Cross Connection of Boolean Lattice

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Abstract

It is known that the cross connection semigroup of a complemented modular lattices is a strongly regular Baer semigroup and conversely if $S$ is a strongly regular Baer semigroup then $L_l, L_r$ the poset of principal left and right ideals of $S$ are dually isomorphic complemented modular lattices. In this paper we discuss the cross connection of complemented distributive lattice and showns that the cross connection determines a Boolean ring, which is a strongly regular Baer ring. Further it is also proved that the ideals of such a Boolean ring form a complemented distributive lattice.

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1. Introduction

A cross connection is a categorical duality which turns out to be very significant in the study of the structure of the objects under consideration. In [3] P.A.Grillet described the cross connection of partially ordered sets and obtained the fundamental regular semigroup as cross connection semigroup. K.S.S.Nambooripad and F.J.Pastijn together established the cross connection of a complemented modular lattice and obtaied the strongly regular Baer semigroup in [1]. Further in [5] K.S.S.Nambooripad generalized the cross connection theory to include arbitrary regular semigroups by replacing the partially ordered sets with normal categories. In the following we describe the cross
connection of boolean lattice and obtained its representation as a cross connection ring.

2. Preliminaries

A lattice is a partially ordered set (poset) $L$ such that every finite subset of $L$ has a least upper bound and a greatest lower bound called the join and meet respectively. A lattice is distributive if:

$$a \land (b \lor c) = (a \land b) \lor (a \land c), \quad a \lor (b \land c) = (a \lor b) \land (a \lor c), \quad \forall a, b, c \in L.$$ 

If a lattice $L$ contains a smallest (greatest) element, then it is uniquely determined and is denoted by $0$ ($1$). A lattice in which every subset has meet and join is a complete lattice.

Definition 1. Let $L$ be a distributive lattice, if for every $a \in L$ there exists an element $b \in L$ such that $a \land b = 0$ and $a \lor b = 1$, then $b$ is a complement of $a$. If every $a \in L$ has a complement then $L$ is a complemented distributive lattice (a Boolean lattice).

Definition 2. Let $A$ and $B$ be Boolean lattices. A (boolean) homomorphism is a mapping $f : A \to B$ such that, for all $p, q \in A$:

1. $f(p \land q) = f(p) \land f(q)$,
2. $f(p \lor q) = f(p) \lor f(q)$ and
3. $f(a^c) = f(a)^c$.

A boolean ring is a ring $(R, +, \cdot)$ with unit in which every element is a multiplicative idempotent.

Example 1. Let $(X, \Sigma)$ be a measurable space and $R = \{\chi_A : A \in \Sigma\}$, where $\chi_A$ denotes the characteristic function of $A$. Then $R$ together with Boolean operations

$$\chi_A \oplus \chi_B = \chi_{A \Delta B} \quad \text{and} \quad \chi_A \cdot \chi_B = \chi_{A \cap B} \quad \forall A, B \in \Sigma$$

and

$$A \Delta B = (A \cap B^c) \cup (A^c \cap B)$$

is a Boolean ring.

An element $a$ in a ring $R$ is regular (von Neumann regular) if there exists $x \in R$ such that $axa = a$. The ring $R$ is regular if every $a \in R$ is regular and $R$ is strongly regular if each $a$ satisfies $a = xa^2 = a^2x$ for some $x \in R$. A Boolean ring is von Neumann regular and strongly regular. The subset $E^l = \{y \in R \mid yx = 0\}, \forall x \in E$ is called the left annihilator of $R$ and $E^r = \{y \in R \mid xy = 0\}, \forall x \in E$ is the right annihilator.

Definition 3. A ring $R$ is said to be Baer if every left (equivalently, right) annihilator ideal is generated by an idempotent.

A strongly regular ring $R$ is Baer if and only if the Boolean lattice $B(R)$ of all idempotents in $R$ is complete.
Definition 4. An ideal in a partially ordered set \( P \) is a subset \( I \) of \( P \) such that \( x \leq y \in I \) implies \( x \in I \). The principal ideal \( P(x) \) of \( P \) generated by \( x \in P \) is \( P(x) = \{ y \in P \mid y \leq x \} \).

Definition 5. Let \( P \) and \( Q \) be two partially ordered sets. A mapping \( f : P \to Q \) is said to be normal mapping if it is order preserving, \( \text{im}f = Q(a) \) for some \( a \) in \( Q \) and \( \forall x \in P \), there exists \( z \leq x \) such that \( f|_{P(z)} \) is an isomorphism from \( P(z) \) onto \( Q(xf) \).

If \( f : P \to Q \) is normal, then there exists at least one element \( b \) in \( P \) such that \( f \) is an isomorphism of \( P(b) \) onto \( Q(a) = \text{im}f \) and we denote by \( M(f) \) the set of all elements \( b \in P \) with this property. The set of all normal mappings from \( P \) to \( P \), denoted by \( S(P) \) is a semigroup under composition. Idempotent normal mappings are called normal retractions and a principal ideal \( P(a) \) is called normal retract if \( P(a) = \text{im e} \) where \( e \) is some normal retraction. \( P \) is called regular poset if every principal ideal of \( P \) is a normal retract.

An equivalence relation \( \rho \) on a poset \( P \) is said to be normal if there exists a normal mapping \( f \in S(P) \) such that \( \text{ker}f = \rho \). The poset (under the reverse of inclusion) of all normal equivalences on \( P \) is denote by \( P^0 \) and \( P^0 \) is regular whenever \( P \) is. For each \( \rho \in P^0 \) we may associate the subset \( M(\rho) \) defined by \( M(\rho) = M(f) \), where \( f \) is any normal mapping with \( \text{ker}f = \rho \) and \( a \in M(\rho) \) if and only if \( P(a) \) intersects every \( \rho \) - class in exactly one element. Then \( P(a) \cap \rho(x) \) contains a single element which is minimal in its \( \rho \) - class and the mapping \( \varepsilon_p(\rho, a) \) which sends each \( x \) in \( P \) to the unique element in \( P(a) \cap \rho(x) \) is a normal retraction with \( \text{ker}\varepsilon_p(\rho, a) = \rho \) and \( \text{im}\varepsilon_p(\rho, a) = P(a) \). \( \varepsilon_p(\rho, a) \) is called the projection along \( \rho \) upon \( P(a) \).

If \( f \) is a normal mapping with \( \text{dom}f = P \) and \( \text{ker}f = \rho \) then for any \( a \in M(f) = M(\rho) \), the map \( f \) is factorized as \( f = \varepsilon_p(\rho, a)\alpha \) where \( \alpha = f|_{P(a)} \) is an isomorphism of \( P(a) \) onto \( \text{im}f \) and this factorization is called a normal factorization of \( f \). The following proposition is from [1].

Proposition 1. Let \( I \) and \( \Lambda \) be regular partially ordered sets and \( f : I \to \Lambda \) be a normal mapping. For \( \sigma \in \Lambda^0 \), define \( f^0(\sigma) = \ker(f\varepsilon_\Lambda(\sigma, u)) = \sigma f^{-1} \) where \( u \in M(\sigma) \). Then \( f^0 : \Lambda^0 \to I^0 \) is a normal mapping such that \( \text{im}f^0 = I^0(\text{ker}f) \) and \( M(f^0) = \{ \rho \in \Lambda^0 \mid b \in M(\rho) \} \), where \( \text{im}f = \Lambda(b) \). If \( P, Q \) and \( R \) are regular partially ordered sets and if \( f : P \to Q \) and \( g : Q \to R \) are normal mappings, then \( (fg)^0 = f^0g^0 \).

Let \( I \) and \( \Lambda \) be regular partially ordered sets \( \Gamma : \Lambda \to I^0 \) and \( \Delta : I \to \Lambda^0 \) are order preserving mappings, then \((f, g) \in N(I)^{op} \times N(\Lambda) \) is compatible with \((\Gamma, \Delta) \) if the following conditions hold:

(c1) \( \text{im}f = I(x) \), \( \text{im}g = \Lambda(y) \) \( \Rightarrow \) \( \text{ker} f = \Gamma(y) \), \( \text{ker} g = \Delta(x) \)

(c2) the diagram below commutes
Theorem 1. (Theorem 2 cf. [1]) Let $I$, $\Lambda$ be regular partially ordered sets and $\Gamma : \Lambda \to I^\circ$, $\Delta : I \to \Lambda^\circ$ be order preserving mappings. Then $[I, \Lambda; \Gamma, \Delta]$ is a cross connection if and only if the following conditions are satisfied:

1. $x \in M(\Gamma(y)) \iff y \in M(\Delta(x))$, $x \in I, y \in \Lambda$
2. If $x \in M(\Gamma(y))$ then the pair $(\varepsilon_I(\Gamma(y), x), \varepsilon_\Lambda(\Delta(x), y)$ is compatible with $(\Gamma, \Delta)$.

Remark 1. $U = U(I, \Lambda; \Gamma, \Delta)$ consisting of all the pairs $(f, g)$ that are compatible with $(\Gamma, \Delta)$ is a regular semigroup under the composition of mappings.

Definition 6. An order preserving mapping $f : P \to Q$ of posets is said to be residuated if there exists an order preserving mapping $f^+ : Q \to P$ such that $yf^+f \leq y$ and $x \geq ff^+f$ for all $x \in P$ and $y \in Q$. The mapping $f^+$ is called the residual of $f$.

The set $ResP$ of all residuated maps of $P$ is a semigroup and $f \rightarrow f^+$ is a dual isomorphism.

An $f \in ResP$ is totally range closed if $f$ maps principal ideals onto principal ideals. Observe that a residuated map that is also normal must be totally range closed, further $f \in ResP$ is strongly range closed if $f$ and $f^+$ are totally range closed. The set $B(P)$ of all strongly range closed transformations of $P$ is a sub semigroup of $ResP$. If $f \in ResP$ and if both $f$ and $f^+$ are normal, then $f$ is binormal and $f \in B(P)$.

3. Cross connection of Boolean lattice

Recall that an ideal of a boolean lattice $L$ is a subset $I$ of $L$ such that

1. $0 \in I$,
2. if $a \in I$ and $b \in I$, then $a \lor b \in I$, and
3. if $a \in I$ and $b \in L$, then $a \land b \in I$.

The principal ideal of $L$ generated by $a$ is $L(a) = \{b \in L : b \leq a\}$ and an ideal $I$ of $L$ is a complete ideal if $\{a_i\}$ is a family in $I$ with a supremum $a$ in $L$, then $a \in I$. Note that principal ideals are examples of complete ideals.

Theorem 2. (Theorem 21 cf. [2]) The class of all complete ideals in a Boolean algebra $L$ is itself a complete Boolean algebra with

1. $0 = \{0\}$,
2. $1 = L$,
3. $M \land N = M \cap N$,
4. $M \lor N = \bigcap\{I : I$ is a complete ideal in $L$ and $M \cup N \subseteq I\}$,
5. $M^c = \{p \in L : p \land q = 0$ for all $q \in M\}$. 
A subset of a Boolean lattice is a Boolean ideal if and only if it is an ideal in the corresponding Boolean ring [2].

**Proposition 2.** Let $L$ be a boolean lattice. For $a \in L$, let $a^c$ denotes the unique complement of $a$ and $f_a : L \to L$ and $f_a^+ : L^{op} \to L^{op}$ defined by

$$f_a(x) = x \land a$$

and

$$f_a^+(y) = y \lor a^c, \forall x \in L, y \in L^{op}.$$ 

Then $f_a$ and $f_a^+$ are normal mappings.

**Proof.** For $x \leq y$, in $L$, clearly $f_a(x) \leq f_a(y)$ and $\text{im } f_a = L(a)$, principal ideal generated by $a$. For every $x \in L$, there exists $z = x \land a \leq x$ such that $L(z) = L(f_a(x))$ and $f_a(z) = f_a(x)$. Thus $f_a$ acts as an identity morphism from $L(z) \to L(f_a(x))$. Thus $f_a$ is normal mapping. In a similar way it is seen that $f_a^+$ is also normal.

$f_a \in \text{Res}_L L$ is a binormal idempotent map which is residuated with residual $f_a^+ \in \text{Res}_L L^{op}$. Also $f_a$ and $f_a^+$ are totally range closed mappings such that $f_a \in B(L)$ and $f_a^+ \in B(L^{op})$. Further

$$ker f_a = \{(x, y) \mid x \land a = y \land a\} \text{ and } ker f_a^+ = \{(x, y) \mid x \lor a = y \lor a\}$$

**Theorem 3.** Let $L$ be a boolean lattice and $a \in L$. Define

$$\Gamma(a) = ker f_a \text{ and } \Delta(a) = ker f_a^{a^c}$$

so that $\Gamma, \Delta$ are order preserving embedding from $L \to (L^{op})^o$ and $L^{op} \to L^o$ respectively. Then $\Gamma(a) = \Delta(a^c)$ for all $a \in L$ and $[L, L^{op}; \Gamma, \Delta]$ is a cross connection.

**Proof.** We have $\text{im } f_a = L(a)$, $\text{im } f_a^+ = L^{op}(a^c)$, $ker f_a^+ = \Gamma(a)$, $ker f^+ = \Delta(a^c)$ and $\Gamma(a) = \Delta(a^c)$ hence $ker f_a = ker f_a^+$ and the following diagrams commutes.

$$L \xrightarrow{\Gamma} (L^{op})^o \quad L^{op} \xrightarrow{\Delta} L^o$$

$$f \uparrow \quad (f^+)^o \quad f^+ \quad f^o$$

$$L \xrightarrow{\Gamma} (L^{op})^o \quad L^{op} \xrightarrow{\Delta} L^o$$

For $(u, v) \in \Gamma(xf) \Rightarrow u \land xf = v \land xf, f^+(u \land xf) = f^+(v \land xf)$, then $f^+u \land x = f^+v \land x$ [follows from [6]]. Hence $(f^+u, f^+v) \in \Gamma(x)$ thus $(u, v) \in (f^+)^{-1}(\Gamma(x))$ thus

$$\Gamma(xf) \subseteq (f^+)^{-1}(\Gamma(x)).$$

Similarly, $(f^+)^{-1}(\Gamma(x)) \subseteq \Gamma(xf)$ and so $\Gamma(xf) = (f^+)^{o}\Gamma(x)$ and $\Delta(f^+y) = (\Delta y)f^o = (\Delta y)f^{−1}$, thus $(f, f^+)$ is compatible with $(\Gamma, \Delta)$.

For $a \in M(\Gamma(a))$ there exists $f_a$ such that $ker f_a = \Gamma(a)$ and $a \in M(f_a)$ that is $\text{im } f_a = L(a)$, principal ideal generated by $a$. Dually, $a^c \in M(\Delta(a^c))$ means there exists $f_a^+$ such that $ker f_a^+ = \Delta(a^c)$ and $a^c \in M(f_a^+)$, i.e., $\text{im } f_a^+ = L^{op}(a^c)$, principal ideal generated by $a^c$. Thus $a \in M(\Gamma(a)) \Leftrightarrow a^c \in M(\Delta(a^c))$, for every $a \in L$ and being binormal idempotent maps $(f, f^+)$ is compatible with $(\Gamma, \Delta)$. Hence $[L, L^{op}; \Gamma, \Delta]$ is a cross connection.
Theorem 4. Let $U = U(L, L^\text{op}; \Gamma, \Delta) = \{(f_a, f_a^+) \mid f_a \in B(L)\}$ the set of all pairs of normal maps $(f_a, f_a^+)$ with $f_a : L \to L, f_a^+ : L^\text{op} \to L^\text{op}$, that are compatible with $(\Gamma, \Delta)$. Clearly $U \subseteq B(L) \times B(L^\text{op})$ and $U$ together with product $\cdot$ and $\oplus$ defined by
\[ (f_a, f_a^+) \cdot (f_b, f_b^+) = (f_a \land b, f_a^+ \land f_b^+) \]
and
\[ (f_a, f_a^+) + (f_b, f_b^+) = (f_a \oplus b, f_a^+ \oplus f_b^+) \]
where $\oplus$ is the symmetric difference defined by $a \oplus b = (a \land b^c) \lor (b \land a^c)$. Then $U(L, L^\text{op}; \Gamma, \Delta)$ is a Boolean ring.

Proof. $(f_a, f_a^+) \cdot (f_a, f_a^+) = (f_a, f_a^+)$, i.e., $U$ is a band and $(f_0, f_0^+) \in U$ is the additive identity [since $L$ and $L^\text{op}$ are boolean rings, with additive identity $0$.]

Every element in $U$ is its own inverse, the addition in $U$ is commutative and for $(f_a, f_a^+), (f_b, f_b^+), (f_c, f_c^+) \in U$,
\[ (f_a, f_a^+) \cdot [(f_b, f_b^+) + (f_c, f_c^+)] = (f_a, f_a^+) \cdot (f_b \land c, f_b^+ \land c) \]
\[ = (f_a \land (b \land c), f_a^+ \land (b \land c)) \]
\[ = (f_a \land (b \land (a \land c)), f_a^+ \land (a \land c)) \]
\[ = (f_a \land (b \land a \land c), f_a^+ \land (f_a \land c)) \]
\[ = [(f_a, f_a^+) \cdot (f_b, f_b^+)] + [(f_a, f_a^+) \cdot (f_c, f_c^+)]. \]

and
\[ [(f_a, f_a^+) + (f_b, f_b^+)] \cdot (f_c, f_c^+) = [(f_a, f_a^+) \cdot (f_c, f_c^+)] + [(f_b, f_b^+) \cdot (f_c, f_c^+)]. \]
Hence $U$ is a Boolean ring. \qed

Note that $U$ is a Boolean lattice with respect to
\[ (f_a, f_a^+) \land (f_b, f_b^+) = (f_a, f_a^+) \cdot (f_b, f_b^+) \]
and
\[ (f_a, f_a^+) \lor (f_b, f_b^+) = (f_a, f_a^+) + (f_b, f_b^+) + (f_a, f_a^+) \cdot (f_b, f_b^+) \]

Lemma 1. $(f_a, f_a^+)^c = (f_a^c, f_a^c)^+$

Proof. $(f_a, f_a^+) \land (f_a^c, f_a^c)^+ = (f_0, f_0^+)$
and
\[ (f_a, f_a^+) \lor (f_a^c, f_a^c)^+ = (f_a \lor a^c, f_a^+ \lor a^c) + (f_0, f_0^+) \]
\[ = (f_0, f_0^+) \]
\[ = (0, 0^+) \]
\[ = (f_1, f_1) \]

hence $(f_a, f_a^+)^c = (f_a^c, f_a^c)^+$ \qed
Further, $U$ is a strongly regular Baer ring and the ideals $I_a = \langle (f_a, f_a^+) \rangle$ of $U$ are complete ideals generated by $(f_a, f_a^+)$. The complete ideals of $U$ form a complete Boolean lattice $L^*$, and the unique complement of $I_a$ is $I_{a^c}$, for every $a, a^c$ in $L$.

**Theorem 5.** Let $L$ be a boolean lattice and $U$ the cross connection ring obtained. Define a mapping $\psi : L \to U$ as $a \to \langle (f_a, f_a^+) \rangle, \forall a \in L$. Then $\psi$ is a representation. Further it is one-one and onto homomorphism. Hence $L \cong U$.

**Proof.** $\psi(a \wedge b) = (f_{a \wedge b}, f_{a \wedge b}^+) = (f_a, f_a^+) \wedge (f_b, f_b^+) = \psi(a) \wedge \psi(b)$
$\psi(a \oplus b) = (f_{a \oplus b}, f_{a \oplus b}^+) = (f_a \oplus f_b, f_a^+ \oplus f_b^+) = (f_a, f_a^+) \oplus (f_b, f_b^+) = \psi(a) \oplus \psi(b)$
and $\psi(a^c) = ((f_a, f_a^+)^c) = (f_a, f_a^+)^c = \psi(a)^c$

Thus $\psi$ is a boolean isomorphism and $L \cong U$. Similarly we get $U \cong L^*$.

Hence $L \cong U \cong L^*$.

**Example 2.** Let $X = \{a, b, c\}$, then $L = (P(X), \subseteq)$ is a boolean lattice with elements $\phi, A = \{a\}, B = \{b\}, C = \{c\}, A^c = \{b, c\}, B^c = \{a, c\}, C^c = \{a, b\}$ and $X$. For $S \subseteq L$, define $f_S : L \to L$ as $x \to \ x \cap S, \forall x \in L$ and $f_S^+ : L^d \to L^d$ as $x \to x \cup S^c, \forall x \in L^d$, where $L^d$ is the dual lattice of $L$. Then $\text{im} f_S = L(S)$ and $\text{im} f_S^+ = L^d(S^c)$, the principal ideals generated by $S$ and $S^c$ respectively. $f_S$ and $f_S^+$ are idempotent normal mappings and $f_S$ is a residuated mapping with residual $f_S^+$. $B(L) = \{f_S, \forall S \in L\}$ and $B(L^d) = \{f_S^+, \forall S \in L^d\}$.

\[
\begin{array}{ccc}
X & \downarrow & C^c \\
& C & B^c \downarrow & A^c \\
A & B & C \downarrow & \phi
\end{array}
\]

Define $\Gamma(S) = \ker f_S$ and $\Delta(S) = \ker f_S^+$ the normal equivalence relations. Let $L^d = \{\ker f_S | f_S \in B(L)\}$ and $(L^d)^d = \{\ker f_S^+ | f_S^+ \in B(L^d)\}$ then $\Gamma(S) = \Delta(S^c)$ and $(f_S, f_S^+)$ is compatible with $(\Gamma, \Delta)$. Further it is easily seen that the conditions of cross connection are satisfied. Hence $U = U([L, L^d; \Gamma, \Delta]) = \{((f_S, f_S^+) \in B(L) \times B(L^d)\} \text{ is the cross connection ring under the operations } (f_S, f_S^+) \cdot (f_T, f_T^+) = (f_{S\Delta T}, f_{S\Delta T}^+) \text{ and } (f_S, f_S^+) + (f_T, f_T^+) = (f_{S\oplus T}, f_{S\oplus T}^+). U$ is
a strongly regular Baer ring isomorphic to the powerset \((\{a, b, c\}, \subseteq)\).

\[
(f_X, f_X^\perp) \quad (f_C^c, f_C^c) \quad (f_B^c, f_B^c) \quad (f_A^c, f_A^c)
\]

\[
(f_A, f_A^\perp) \quad (f_B, f_B^\perp) \quad (f_C, f_C^\perp)
\]

\[
(f_\phi, f_\phi^\perp)
\]

Again the complete ideals \(I_S = \langle (f_S, f_S^\perp) \rangle\) of \(U\) for every \(S \in L\) form a Boolean lattice \(L^*\) below and is isomorphic to \(U\).

\[
I_X
\]

\[
I_C^c \quad I_B^c \quad I_A^c
\]

\[
I_A \quad I_B \quad I_C
\]

\[
I_\phi
\]

References


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