $x \notin A$ and $y \notin A$ then $x, y \in F$ and $(\langle x \rangle \cap \langle y \rangle \rangle \cap F \neq \emptyset$, we obtained $(\langle x \rangle \cap \langle y \rangle \rangle \cap F \neq \emptyset$. Hence, it must be $x \in A$ or $y \in A$ and from Theorem 3.9 concluded that $A$ be a prime ideal of $X$.

(ii) Let $A$ is a closed ideal of $X$ and $B$ is an ideal of $X$ containing $A$, such that $A \subseteq B$. Since $B$ is an ideal of $X$, then from Lemma 3.6(ii) $B/A$ is a ideal of $X/A$. Let $I$ and $J$ are ideals of $X/A$ such that $I \cap J \subseteq B/A$, then there exists $K$ and $L$ be ideals of $X$, so that $I = K/A$ and $J = L/A$, thus $K/A \cap L/A = (K \cap L)/A \subseteq B/A$. If $B$ is a prime ideals of $X$ then $K \cap L \subseteq B$, so that $K \subseteq B$ or $L \subseteq B$, then $K/A \subseteq B/A$ or $L/A \subseteq B/A$. Hence, $B/A$ is a prime ideal of $X/A$.

Corollary 3.12 Let $x \in X - \{0\}$, such that $x \cdot y = x$, for all $y \in X - \{x\}$. Then, there exists a prime ideal $P$ of $X$, such that $x \notin P$.

Proof Let $P = X - \{x\}$, then $0 \in P$. If $b \in P$ and $a \cdot b \in P$, then $a \neq x$ such that $a \in P$. Since, $P$ is a ideal of $X$. Clearly, that $X - P$ ia a finite $\cap$-structure. From Theorem 3.11(i) obtained $P$ is a prime ideal of $X$. Hence, there exists a prime ideal $P$ of $X$, such that $x \notin P$. 

Table 4: Cayley table for \((X; \ast, 0)\)

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 3.13 Let \(X = \{0, a, b, c\}\). Define the binary operation \(\ast\) of \(X\) with following table:

Then \((X; \ast, 0)\) is \(B\)-algebra. Since \(c \ast y = c\), for all \(y \in X - \{c\}\), then from Corollary 3.12 obtained \(P = X - \{c\}\) is a prime ideal of \(X\), such that \(c \notin P\).

Theorem 3.14 Let \(A\) be an ideal of \(X\).

(i) Let \(A\) is a prime ideal of \(X\), then \(A/A_0\) is a prime ideal of \(X/A_0\).

(ii) Let \(A\) is a closed prime ideal of \(X\), then \(A_0\) is a closed prime ideal of \(X/A\).

Proof

(i) Let \(A\) is an ideal of \(X\), then \(A_0 \subseteq A \subseteq X\). Since \(A_0\) is a closed ideal of \(X\), then from Lemma 3.6 (ii) \(A/A_0\) is an ideal of \(X/A_0\). Let \(P\) and \(Q\) are two ideals of \(X/A_0\), so that \(P \cap Q \subseteq A/A_0\). Then there exists \(S\) and \(T\) be two ideals of \(X\) containing \(A_0\). where \(P = S/A_0\) and \(Q = T/A_0\), so \((S \cap T)/A_0 = S/A_0 \cap T/A_0 = P \cap Q \subseteq A/A_0\). If \(A\) be a prime ideal of \(X\) obtained \(S \cap T \subseteq A\), then \(S \subseteq A\) or \(T \subseteq A\) and \(P \subseteq A/A_0\) or \(Q \subseteq A/A_0\). Therefore, \(A/A_0\) is a prime ideal of \(X/A_0\).

(ii) Let \(A\) is an ideal of \(X\). If \(A\) is closed, then \(A = A_0\) and \(A/A_0 = A_0\) implies \(X/A_0 = X/A\). Since \(A\) is a closed prime ideal of \(X\), then from (i) obtained \(A_0\) is a closed prime ideal of \(X/A\).

From definition of prime ideal and irreducible ideal obtained that every prime ideal is an irreducible ideal in \(B\)-algebra, but the convers may not true in general. In the following example, we will show that there exists an irreducible ideal which is not prime ideal in \(B\)-algebra.

Example 3.15 Let \(X = \{0, 1, 2, a\}\) be a set with the following table:
Table 5: Cayley table for \((X; *, 0)\)

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>a</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>a</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; *, 0)\) is a \(B\)-algebra and \(\{\{0\}\}, \{0, 1\}, \{0, 2\}, \{0, a\}\) are all proper ideals of \(X\). We obtain \(\{0, 1\}, \{0, 2\}\) and \(\{0, a\}\) are irreducible ideals of \(X\). Since \(\{0, 1\} \cap \{0, 2\} \subseteq \{0, a\}\), then \(\{0, a\}\) not a prime ideal of \(X\) and \(\{0\}, \{0, 1\}\), and \(\{0, 2\}\) not also a prime ideal of \(X\). Therefore, \(X\) has not any prime ideal. Furthermore, let \(A = \{0, 1\}\) be an irreducible ideal of \(X\). Since \(2, a \in X - A\) and \(\langle 2 \cap a \rangle = \{0, 2\} \cap \{0, a\} = \{0\}\), then \(\langle 2 \cap a \rangle \cap (X - A) = \emptyset\). Therefore, \(X - A\) not a finite \(\cap\)-structure.

References


Received: September 11, 2017; Published: October 2, 2017