On the Diophantine Equation $8^x + 113^y = z^2$

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Abstract

In this paper we have shown that the Diophantine equation $8^x + 113^y = z^2$ has exactly three non-negative integer solutions for $x$, $y$ and $z$. The solutions are $(1, 0, 3)$, $(1, 1, 11)$ and $(3, 1, 25)$ respectively.

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1 Introduction

There is a lot of studies about Diophantine equations of the type $a^x + b^y = c^z$ by a number of mathematicians in the field of number theory. In 1999, Cao [1], proved that this equation has at most one solution with $c > 1$. In 2012, Peker and Cenberci [2] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. However, in the same year, Sroysang [3] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution in $(x, y, z)$ which is $(1, 0, 3)$. Same author [4, 5] also solved the Diophantine equations $8^x + 13^y = z^2$ and $8^x + 7^y = z^2$, respectively. He
found that the solution of these equations is \( (1, 0, 3) \) in non-negative integers \((x, y, z)\). Rabago [6] showed that the Diophantine equation \( 8^x + 17^y = z^2 \) has four solutions in non-negative integers \((x, y, z)\). Further, several Diophantine equations of different types have been studied by different workers [7, 8, 9, 10, 11, 12, 13]. Recently, Qi and Li [14] established that the Diophantine equation \( 8^x + p^y = z^2 \), \( x, y, z \) belong to natural numbers and \( p \) is an odd integer, with \( p \equiv 1(\text{mod } 8) \) and \( p \neq 17 \), have at most two positive integer solutions in \((x, y, z)\) where \( p \) is an odd prime. Hence, it is a matter of further investigation to examine that, apart from \( p = 17 \), how many other such Diophantine equations are there which do not obey Qi and Li’s [14] generalization. Although a number of other Diophantine equations has been solved by several other authors, yet it is imperative to search many more Diophantine equations violating Qi and Li’s generalization [14], we have made an attempt to solve the new Diophantine equation containing \( p = 113 \), hitherto uninvestigated by any researcher to the best of our knowledge, and have found that it has three exact solutions in non-negative integers \((x, y, z)\). This problem constitutes second exception, apart from that of the Diophantine equation \( 8^x + 17^y = z^2 \) by Rabago[6] to the Qi and Li’s generalizations regarding solutions of Diophantine equations of the type \( 8^x + p^y = z^2 \) with \( p \equiv 1(\text{mod } 8) \) where \( p \) is a prime.

2 Preliminaries

The Catalan’s conjecture is an important well known conjecture and plays an important role in solving Diophantine equations. According to this conjecture, \((3, 2, 2, 3)\) is a unique solution \((a, x, b, y)\) for the Diophantine equation \( a^x - b^y = 1 \) where \( a, b, x \) and \( y \) are integers with \( \min\{a, b, x, y\} > 1 \). This conjecture was proved by Mihăilescu in 2004 [15].

2.1 Proposition

\((3, 2, 2, 3)\) is a solution \((a, b, x, y)\) for the Diophantine equation \( a^x - b^y = 1 \) where \( a, b, x \) and \( y \) are integers with \( \min\{a, b, x, y\} > 1 \).

Now we will prove two Lemma’s by Proposition 2.1.

**Lemma 2.1.** \((1, 3)\) is a unique solution \((x, z)\) for the Diophantine equation \( 8^x + 1 = z^2 \) where \( x \) and \( z \) are non-negative integers.

**Proof.** Let \( x \) and \( z \) be non-negative integers such that \( 8^x + 1 = z^2 \). First we consider the case \( x = 0 \) and \( z = 0 \). If \( x = 0 \), then \( z^2 = 2 \) which is impossible. If \( z = 0 \), then \( 8^x = -1 \) which is impossible. Now we consider the case \( x, z > 0 \). Then \( 8^x + 1 = z^2 \) or \( 8^x = z^2 - 1 \). Then \( 2^{3x} = (z - 1)(z + 1) \). Thus, \( (z - 1) = 2^u \) where \( u \) is a non-negative integer. Then \( (z + 1) = 2^{3x-u} \). Thus, \( 2 = 2^{3x-u} - 2^u \) or \( 2^u(2^{3x-2u} - 1) = 2 \). We have two possibilities
i) $2^u = 2^0$ which implies that $u = 0$ and $2^{3x} - 1 = 2$ which implies that $2^{3x} = 3$, which is impossible.

ii) $2^u = 2$ which implies that $u = 1$ and $2^{3x-2} - 1 = 1$ or $2^{3x-2} = 2$ which implies that $x = 1$.

Putting $x = 1$ in $8^x + 1 = z^2$, we have $z = 3$. Hence, $(1, 3)$ is a unique solution $(x, z)$ for the Diophantine equation $8^x + 1 = z^2$ where $x$ and $z$ are non-negative integers.

\[ \square \]

**Lemma 2.2.** The Diophantine equation $1 + 113^y = z^2$ has no non-negative integer solution.

**Proof.** Suppose that there are non-negative integers $y$ and $z$ such that $1 + 113^y = z^2$. If $y = 0$, then $1 + 1 = z^2$ i.e. $z^2 = 2$ which is impossible. If $z = 0$, then $113^y = -1$ which is not possible. We consider the case when $y, z > 0$. Then $z^2 = 1 + 113^y$ or $113^y = z^2 - 1 = (z - 1)(z + 1)$. Thus, $z - 1 = 113^v$, where $v$ is a non-negative integer. Then $z + 1 = 113^{v+1}$. Thus, $2 = 113^{y-v} - 113^v = 113^v(113^{y-v} - 1)$ which implies that $v = 0$ and $113^{y-1} = 2$ or $113^y = 3$ which is not possible. Hence, the Diophantine equation $1 + 113^y = z^2$ has no non-negative integer solution. \[ \square \]

## 3 Main Result

**Theorem 3.1.** The Diophantine equation $8^x + 113^y = z^2$ has exactly three solutions in non-negative integers $(x, y, z) \in \{(1, 0, 3), (1, 1, 11), (3, 1, 25)\}$.

**Proof.** Let $x, y$ and $z$ be non-negative integers such that $8^x + 113^y = z^2$. We first consider the case when $y$ is zero. By Lemma 2.1, we have $(x, y, z) = (1, 0, 3)$. From Lemma 2.2, $x \geq 1$. This implies that $z$ is odd. Now we will divide $y$ into two cases when $y \geq 1$.

**Case(i)** If $y$ is even i.e. $y = 2l$ for some positive integer $l$, then $8^x = z^2 - 113^{2l} = (z - 113^l)(z + 113^l)$ or, $2^{3x} = (z - 113^l)(z + 113^l)$. This implies that $2 \cdot 113^k = 2^w(2^{3x-2w} - 1)$ where $z - 113^l = 2^w$ and $z + 113^l = 2^{3x-w}$, $w$ is a non-negative integer. We have two subcases:

a) $w = 0$. Then $z - 113^l = 1$. This implies that $z$ is even. This is a contradiction.

b) $w = 1$. Then $2^{3x-2} - 1 = 113^k$. Then $2^{3x-2} - 113^l = 1$. If $x = 1$, then $113^l = 1$ i.e. $l = 0$ so $y = 0$. Thus, $x \geq 2$. By Proposition 2.1, we have $l = 1$. Then $2^{3x-2} = 114$. This is impossible.

**Case(ii)** When $y$ is odd. Let $y = 2l + 1$ where $l$ is a non-negative integer. We will divide this case into two parts i.e. Part(1) and Part(2).

**Part(1)** $8^x + 113^{2l+1} = z^2$

or, $8^x + (13 + 100) \cdot 113^{2l} = z^2$

or, $8^x + 13 \cdot 113^{2l} = z^2 - 100 \cdot 113^{2l}$


or, $8^x + 13 \cdot 113^{2l} = (z - 10 \cdot 113^l)(z + 10 \cdot 113^l)$

There are two possibilities for this equation

\[
\begin{cases}
  z - 10 \cdot 113^l = 1 \\
  z + 10 \cdot 113^l = 8^x + 13 \cdot 113^{2l}
\end{cases}
\]

or

\[
\begin{cases}
  z + 10 \cdot 113^l = 1 \\
  z - 10 \cdot 113^l = 8^x + 13 \cdot 113^{2l}
\end{cases}
\]

Solving first set of equalities, we have $113^l(20 - 13 \cdot 113^l) = 8^x - 1$. It implies that $l = 0$ and $20 - 13 = 8^x - 1$. Hence we obtain $x = 1$, $y = 1$ and $z = 11$. Hence, the solution of the Diophantine equation in non-negative integers $(x, y, z) = (1, 1, 11)$. Solving second set of equalities, we have $113^l(20 - 13 \cdot 113^l) = 1 - 8^x$ which implies that $l = 0$ and $20 - 13 = 1 - 8^x$ or $-8^x = 6$ which is not solvable.

Part(2) Again we have, $8^x + 113^{2l+1} = z^2$

or, $8^x + 113 \cdot 113^{2l} = z^2$

or, $8^x + (576 - 463) \cdot 113^{2l} = z^2$

or, $8^x - 463 \cdot 113^{2l} = z^2 - 576 \cdot 113^{2l}$

or, $8^x - 463 \cdot 113^{2l} = (z - 24 \cdot 113^l)(z + 24 \cdot 113^l)$

There are two possibilities for this equation

\[
\begin{cases}
  z - 24 \cdot 113^l = 1 \\
  z + 24 \cdot 113^l = 8^x - 463 \cdot 113^{2l}
\end{cases}
\]

or

\[
\begin{cases}
  z + 24 \cdot 113^l = 1 \\
  z - 24 \cdot 113^l = 8^x - 463 \cdot 113^{2l}
\end{cases}
\]

Solving first set of equalities, we have $113^l(48 + 463 \cdot 113^l) = 8^x - 1$. It implies that $l = 0$ and $48 + 463 = 8^x - 1$ or $511 + 1 = 8^x$ or $512 = 8^x$ or $8^x = 8^3$ which implies that $x = 3$. Hence we obtain $x = 3$, $y = 1$ and $z = 25$. Therefore the solution of this Diophantine equation is $(3, 1, 25)$. Solving second set of equalities, we have $113^l(48 - 463 \cdot 113^l) = 1 - 8^x$. It implies that $l = 0$ and $48 - 463 = 1 - 8^x$ or $-416 = -8^x$ or $416 = 8^x$ which is not solvable.

Corollary 3.2 The Diophantine equation $8^x + 113^y = w^4$ has a unique non-negative integer solution $(x, y, z) = (3, 1, 5)$.

**Proof.** Suppose that there are non-negative integers $x$, $y$ and $w$ such that $8^x + 113^y = w^4$. Let $z = w^2$. Then $8^x + 113^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 1, 25)$. Then $w^2 = z = 25$ i.e $w = 5$. Hence, the equation $8^x + 113^y = w^4$ has unique solution in non-negative integers $(x, y, w) = (3, 1, 5)$. 

\[
\square
\]
4 Conclusion

In this Theorem, we have solved the Diophantine equation $8^x + 113^y = z^2$ where 113 is a prime number. We have shown that the entitled equation has three non-negative integer solutions in $(x, y, z)$ i.e. $(1,0,3), (1,1,11)$ and $(3,1,25)$. Thus, we have proved that the entitled equation offers an exception to the generalization made by Li and Qi that the prime numbers satisfying the congruence equation $p \equiv 1 \,(mod \, 8)$ have at most two positive integer solutions $(x, y, z)$ only.

References


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