Nil Elements and Even Square Rings

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Abstract

In this article we introduce the notion of nil elements and even square rings and provide some examples. Nil elements are special type of nilpotent elements and possess interesting properties. Like nilpotent elements the set of all nil elements in a finite commutative ring forms an ideal of the ring. We provide some interesting results on commutative as well as non-commutative even square rings. It is noticed that if \( a \) is a unique non-zero nil element of a finite commutative even square ring \( R \) then \( a \) annihilates \( R \). In addition it is seen that each nil element of a finite commutative even square ring \( R \) does not necessarily annihilate \( R \) if it contains more than two nil elements.

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1 Introduction

The notion of nilpotent elements is very old and can be found in each textbook of modern algebra (one may refer [1-5]). In this article we introduce the notion of nil elements and even square rings and provide some examples.

It may be noted that nil elements are distinct from the zero divisors and nilpotent elements in the sense that every nil element is nilpotent however a nilpotent element is not necessarily a nil element. Similarly every non-zero nil element is a zero divisor but a zero divisor need not be a nil element.
It is worth to note that $Z_m = \{0,1,2,3,4,...,m-1\}$, the set of first $m$ non-negative integers is a commutative ring under addition and multiplication modulo $m$ and this ring contains a unique non-zero nil element for $m = 4(y + 1)$, where $y$ is any non-negative integer. But in $Z_m$ the non-zero nil element does not annihilate $Z_m$. However we notice that if $R$ is a finite commutative even square ring containing a unique nonzero nil element $a$ then $a$ annihilates $R$.

In addition we consider some examples of an even square ring $R$ containing more than two nil elements each nil element of which annihilates $R$. However it is noticed that each nil element of a finite commutative even square ring $R$ does not necessarily annihilate $R$ if it contains more than two nil elements.

### 2 Nil Elements and Even Square Rings

**Definition 1.** Let us call an element $a$ of a ring $R$ a nil element if $a + a = 0$ and $a.a = 0$. From this definition it is clear that every nil element is nilpotent but a nilpotent element need not be a nil element. It is also obvious that each non-zero nil element is a zero divisor however a zero divisor need not be a nil element.

**Definition 2.** An element $a$ of a ring $R$ is said to annihilate $R$ if $\{ax = xa = 0, \forall x \in R\}$.

**Definition 3.** Let us call an element $a$ of a ring $R$ an even element if $a = 2b$ for some $b \in R$.

**Definition 4.** Let $R$ be a ring. Let us call an element $a \in R$ an even square element if $a^2 = 2b$ for some $b \in R$.

**Definition 5.** Let us call a ring $R$ an even ring if every element of $R$ is an even element.

**Definition 6.** Let us call a ring $R$ an even square ring if every element of $R$ is an even square element.

**Remark.** Every even ring is an even square ring however an even square ring is not necessarily an even ring.

**Proposition 1.** If $a$ and $b$ are any non-nil nilpotent elements of a finite commutative ring $R$ containing nil elements then $a + b$ and $ab$ are not necessarily non-nil nilpotent elements. Refer example 2.

**Proposition 2a.** In a commutative ring $R$, the set of all nil elements forms an ideal of $R$. 
Proposition 2b. In a ring $R$, the set of all even elements forms an ideal of $R$.

Proposition 2c. In a commutative ring $R$, the set of all even square elements forms an ideal of $R$.

Corollary 1. If $R$ is a commutative ring and $a$ is a nil element of $R$ then for each $b \in R$, $ab$ is a nil element of $R$.

Proposition 3. Let $R$ be a finite even square ring (commutative or noncommutative) and let $R$ has $n > 2$ nil elements. Then $R$ has at least $n + 2$ ideals provided each nil element annihilates $R$.

Proof. Trivial. See example 5.

Remark. If each element of $R$ is a nil element and it contains $n > 2$ nil elements then $R$ has $n + 1$ ideals.

Proposition 4. There does not exist any noncommutative even square ring of order four.

Proof. Let $R$ be an even square ring of order four. If the order and characteristic of $R$ are equal, then it will be a commutative ring. If the order and characteristic of $R$ are not equal, then its characteristic will be two. But every even square ring of characteristic two is commutative. Hence there does not exist any noncommutative even square ring of order four.

Corollary 2. There does not exist any noncommutative even square ring whose every element is a nil element.

Corollary 3. There does not exist any noncommutative even square ring of characteristic two.

Proposition 5. Let $R$ be a finite even square ring of order $2^n, n \in \mathbb{Z}^+$ then each element of $R$ is nilpotent.

Proof. Trivial. See example 5.

Proposition 6. In a finite commutative even square ring $R$ of characteristic two every element is a nil element.

Remark: There is a finite commutative even square ring $R$ of characteristic two having $2^t$ elements for each positive integer $t$. This ring gives a finite commutative ring of order $2^t$ in which $x^2 = 0, \forall x \in R$ ($R$ is a vector space of dimension $t$ over $GF(2)$).
Proposition 7. Every finite commutative even square ring $R$ of even order contains a unique non-zero nil element provided its order and characteristic are equal.

Proof. Let $R$ be a finite commutative even square ring of even order. Let the order and characteristic of $R$ are equal. Clearly the additive group of $R$ is cyclic and there is a unique non-zero element $a$ in $R$ such that $a + a = 0$. This implies that $2a = 0$. If $a$ is an even element, then $a^2 = 0$. Therefore $a$ is a nil element. If $a$ is not an even element then since $a$ is an even square element therefore $a^2 = 2c$ for some $c \in R$. This gives $2a^2 = 4c$. Let $a^2 \neq 0$ then $2c \neq 0$. $4c = 0 \Rightarrow 2(2c) = 0$. But $a$ is the only element such that $a \neq 0$ and $2a = 0$. Hence $a^2 = 0$.

Proposition 8. Let $R$ be a finite commutative ring and its characteristic and order are equal. If it contains a unique non-zero nil element then $R$ is not necessarily an even square ring.

Note: $\{0, 1, 2, \ldots, m-1\}$ gives such a ring for each $m = 4(y + 1)$, where $y$ is any non-negative integer.

Proposition 9. Each nil element of a finite commutative even square ring $R$ containing a unique non-zero nil element annihilates $R$.

Proof. Let $R$ be an even square ring. Let $a$ is any nil element then $2a = 0$. If $a = 0$ then $ab = 0, \forall b \in R$. If $a \neq 0$ and $b = 0$ then $ab = 0$. If $a \neq 0$ and $b \neq 0$ and $R$ contains a unique non-zero nil element then $ab = 0$.

Proposition 10. Each nil element of a finite commutative even square ring $R$ whose every element is an even element annihilates $R$.

Proof. Trivial.

Corollary 4. If a non-zero element annihilates a ring $R$ then $R$ is a ring without identity.

Corollary 5. If a non-zero element annihilates a ring $R$ then every non-zero element of $R$ is a zero divisor.

Proposition 11. Let $R$ be a finite commutative even square ring containing more than two nil elements then each nil element of $R$ does not necessarily annihilate $R$. 
Proof. See example 6.

EXAMPLE 1.

\[ R = \begin{pmatrix}
0 & 0 & 2 & 0 & 3 & 0 & 0 & 2 & 0 & 3 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 2 & 2 \\
3 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 3 & 1 & 3 & 3 \\
2 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 1 & 3
\end{pmatrix} \]

One can verify that \( R \) is a finite commutative ring under matrix addition and multiplication modulo 4. Let \( a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \), \( b = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \) and \( c = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \) then \( a, b \) and \( c \) are non-zero nil elements of \( R \). \( R \) is not an even square ring. The set of all nil elements of \( R \) is given by \( N = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 2 & 2 & 0 & 2 \end{pmatrix} \).

EXAMPLE 2. Let

\[ R = \begin{pmatrix}
0 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 3 & 1 & 3 & 3 \\
0 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 3 & 1 & 3 & 3 \\
3 & 2 & 2 & 1 & 3 & 2 & 3 & 2 & 2 & 1 & 3 & 3 \\
1 & 3 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 1
\end{pmatrix} \]

Then \( R \) is a finite commutative ring under matrix addition and multiplication modulo 4. It is easy to see that \( a = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \), \( b = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \), \( c = \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix} \) and \( d = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix} \) are non-nil nilpotent elements of \( R \). One can verify that the sum and product of any two non-nil nilpotent elements of \( R \) are nil elements.

EXAMPLE 3. Let \( m \) be an even positive integer and \( R_m = \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in Z_m \), where \( Z_m = \{0, 1, 2, 3, 4,..., m-1\} \), is the set of first \( m \) non-negative integers. It is a ring under addition and multiplication modulo \( m \). One can easily verify that \( R_m \) is a finite commutative even square ring of order \( m \) under matrix addition and multiplication modulo \( m \). Let \( m = 2b \) then \( b \in Z_m \) and \( B = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \in R_m \) is the non-zero nil element of \( R_m \). It is easy to see that \( B \) annihilates \( R_m \).
EXAMPLE 4.

Let \( D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in R \right\} \). Here \( R \) be the set of all even square elements of \( \mathbb{Z}_m \) and \( 2n = m, m = 4(y+1) \), where \( y \) is any non-negative integer. One may see that \( D \) gives an even square ring of order \( n^2 \) under addition and multiplication of matrices modulo \( m \). This ring contains four nil elements and each nil element annihilates \( D \).

EXAMPLE 5. Let

\[
R = \left\{ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 6 & 0 & 0 & 6 \\ 0 & 6 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 4 & 2 & 0 & 4 \\ 6 & 2 & 0 & 6 \\ 4 & 6 & 0 & 4 \end{pmatrix} \right\}
\]

\( R \) is a non-commutative even square ring under addition and multiplication of matrices modulo \( 8 \) having the following properties.
1. \( R \) has 4 nil elements.
2. Each nil element of \( R \) annihilates \( R \).
3. \( R \) has at least 6 ideals
4. Each element of \( R \) is a nilpotent element.

EXAMPLE 6.

Let \( a \) and \( b \) are any two non-zero distinct nil elements in a commutative ring \( R \). Then \( R_6 = \{0, a, b, ab, a+b, b+ab, a+b+ab\} \) is a finite commutative even square ring of order 8 under the operations defined in \( R \). Each nil element of this ring does not annihilate the ring.

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References


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