Determining of a Finite Abelian $p$-Group up to Isomorphism

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Abstract

Analogically to the invariants of Ulm-Kaplansky we find a new complete system of invariants of a finite Abelian $p$-group. We come up with formulas which give the cut out of each of those invariant systems through the other. We prove that invariants we have found determine a finite Abelian $p$-group up to isomorphism. We give the necessary and sufficient condition for the existence of a finite Abelian $p$-group through all of values of those invariants. We come up with criterion, that given only some of the values of those invariants, guarantees the existence of a finite Abelian $p$-group, but in this case the group is not uniquely determined.

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1. Introduction

If $G$ is Abelian $p$-group, then the sets $G^{p^i} = \{g^{p^i} | g \in G\}$ and $G[p^i] = \{g \in G | g^{p^i} = 1\}$ are subgroups of the group $G$ for every $i \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integer. For those subgroups we have
$G^{p^{i+1}} \leq G^{p^{i}}$ and $G[p^i] \leq G[p^{i+1}]$.  

(1)

If $G$ is finite then there exists a natural number $k$, such that $G^{p^k} = 1$. Let $k$ be the smallest number with this property. Then the number $p^k$ is called the exponent of the group $G$ and we shall denote it with $\exp G$. The inclusions in (1) are strong if and only if $p^k < \exp G$.

Consider the factor-groups $G^{p^{i+1}}[p]/G^{p^i}[p]$ for each $i \in \mathbb{N}$. These factor-groups are elementary Abelian $p$-groups and therefore they are linear spaces over the Galois field $GF(p)$. Put $\alpha_i = \dim_{GF(p)}(G^{p^{i+1}}[p]/G^{p^i}[p])$, $i \in \mathbb{N}$.

The number $\alpha_i$ is called $i$-th Ulm-Kaplansky invariant of the group $G$. Let $r(G)$ be the rank of the Abelian $p$-group $G$ [1]. Then it holds $\alpha_i = r(G^{p^{i+1}}[p]/G^{p^i}[p])$, $i \in \mathbb{N}$. Therefore $\alpha_i$ is the number of direct factors of order $p^i$ in the direct decomposition of $G$ of cyclic $p$-subgroups.

I. Kaplansky and G. Mackey show [3] that a countable periodic reduced Abelian $p$-group is uniquely determined up to isomorphism by its invariants (2) for all primes $p$ and countable ordinals $i$. E. Walker proves [6] that the largest class of Abelian $p$-groups, for which the Ulm-Kaplansky invariants form a complete system of invariants, are totally-projective groups, introduced by R. Nunke [5]. For groups outside of this class this is not the case [2].

For the finite Abelian $p$-groups the Ulm-Kaplansky invariants not only do form a complete system of invariants, but they are also independent of each other. That means that if we chose $\alpha_1, \alpha_2, \ldots, \alpha_n$ to be arbitrary non-negative integers then there shall exists a finite Abelian $p$-group $G$, for which the chosen numbers shall be the Ulm-Kaplansky invariants of $G$. This group is unique up to isomorphism and for $\alpha_n \neq 0$ we have $\exp G = p^n$. This result has different generalisations (for example [4, 1]).

2. New complete system of invariants of a finite Abelian $p$-group

Let $G$ be a finite Abelian $p$-group, where $p$ is prime. Put

$$|G[p^i]| = p^{\beta_i}. \hspace{1cm} (3)$$

**Lemma 1.** As per (3) if $p^i \leq \exp G$, then $\beta_i \geq i$. An equality is achieved if and only if $G$ is a cyclic group.
The proof follows from (3) and from the second inequalities of (1).

In the next lemma we shall denote with $p^\alpha$ the exponent of the finite Abelian $p$-group $G$.

**Lemma 2.** The numbers $\beta_i$ from (3) are determined from the Ulm-Kaplansky invariants by

$$\beta_i = \sum_{j=1}^i j \alpha_j + i \sum_{j=i+1}^n \alpha_j = \alpha_1 + 2\alpha_2 + \ldots + i\alpha_i + i\alpha_{i+1} + \ldots + i\alpha_n$$

(4)

for each $i=1, 2, \ldots, n$ where $n = \log_p(\exp G)$.

**Proof.** Decompose $G$ in a direct product of cyclic $p$-groups. This decomposition implies a respective decomposition of $G[p^s]$. It contains all cyclic direct factors of $G$, whose orders do not exceed $p^s$. When $i < n$ each of the rest of the factors contain exactly one subgroup of order $p^i$. The number of these factors is equal to $s = \alpha_{i+1} + \alpha_{i+2} + \ldots + \alpha_n$ and the order of their direct product is $p^{s+i}$. The order of the direct product of the cyclic factors of $G$, whose orders do not exceed $p^i$ is $p^t$, where $t = \alpha_i + 2\alpha_2 + \ldots + i\alpha_i$. Then the order of $G[p^i]$ is $p^{t+i}$, from where (4) follows.

Lemma 2 gives an expression of the numbers $\beta_i$ by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Now we shall find $\alpha_i$, expressed by $\beta_1, \beta_2, \ldots, \beta_n$.

**Lemma 3.** For the numbers $\alpha_i$ from (4) we have

$$\alpha_i = 2\beta_i - \beta_2, \quad \alpha_i = 2\beta_i - \beta_{i-1} - \beta_{i+1} \text{ for } i = 2, 3, \ldots, n-1,$$

$$\alpha_n = \beta_n - \beta_{n-1}, \quad \alpha_{n+k} = 0 \text{ for } k \in \mathbb{N}. \quad (5)$$

**Proof.** For $i = 1$ formula (4) implies

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \beta_1,$$

and for $i = 2$ we have

$$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n = \beta_2.$$

Multiplying the first equality with 2 and subtracting the second equality from it we get $\alpha_1 = 2\beta_1 - \beta_2$. For $i = 2, 3, \ldots, n-1$ let us get from (4) the formulas for $i-1, i, i+1$. They are

$$\alpha_i + 2\alpha_2 + \ldots + (i-1)\alpha_{i-1} + (i-1)\alpha_i + \ldots + (i-1)\alpha_n = \beta_{i-1}.$$
\[ \alpha_1 + 2\alpha_2 + \ldots + i\alpha_i + i\alpha_{i+1} + \ldots + i\alpha_n = \beta_i; \]

\[ \alpha_1 + 2\alpha_2 + \ldots + (i+1)\alpha_{i+1} + (i+1)\alpha_{i+2} + \ldots + (i+1)\alpha_n = \beta_{i+1}. \]

Multiplying the second of these equalities with 2 and subtracting the first and the third equalities from the second we obtain \( \alpha_i = 2\beta_i - \beta_{i-1} - \beta_{i+1} \). Finally by subtracting from the equality

\[ \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = \beta_n \]

the equality

\[ \alpha_1 + 2\alpha_2 + \ldots + (n-1)\alpha_{n-1} + (n-1)\alpha_n = \beta_{n-1} \]

we obtain \( \alpha_n = \beta_n - \beta_{n-1} \). #

We know that in order to exist an Abelian \( p \)-group with invariants \( \alpha_1, \alpha_2, \ldots, \alpha_n \), these numbers can be arbitrary non-negative integers. We see from Lemma 3 that the numbers \( \beta_1, \beta_2, \ldots, \beta_n \) can not be arbitrary non-negative because some of \( \alpha_i \) could obtain negative values. Now we shall establish the conditions which the numbers \( \beta_i \) must satisfy so that there exists a finite Abelian \( p \)-group \( G \), for which \( [G[p^i]] = p^{\beta_i}, \quad i = 1, 2, \ldots, n = \log_p |G| \).

**Theorem 4.** There exists a finite Abelian \( p \)-group \( G \) with \( \exp G = p^n \) and \( |G[p^i]| = p^{\beta_i} \) if and only if the numbers \( \beta_1, \beta_2, \ldots, \beta_n \) satisfy the inequalities

\[ \beta_1 \geq \beta_2 \geq \beta_3 - \beta_2 \geq \ldots \geq \beta_n - \beta_{n-1} > 0. \quad (6) \]

**Proof.** Let there exists such a group \( G \) so that the Ulm-Kaplansky invariants of \( G \) are \( \alpha_1, \alpha_2, \ldots, \alpha_n \). Then they shall be determined by the formulas (5) of Lemma 3. Since the invariants \( \alpha_i \) are non-negative integers then (5) implies the inequalities (6).

Conversely if the inequalities (6) are satisfied, then (5) determine \( \alpha_i \) and we have \( \alpha_i \geq 0 \) for each \( i = 1, 2, \ldots, n-1 \) and \( \alpha_n > 0 \). Then for the group \( G \), determined by these Ulm-Kaplansky invariants, we have \( \exp G = p^n \) and \( |G[p^i]| = p^{\beta_i}, \quad i = 1, 2, \ldots, n. \) #

For the numbers \( \beta_i \) there is one more inequality, which we will need later.
Lemma 5. If $1 \leq i < j \leq n$ and the numbers $\beta_i, \beta_j$ are from (3), then $i \beta_j \leq j \beta_i$.

Proof. The formulas (4) imply

$$\beta_i = \alpha_i + 2\alpha_2 + \ldots + i\alpha_i + \ldots + i\alpha_n,$$

$$\beta_j = \alpha_i + 2\alpha_2 + \ldots + j\alpha_j + \ldots + j\alpha_n.$$  

Multiplying the first equality with $j$ and the second with $i$ and subtracting them we get

$$j\beta_i - i\beta_j = (j-i)\alpha_i + 2(j-i)\alpha_2 + \ldots + i(j-i)\alpha_i + i(j-i-1)\alpha_{i+1} + \ldots + i\alpha_{j-1} \geq 0.$$ 

3. Incomplete system of invariants of a finite Abelian $p$-group

Now we shall deduce a criterion that would ensure the existence of a finite Abelian $p$-group $G$, if only some of the values of $[G[p^l]]$ are given. To this aim we shall give the following definitions.

Definition 1. Let $m < n$ be natural numbers and let $\beta_m, \beta_{m+1}, \ldots, \beta_n$ be a system of natural numbers. We shall call this system normal of the first type if it satisfies the following inequalities

$$\frac{1}{m} \beta_m \geq \beta_{m+1} - \beta_m \geq \beta_{m+2} - \beta_{m+1} \geq \ldots \geq \beta_n - \beta_{n-1} > 0.$$  

Definition 2. Let $m < n$ be natural numbers and let $\beta_1, \beta_n, \beta_{n-1}, \ldots, \beta_n$ be a system of natural numbers. We shall call this system normal of the second type if it satisfies the following inequalities

$$\beta_1 \geq \frac{1}{m} \beta_m \geq \beta_{m+1} - \beta_m \geq \beta_{m+2} - \beta_{m+1} \geq \ldots \geq \beta_n - \beta_{n-1} > 0.$$ 

Theorem 6. Let $m < n$ be natural numbers and let a normal system of the first or the second type be given. Then there exists a finite Abelian $p$-group $G$, such that $[G[p^l]] = p^\beta$, where $\beta_i$ are the numbers of the given normal system.

Proof. 1). Let the given normal system be of the first type. From the first inequality in (7) we get $(m+1)\beta_m - m\beta_{m+1} \geq 0$. Consequently there exists
Abelian $p$-group $A$, for which $|A| = p^{(m+1)\beta_n - m\beta_{n+1}}$. Choose $A$ such that $\exp A \leq p^m$. This is possible because the limit of $\exp A$ above does not affect on the choice of $A$. Let now us make an Abelian group $B$, for which $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, $\alpha_i = 2\beta_i - \beta_{i-1} - \beta_{i+1}$ for $i = m+1, m+2, \ldots, n-1$ when $m+1 < n$, $\alpha_n = \beta_n - \beta_{n-1}$. If $m+1 = n$, then we put $\alpha_i = 0$ for $i \leq n-1$. In view of (7) these settings are possible. Now let us put $G = A \times B$. It can be immediately verified that the group $G$ satisfies the conditions of the theorem.

2) Now let the normal system be of the second type. From the first two inequalities of (8) follows $\beta_1 + \beta_m - \beta_{m+1} \geq 0$. Then there exists an Abelian $p$-group $A$, for which $|A[p]| = p^{\beta_1 + \beta_m - \beta_{m+1}}$ and $\exp A \leq p^m$. The maximum order of such group is $p^s$, where $s = m(\beta_1 + \beta_m - \beta_{m+1})$. We put $t = (m+1)(\beta_m - m\beta_{m+1})$. From the first inequality of (8) we have $s \geq t$ and from the second we have $t \geq 0$. Then $A$ can be chosen such that $|A| = p^t$. Further we choose an Abelian group $B$ as in case 1). This is possible, because all the inequalities in (7) participate in (8). Then the group $G = A \times B$ satisfies the required conditions.

4. Discussion

The group $G$, defined in the proof of Theorem 6 is not unique, because $A$ is not determined uniquely. In case 1) $A$ will be unique if and only if in the first inequality of (7) we have equality and then we obtain $A = 1$. In case 2) $A$ is unique if and only if in the first two inequalities of (8) we have equality and then we obtain $A = 1$.

References


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