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Forbidden Structures in Heyting Algebras with Respect to Sublattices

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Abstract

In this paper, we have obtained forbidden structures of varieties of Heyting algebras namely $H_2, H_3, H_4, H_5, H_6, H_7$ with respect to sublattices.

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Keywords: Brouwerian lattice, Heyting Algebra, Distributive lattice, Sublattice, Forbidden stucture

1 Introduction

The study of forbidden structures has a long history. In fact, it has been noted and studied in depth by many researchers; see [1], [3], [4], [5], [6], [13], [17], [19], [20], [21], [22], [23], etc. A *Boolean lattice* is a complemented distributive lattice. It is well known that complements (if exist) are unique in any distributive lattice. It follows that any Boolean lattice is dually isomorphic with itself (self-dual); see [1]. A Boolean lattice can also be regarded as an algebra with two binary operations $\wedge$, $\vee$, an unary operation $'$ (complementation) and two nullary operations of picking up special elements namely 0 and 1. Thus a Boolean algebra is an algebra of the type $< B; \wedge, \vee, ', 0, 1 >$; see [6]. Boolean lattices considered as algebras (2, 2, 1, 0, 0) are called Boolean algebras. In a Boolean algebra A, for an element $a \in A$, there is a complement of a, denoted by $a' \in A$ which is the largest element $x \in A$ such that $a \wedge x = 0$. More generally, for $a, b, x \in A$, $a \wedge x \leq b$ if and only if $a \wedge x \wedge b' = 0$, i.e., $x \wedge (a \wedge b') = 0$ or $x \leq (a \wedge b')' = b \vee a'$. Therefore, given $a, b \in A$, there exists a largest element $c \in A$, $c = b \vee a'$ such that $a \wedge c \leq b$. Brouwer and Heyting characterized an important generalization of Boolean algebras through an extension of the preceding property as stipulated in the definition given below.

Definition 1.1 *A Brouwerian lattice L is a lattice in which, for any elements a and b, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element. Such greatest element is called relative pseudocomplement of a in b, denoted as a_b^* or $a \to b$ or $b : a$, and so the lattice is also called relatively pseudocomplemented lattice. The operation of getting relative pseudocomplements is called Heyting operation.*

From the definition, it is easy to observe that every Brouwerian lattice has the unit element. In a Brouwerian lattice L with 0, the relative pseudocomplement of a in 0 is nothing but the pseudocomplement a^* of a. The study of different types of algebras can be seen in [7], [8], [9], [10] and [2]. The following result is proved in [1].

Theorem 1.2 ([1]) *Any Brouwerian lattice is distributive.*

According to [7], a Brouwerian lattice which is bounded, is called a *Heyting algebra*. The class of Heyting algebras is a subclass of the class of distributive lattices and also the class of pseudocomplemented lattices. A Heyting algebra H is said to be of order 3 if every interval in $D(H)$ is complemented, where $D(H) = \{x \in H : x^* = 0\}$. Heyting algebras are well studied in the literature. Especially, Heyting algebras have been found useful to analyze qualitative relations in biological systems and different biological processes (see [14], [15],

[16]). Qualitative relationships concern with those parts of biological systems that determine functional properties, and also different relations among them.

We have the following definitions.

A Heyting algebra is of order 3 (H_3) if and only if it satisfies the identity $x \vee x^*_{y \vee y^*} = 1$. (see [7])

A Heyting algebra is of order 2 (H_2) if and only if it satisfies the identity $x^* \vee x^{**} = 1$. This is also called as Stone algebra.

A Heyting algebra is of order 4 (H_4) if and only if it satisfies the identities $x \vee x^*_{y \vee y^*} = 1$ and $x^*_y \vee y^*_x \vee [x^*_{y^*} \wedge (y^*)^*_x] = 1$.

A Heyting algebra is of order 5 (H_5) if and only if it satisfies the identities $x \vee x^*_{y \vee y^*} = 1$ and $x^* \vee x^{**} = 1$.

A Heyting algebra is of order 6 (H_6) if and only if it satisfies the identity $x^*_y \vee y^*_x = 1$.

A Heyting algebra is of order 7 (H_7) if and only if it satisfies the identity $x \vee x^* = 1$. This is also called as Boolean algebra.

These varieties satisfy following inclusion:

$$H_7 \subset H_5 \subset H_6 \subset H_2,$$

and

$$H_7 \subset H_5 \subset H_4 \subset H_3.$$

2 Main Results

In this paper, all lattices considered are finite.

Lemma 2.1 *Let L be a Brouwerian lattice with* 0 *and* 1*. If $a, b \in L$ such that b is meet irreducible with $b < a$, then $a^*_b = b$.*

Proof. Suppose that L is a Brouwerian lattice with 0 and 1 and $a, b \in L$ are such that b is meet irreducible with $b < a$, with $a^*_b \neq b$. Since $b \leq a^*_b$, we must have $b < a^*_b$. Suppose there exists an element z in L such that $b \prec z \leq a$. Since b is meet irreducible, we get that $z \leq a^*_b$, which implies that $z \wedge a \leq a \wedge a^*_b$, and so $z \wedge a \leq b$, i.e., $z \leq b$, a contradiction to the fact that $b \prec z$. We conclude that, whenever b is meet irreducible and $b < a$, we must have $b = a^*_b$. Hence $a^*_b = b$. □

The following results are given in [6].

Theorem 2.2 ([6]) *Identities are preserved under the formation of sublattices, homomorphic images, direct products, and ideal lattices.*

Distributivity criteria in terms of forbidden structure is given by Birkhoff and modularity criteria in terms of forbidden structure is given by Dedekind.

Theorem 2.3 (**[6]**) *A lattice L is distributive if and only if L does not contain a pentagon (N_5) or a diamond (M_3).*

Theorem 2.4 *A Heyting algebra L is in H_7 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-1.*

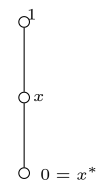

Figure 1:

Proof. If L is in H_7, then by Theorem 2.2, every sublattice of L is in H_7 and the lattice as depicted in the Figure-1 is not in H_7.

Conversely, suppose $L \notin H_7$. Then there exists an element, say $x \in L$ such that $x \vee x^* \neq 1$. Suppose that $x^* = 0$, then the sublattice $\{0 = x^*, x = x \vee x^*, 1\}$ of L is isomorphic to the lattice depicted in the Figure-1. Now, suppose $x^* \neq 0$, then $0, x, x^*, x \vee x^*, 1$ are distinct elements of L. Take $y = x \vee x^*$, then the sublattice $\{0 = y^*, y = y \vee y^*, 1\}$ of L is isomorphic to the lattice depicted in the Figure-1. □

Theorem 2.5 *A Heyting algebra L is in H_2 if and only if it does not contain a sublattice containing* 0 *of L isomorphic to the lattice as depicted in the Figure-2.*

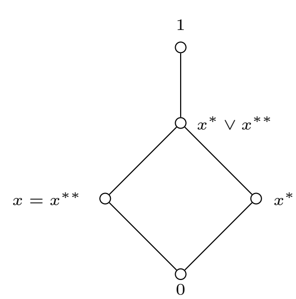

Figure 2:

Proof. If L is in H_2, then by Theorem 2.2, every sublattice of L is also in H_2 and the lattice as depicted in the Figure-2 is not in H_2.

Conversely, suppose $L \notin H_2$. Then there exists an element, say $x \in L$ such that $x^* \vee x^{**} \neq 1$. Suppose $x = x^{**}$, then $\{0, x, x^*, x \vee x^*, 1\}$ forms a sublattice isomorphic to the lattice depicted in Figure-2. Now, suppose that $x \neq x^{**}$ But then, $x < x^{**}$ and $0, x, x^*, x^{**}, 1$ are distinct elements of L. Note that $x \vee x^* \neq x^* \vee x^{**}$; otherwise, $\{0, x, x^*, x^{**}, x \vee x^* = x^* \vee x^{**}, 1\}$ forms a sublattice isomorphic to N_5, contradicting distributivity of L. Also, we have $x^{**} \wedge (x \vee x^*) = (x^{**} \wedge x) \vee (x^{**} \wedge x^*) = x \vee 0 = x$. Take $y = x^*$, then the sublattice $\{0, y = y^{**}, y^*, y^* \vee y^{**}, 1\}$ of L is isomorphic to the lattice depicted in Figure-2. □

Theorem 2.6 *A Heyting algebra L is in H_6 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-2.*

Proof. If L is in H_6, then by Theorem 2.2, every sublattice of L is in H_6 and the lattice as depicted in the Figure-2 is not in H_6.

Conversely, suppose $L \notin H_6$. Then there exists a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \neq 1$. Note that $x \parallel y$; otherwise, if $x \nparallel y$, then either $x \leq y$ or $y \leq x$ and $x \leq y$ gives $x_y^* = 1$, a contradiction to the fact $x_y^* \vee y_x^* \neq 1$ and similarly for $y \leq x$. Also, we must have $x \nleq x_y^*$ and $y \nleq y_x^*$. Indeed, if $x \leq x_y^*$, then $x = x \wedge x \leq x \wedge x_y^* \leq y$, a contradiction to the fact that $x \parallel y$ and similarly for $y \nleq y_x^*$. Now, by definition, $y \leq x_y^*$ and $x \leq y_x^*$ and accordingly we have the following cases.

Case **1** Suppose $y = x_y^*$ and $x = y_x^*$. Then $\{x \wedge y, x, y, x \vee y, 1\}$ forms a sublattice isomorphic to the lattice depicted in Figure-2.

Case **2** Suppose $y = x_y^*$ and $x \neq y_x^*$. Then $x \wedge y, x, y, y_x^*, x \vee y, 1$ are distinct elements of L. Also, we have the following.

(**2-I**) We claim that $y_x^* \wedge (x \vee y) = x$. Indeed, if $y_x^* \wedge (x \vee y) = z \neq x$, then $x < z < x \vee y$ and the set $\{x \wedge y, x, y, z, x \vee y\}$ forms a sublattice of L isomorphic to N_5.

(**2-II**) We claim that $y_x^* \wedge y = x \wedge y$. If $y_x^* \wedge y = z$, then $x \wedge y \leq z$ since $x \leq y_x^*$. By definition we get, $y \wedge y_x^* \leq x$, i.e., $z \leq x$. Also, we have $z \leq y$ and so $z \leq x \wedge y$ and consequently $z = x \wedge y$.

(**2-III**) We claim that $x \vee y \neq (y_x^*) \vee y$. Indeed, if $x \vee y = (y_x^*) \vee y$, then $x \wedge y, x, y, y_x^*, x \vee y$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5.

Now, the set $\{x \wedge y, x, y, x \vee y, y_x^*, y_x^* \vee y, 1\}$ of distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

Case **3** Similar to the Case **2**, if $y \neq x_y^*$ and $x = y_x^*$, we get a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

Case **4** Suppose $y \neq x_y^*$ and $x \neq y_x^*$, then $y < x_y^*$ and $x < y_x^*$. Consider the set $\{x \wedge y, x, y, y_x^*, x_y^*, 1\}$ of distinct elements of L. Now, as in Case **2**, we have $y_x^* \wedge y = x \wedge y$ and $x_y^* \wedge x = x \wedge y$. Also, $x_y^* \vee (x \vee y) = (x_y^* \vee y) \vee x = x_y^* \vee x$. Similarly, $y_x^* \vee (x \vee y) = y_x^* \vee y$.
Next, note that $y_x^* \vee y \neq x \vee y$. Indeed, if $y_x^* \vee y = x \vee y$, then the set $\{x \wedge y, x, y, y_x^*, x \vee y\}$ of distinct elements of L is a sublattice isomorphic to N_5. Similarly $x_y^* \vee x \neq x \vee y$.
Also, $y_x^* \vee y \neq y_x^*$. Indeed, if $y_x^* \vee y = y_x^*$, then $y \leq y_x^*$, a contradiction to the fact that $y \nleq y_x^*$. Similarly $x_y^* \vee x \neq x_y^*$.
Now, we have $x_y^* \vee x \neq x_y^*, y_x^* \vee y \neq y_x^*, y_x^* \vee y \neq x \vee y, x_y^* \vee x \neq x \vee y$ and the following subcases.

(4-I) Suppose $x_y^* \vee x = y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$, then $x \wedge y, x, y, x \vee y, x_y^*, y_x^*, y_x^* \vee x_y^*, y_x^* \wedge x_y^*, 1$ are distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

(4-II) Suppose $x_y^* \vee x \neq y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$. Then $y_x^*, x \vee y, y_x^* \wedge (x \vee x_y^*), x \vee x_y^*, y_x^* \vee x_y^*$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5.

(4-III) Suppose $x_y^* \vee x \neq y_x^* \vee y \neq y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$. We claim that $x \wedge x_y^* = x \wedge y$. Indeed, if $x \wedge x_y^* = z \neq x \wedge y$, then $x \wedge y, z, y_x^* \vee x_y^*, y$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5. Next, consider $x_y^* \wedge (x \vee y) = (x_y^* \wedge x) \vee (x_y^* \wedge y) = (x \wedge y) \vee y = y$ and similarly $y_x^* \wedge (x \vee y) = x$ which implies that $y_x^* \wedge x_y^* = x \wedge y$, a contradiction to the fact that $y_x^* \wedge x_y^* \neq x \wedge y$, so this case will not arise.

(4-IV) Suppose $x_y^* \vee x = y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x_y^*, y_x^*, y, y_x^* \vee x_y^*\}$ is a set of distinct elements and is a sublattice of L that is isomorphic to N_5.

(4-V) Suppose $x_y^* \vee x \neq y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x_y^*, y_x^*, y, y_x^* \vee x_y^*\}$ is a set of distinct elements and is a sublattice of L that is isomorphic to N_5.

(4-VI) Suppose $x_y^* \vee x \neq y_x^* \vee y \neq y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x, y, x \vee y, x_y^*, y_x^*, y_x^* \vee x_y^*, x_y^* \vee x, y_x^* \vee y, 1\}$ is a set of distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2. $\square$

Theorem 2.7 *A Heyting algebra L is in H_3 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-3.*

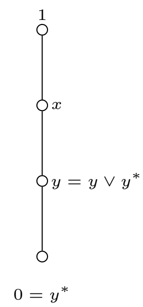

Figure 3:

Proof. If L is in H_3, then by Theorem 2.2, every sublattice of L is in H_3 and the lattice as depicted in the Figure-3 is not in H_3.

Conversely, suppose that $L \notin H_3$. Then there exists a pair of elements $x, y \in L$ such that $x \vee x^*_{y \vee y^*} \neq 1$. Now, we have $x \neq 1, x \neq 0, x \nleq y, y \neq 0, y^* \neq 1, x \nleq y^*, x \nleq y \vee y^*, x \neq y \vee y^*, x \neq y, x \neq y^*$. If either of these conditions is violated, then we get $x \vee x^*_{y \vee y^*} = 1$. If $y^* = 0$ and $y < x$, then $y = y \vee y^* = x^*_{y \vee y^*}$ and so $\{x, y^* = 0, y = y \vee y^*, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3. Suppose that $y^* \neq 0$ and $y \not< x$. Therefore $0, y, x, y^*, y \vee y^*, 1$ are distinct elements of L, and accordingly we have the following cases.

Case **1** Suppose $y \vee y^* < x$. Then $\{0, y, x, y^*, y \vee y^*, 1\}$ are distinct elements of L. Take $z = x^*_{y \vee y^*}$, then $\{0, z, x, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

Case **2** Suppose $(y \vee y^*) \parallel x$. We have the following subcases.

(**2-I**) Suppose $y < x$ and $y^* \parallel x$.

(**2-I-i**) If $y = x \wedge (y \vee y^*)$ and $x \wedge y^* = 0$, then $\{0, y, x, y^*, y \vee y^*, x \vee y^*, 1\}$ are distinct elements of L. Take $z = y \vee y^*$, then $\{0, z, x, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(**2-I-ii**) If $y \neq x \wedge (y \vee y^*)$ and $x \wedge y^* \neq 0$, then $\{0, y, x, y^*, y \vee y^*, x \vee y^*, x \wedge y^*, x \wedge (y \vee y^*), 1\}$ are distinct elements of L. Take $z = y \vee y^*, u = x \vee y^*$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(**2-I-iii**) If $y \neq x \wedge (y \vee y^*)$ and $x \wedge y^* = 0$, then $x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y \vee 0 = y$, which implies that $y = x \wedge (y \vee y^*)$ and this case will not arise.

(2-I-iv) If $y = x \wedge (y \vee y^*)$ and $x \wedge y^* \neq 0$, then $y = x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y \vee (x \wedge y^*)$, which implies that $x \wedge y^* \leq y$ and $x \wedge y^* = 0$ and this case will not arise.

(2-II) Suppose $y^* < x$ and $y \parallel x$. But then $x \wedge y \neq 0$.

(2-II-i) If $y^* \neq x \wedge (y \vee y^*)$, then $\{0, y, x, y^*, y \vee y^*, x \wedge y, x \wedge (y \vee y^*), y \vee x, 1\}$ are distinct elements of L. Take $z = y \vee y^*, u = x \vee y$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(2-II-ii) If $y^* = x \wedge (y \vee y^*)$, then $y^* = x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y^* \vee (x \wedge y)$, which implies that $x \wedge y \leq y^*$ and so $x \wedge y = 0$ and this case will not arise.

(2-III) Suppose $y^* \parallel x$ and $y \parallel x$. But then, $x \wedge y \neq 0$ and $x \parallel (y \vee y^*)$. Note that $y^* \vee (x \wedge y) < y \vee y^*$. Indeed, if $y^* \vee (x \wedge y) = y \vee y^*$, then the set of distinct elements, namely $\{0, y, y^*, x \wedge y, y \vee y^*\}$ forms a sublattice of L isomorphic to N_5. Now, $0, y, x, y^*, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), 1$ are distinct elements of L. We claim that $x \vee y, x \vee y^*, y \vee y^*, x \vee (y \vee y^*)$ are mutually distinct elements of L. If $x \vee y = x \vee y^* = y \vee y^* = x \vee (y \vee y^*)$, then $\{x, y, y^* \vee (x \wedge y), x \wedge y, x \vee (y \vee y^*)\}$ is a sublattice of L isomorphic to M_3. If $x \vee y = x \vee y^*$, then $(x \vee y) \vee y^* = (x \vee y^*) \vee y^* = x \vee y^*$, which is not possible. If $x \vee y = y \vee y^*$ or $x \vee y^* = y \vee y^*$, then $x \leq y \vee y^*$, a contradiction to the assumption. Thus $\{0, y, y^*, x, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), 1\}$ is a set of distinct elements of L. Now we have the following two subcases.

(2-III-i) Suppose $x \wedge [y^* \vee (x \wedge y)] = x \wedge y$. Note that $x \wedge y = x \wedge [y^* \vee (x \wedge y)] = (x \wedge y^*) \vee [x \wedge (x \wedge y)] = (x \wedge y^*) \vee (x \wedge y)$ which implies that $x \wedge y^* \leq x \wedge y$ and so $x \wedge y^* = 0$. Also, $y \vee (x \wedge y^*) = y \vee 0 = y$. Hence, $\{0, y, y^*, x, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), 1\}$ is a set of distinct elements of L. Take $z = y \vee y^*, u = x \vee (y \vee y^*)$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(2-III-ii) Suppose $x \wedge [y^* \vee (x \wedge y)] \neq x \wedge y$.
If $x \wedge [y^* \vee (x \wedge y)] \neq x \wedge y$, then $x \wedge y < x \wedge [y^* \vee (x \wedge y)]$. Note that $x \wedge y^* \neq 0$. Indeed, if $x \wedge y^* = 0$, then $x \wedge [y^* \vee (x \wedge y)] = (x \wedge y^*) \vee [x \wedge (x \wedge y)] = (x \wedge y^*) \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y$, a contradiction to the assumption. Therefore $\{0, y, y^*, x, x \wedge y, x \wedge (y \vee y^*), y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), x \wedge [y^* \vee (x \wedge y)], x \wedge y^*, 1\}$ is a set of distinct elements of L. Take $z = y \vee y^*, u = x \vee (y \vee y^*)$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3. $\square$

Theorem 2.8 *A Heyting algebra L is in H_5 if and only if it does not contain a sublattice isomorphic to one of the lattices as depicted in the Figure-2 or 3.*

Proof. If L is in H_5, then by Theorem 2.2, every sublattice of L is in H_5 and the lattices as depicted in the Figure-2 and 3 are not in H_5.

Conversely, suppose $L \notin H_5$. Then it is either not in H_3 or not in H_2. Now, since $L \notin H_3$, by Theorem-2.7, L contains a sublattice isomorphic to the lattice depicted in Figure-3. Also, since $L \notin H_2$, by Theorem-2.5, L contains a sublattice isomorphic to the lattice depicted in Figure-2. $\square$

Theorem 2.9 *A Heyting algebra L is in H_4 if and only if it does not contain a sublattice isomorphic to one of the lattices as depicted in the Figure-3 or 4.*

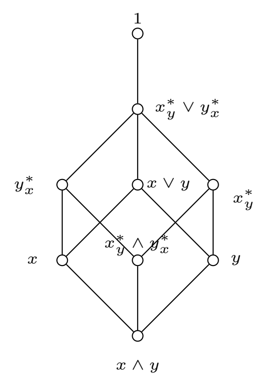

Figure 4:

Proof. If L is in H_4, then by Theorem 2.2 every sublattice of L is in H_4 and the lattices as depicted in the Figure-3 and 4 are not in H_4.

Conversely, suppose $L \notin H_4$. Then it is a either not in H_3 or there exists a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) \neq 1$. Suppose it is not in H_3, then by Theorem-2.7, L contains a sublattice isomorphic to the lattice depicted in Figure-3. Now, suppose it is in H_3, then we have a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) \neq 1$. Now, we have $x \nleq y, y \nleq x, x \neq y^*, y \neq 0, y \neq 1, x \neq 0, x \neq 1$. If either of these conditions is violated, then $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) = 1$. Also, we have $x \parallel y, x \parallel x_y^*, y \parallel y_x^*, x \parallel x_{y^*}^*, y^* \parallel (y^*)_x^*, y \leq x_y^*, x \leq y_x^*, y^* \leq x_{y^*}^*, x \leq (y^*)_x^*$. Therefore, $0, y, x, y^*, y \vee y^*, 1$ are distinct elements of L.

Case **1** Suppose $x < y^*$ and so $x \wedge y = 0, y^* \leq y_x^*$. Also, $x < y^*$ and so $x_{y^*}^* = 1 \neq y^*$.

(**1-I**) Suppose $y^* = y_x^*$. We have the following subcases.

(**1-I-i**) For $x \vee y = (y^*)^*_x$.

(**1-I-i-**A) If $y = x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, 1\}$ are distinct elements of L. Take $u = x \vee y, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**1-I-i-**B) If $y \neq x^*_y$, we claim that $x \vee y \parallel x^*_y$. Indeed, if $x \vee y \leq x^*_y$, then $x = x \wedge (x \vee y) \leq x \wedge (x^*_y) \leq y$, a contradiction to the fact $x \parallel y$. Now, if $x^*_y < x \vee y$, then $\{0, x, y, y \vee x, x^*_y\}$ is a set of distinct elements of L and is a sublattice of L isomorphic to N_5.

(**1-I-i-**B-a) If $x^*_y \wedge y^*_x = 0$, then $\{0, x, y, y \vee x, x^*_y, y^*_x\}$ is a set of distinct elements of L. If $x \vee x^*_y < x^*_y \vee y^*_x$, then $y^* \wedge (x \vee x^*_y) = x$ with $(y^*)^*_x < x \vee x^*_y$ which is not possible and so $x \vee x^*_y = x^*_y \vee y^*_x$. Now, if $y \vee y^*_x < x^*_y \vee y^*_x$, then the set of distinct elements, namely $\{y, x^*_y, x^*_y \vee y^*_x, x \vee y, y \vee y^*_x\}$ is a sublattice of L isomorphic to N_5 and so $x \vee x^*_y = y \vee y^*_x = x^*_y \vee y^*_x$, then the set of distinct elements, namely $\{0, y^*, x^*_y, y, y \vee y^*\}$ is a sublattice of L isomorphic to N_5. Hence this case will not arise.

(**1-I-i-**B-b) If $x^*_y \wedge y^*_x \neq 0$, then similar to previous Case (**1-I-i-**B-a) we have $x \vee x^*_y = y \vee y^*_x = x^*_y \vee y^*_x$, then the sublattice $\{0, y, y^* = y^*_x, x, x \vee y, x^*_y, x^*_y \vee y^*_x = y \vee y^*, 1\}$ of L is isomorphic to the lattice depicted in Figure-4.

(**1-I-ii**) If $x \vee y \neq (y^*)^*_x$, then $(x \vee y) \wedge y^* = (x \wedge y^*) \vee (y \wedge y^*) = x \vee 0 = x$ which implies $x \vee y \leq (y^*)^*_x$ and so $x \vee y < (y^*)^*_x$.

(**1-I-ii-**A) If $y = x^*_y$, then $(x \vee y) \wedge y^* = x$. We claim that $(y^*)^*_x \wedge (y \vee y^*) = x \vee y$. Indeed, if $z = (y^*)^*_x \wedge (y \vee y^*) > x \vee y$, then the set of distinct elements, namely $\{x, y^*, y \vee x, z, y \vee y^*\}$ is a sublattice of L isomorphic to N_5. We claim that $(y^*)^*_x \wedge y^* = x$. Indeed, if $z = (y^*)^*_x \wedge y^* > x$, then the set of distinct elements, namely $\{0, x, y, y \vee x, z\}$ is a sublattice of L isomorphic to N_5. Therefore $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, (y^*)^*_x \vee y^*_x, 1\}$are distinct elements of L. Take $u = x \vee y, v = (y^*)^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**1-I-ii-**B) For $y \neq x^*_y$.

(**1-I-ii-**B-a) If $(y^*)^*_x = (x \vee y) \vee x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, (y^*)^*_x \vee y^*_x, x^*_y, 1\}$ are distinct elements of L. Take $u = x \vee y, v = (y^*)^*_x \vee y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**1-I-ii-**B-b) If $(y^*)^*_x \neq (x \vee y) \vee x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, x \vee x^*_y, (y^*)^*_x \vee y^*_x, x^*_y, x^*_y \vee y^*_x, 1\}$ are distinct elements of L. Take $u = x \vee y, v = (y^*)^*_x \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**1-II**) If $y^* \neq y^*_x$, then $y^* < y^*_x$ and so $0 \leq y \wedge y^*_x \leq x$. If $y \wedge y^*_x = x$, then $x \leq y$, a contradiction to the fact $x \parallel y$ and so $y \wedge y^*_x \neq x$. Now, if $y \wedge y^*_x < x$, then $y \wedge (y \wedge y^*_x) \leq y \wedge x = 0$, which implies that $y \wedge (y \wedge y^*_x) = (y \wedge y) \wedge y^*_x = y \wedge y^*_x = 0$ and so $y^*_x \leq y^*$, a contradiction to the fact $y^* < y^*_x$. Hence this case will not arise.

Case **2** Suppose $y^* < x$ and consequently $x \wedge y \neq 0$ and $(y^*)^*_x = 1 \neq x$. Also, $y^* < x \leq y^*_x$ and $x \vee y \neq y^*_x$.

(**2-I**) For $y = x^*_y$, we have the following subcases.

(**2-I-i**) If $x \neq y^*_x$, then $\{0, y, y^*, x, x \vee y, y^*_x, y \vee y^*_x, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y \vee y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-I-ii**) If $x = y^*_x$, then $\{0, y, y^*, x, x \vee y, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-II**) Suppose $y < x^*_y$. We have the following subcases.

(**2-II-i**) If $x \neq y^*_x$, then $\{0, y, y^*, x, x \vee y, y^*_x, x^*_y, x \vee x^*_y, x^*_y \vee y^*_x, y \vee y^*_x, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-II-ii**) If $x = y^*_x$, then $\{0, y, y^*, x, x \vee y, x^*_y, x \vee x^*_y, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = x \vee y$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

Case **3** Suppose $x \parallel y, x \parallel y^*$ and so $x \wedge y \neq 0, y^* \leq y^*_x$ and $y \leq (y^*)^*_x$. We claim that $y^* < y^*_x$. Indeed, if $y^* = y^*_x$, then $x \leq y^*_x = y^*$, a contradiction to the fact $x \parallel y^*$. Similarly, $y = (y^*)^*_x$. Therefore, $0, 1, x, y, y^*, x \wedge y, y^*_x, (y^*)^*_x$ are distinct elements of L.

(**3-I**) Suppose $y \vee y^* = y^*_x \vee (y^*)^*_x$. We have the following subcases.

(**3-I-i**) If $x \wedge y^* \neq 0$, then the sublattice $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*\}$ are distinct elements of L. Take $u = x, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**3-I-ii**) If $x \wedge y^* = 0$, then the set of distinct elements, namely $\{0, x, y^*, x \wedge y, y^*_x\}$ is a sublattice of L isomorphic to N_5. So this case will not arise.

(**3-II**) Suppose $y \vee y^* \neq y^*_x \vee (y^*)^*_x$. We have the following subcases.

(**3-II-i**) If $x \wedge y^* \neq 0$, then $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*, y^*_x \vee (y^*)^*_x, (x \wedge y) \vee y^*, (x \wedge y^*) \vee y, x \wedge (y \vee y^*)\}$ are distinct elements of L. Take $u = x, v = (y^*)^*_x \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**3-II-ii**) If $x \wedge y^* = 0$, then $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*, y^*_x \vee (y^*)^*_x\}$ are distinct elements of L. Take $u = y \vee y^*, v = x{*}_y$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.
□

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