Abstract

In this paper, we use specific associated hyperplanes in $\mathbb{R}^n$ related to the study of some metric problems. We intend to keep some geometric flavour when working with projections onto lines and hyperplanes. We give special emphasis to the distance from a point to a hyperplane which will be the main tool for finding the distance between two lines in $\mathbb{R}^n$.

Mathematics Subject Classification: 51N10, 51N20

Keywords: projection, distance, inner product, hyperplane associated to a point and a line, hyperplane associated to two skew lines, best approximation pair

1 Introduction

The main concern of this paper is to let be known to a larger audience the idea of Ruiz [6] related to the problems of distances in $\mathbb{R}^n$, for $n \in \mathbb{N}$. Ruiz, in her book [6], uses specific hyperplanes in $\mathbb{R}^n$ associated to the study of some metric
problems, namely, the distance from a point to a line, the distance from a point to a hyperplane, the distance between two lines and the distance between two hyperplanes. Ruiz, in [6], extended to the space $\mathbb{R}^n$ the approaches made by F.G-M. in [4] and Nieweglowski in [5] for similar problems in ordinary space $\mathbb{R}^3$.

We consider the space $\mathbb{R}^n$ endowed with the standard inner product $\cdot$ and with the correspondent norm generated by this inner product. Also we present, in terms of inner product, the best approximation pair of two skew lines in the space $\mathbb{R}^n$.

All results were inspired in [6] and the authors only formalized them. In [2], another approach is found by using the double vector cross product in $\mathbb{R}^7$.

We use two main tools: the hyperplane associated to a point and a line; and the hyperplane associated to two lines.

Some abuse of notations, authorized by adequate isomorphisms, is in this text. The standard basis in $\mathbb{R}^n$ is $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$. For points and vectors in $\mathbb{R}^n$, we put

$$\vec{m} := (m_1, m_2, \ldots, m_n) = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}.$$  

With the standard inner product $\cdot$, we have, for the inner product of two vectors $\vec{m} \cdot \vec{q} = \sum_{i=1}^n m_i q_i$, being $||\vec{m}|| = \sqrt{\vec{m} \cdot \vec{m}}$ the associated Euclidean norm.

The Gram determinant appears several times in this text. We define the Gram determinant of two vectors $\vec{m}, \vec{q} \in \mathbb{R}^n$, as

$$G(\vec{m}, \vec{q}) := \det \begin{bmatrix} \vec{m} \cdot \vec{m} & \vec{m} \cdot \vec{q} \\ \vec{q} \cdot \vec{m} & \vec{q} \cdot \vec{q} \end{bmatrix}.$$  

The Gram determinant is zero if and only if the vectors $\vec{m}, \vec{q}$ are linearly dependent, [[7], page 125, Theorem 4], [[3], page 129, Lemma 7.5].

We discuss orthogonality in various parts of this text. Two vectors $\vec{a}, \vec{b}$ are said to be orthogonal, denoted by $\vec{a} \perp \vec{b}$, if $\vec{a} \cdot \vec{b} = 0$. We say that the set $\mathcal{A}$ is orthogonal to a set $\mathcal{B}$ if all vectors $\vec{a}$ of $\mathcal{A}$ are orthogonal to all vectors $\vec{b}$ of $\mathcal{B}$. [[7], page 128 and page 152].

The concept of orthogonal projection is important in this study. The vector $\vec{s}$ of the linear variety $\mathcal{M} = \vec{m} + \mathcal{M}_0$ is the orthogonal projection of a vector $\vec{q} \in \mathbb{R}^n$ onto the variety $\mathcal{M}$ if and only if the vector $\vec{q} - \vec{s}$ is orthogonal to the associate subspace $\mathcal{M}_0$ of $\mathcal{M}$, [[3], page 215, Theorem 9.26]. Note that the vector $\vec{q} - \vec{s}$ is not orthogonal to the variety $\mathcal{M}$.

In this paper for the sake of simplicity of language, we propose the following concept related with the orthogonality between sets of vectors.

**Definition 1.1** 1. A vector $\vec{a}$ is said to be perpendicular to the linear variety $\mathcal{V} = \vec{v} + \mathcal{M}$ if $\vec{a}$ is orthogonal to the subspace $\mathcal{M}$;
2. Two linear varieties $V_1 = \mathbf{v}_1 + \mathcal{M}_1$ and $V_2 = \mathbf{v}_2 + \mathcal{M}_2$ are said to be perpendicular whenever the respective associated subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal.

We define the distance $d(\mathbf{p}, \mathcal{A})$ from a point $\mathbf{p}$ to a closed set $\mathcal{A}$ as

$$d(\mathbf{p}, \mathcal{A}) := \min\{d(\mathbf{p}, \mathbf{a}) : \mathbf{a} \in \mathcal{A}\} := \min\{||\mathbf{p} - \mathbf{a}|| : \mathbf{a} \in \mathcal{A}\}. $$

This paper is organized in the following way: in the Section 2 some auxiliary concepts and results are presented in order to facilitate the statements and proofs in the rest of the paper. Also the orthogonal projection of a vector onto a non zero vector is referred to, and then we deal with particular cases of hyperplanes in $\mathbb{R}^n$ – the hyperplane associated to a point and a line and the hyperplane associated to two skew lines – which will be a useful tool in the metric problems considered in the next section; in the Section 3 we refer the orthogonal projection of a point onto a line, the distance from a point to a hyperplane, the distance from a point to a line, the best approximation pair of two skew lines and the distance between two skew lines; in the Section 4 we write some final remarks and conclusions.

2 Auxiliary technical concepts and results

Some detail is given to the associated hyperplane to a line and to two lines.

2.1 Orthogonal projection of a vector onto a non zero vector

We show how we can derive the notion of orthogonal projection from the notion of orthogonality between two vectors. We start with the definition of a projection of a vector onto another vector.

From [[6], page 164-165] we may write the following

**Definition 2.1** In $\mathbb{R}^n$, given the vectors $\mathbf{v}$ and $\mathbf{u} \neq 0$, we define a new vector $\mathbf{p}$, called projection of $\mathbf{v}$ onto $\mathbf{u}$ (or onto direction of $\mathbf{u}$), as the vector that satisfies the following two conditions:

1. $(\mathbf{v} - \mathbf{p})$ is orthogonal to $\mathbf{u}$;
2. $\mathbf{p}$ has the same direction of $\mathbf{u}$.

In fact, the vector $\mathbf{p}$ is well defined by these conditions, because if $\mathbf{p} = k \mathbf{u}$ for some $k \in \mathbb{R}$, by linearity of the inner product, we get

$$(\mathbf{v} - k \mathbf{u}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - k \mathbf{u} \cdot \mathbf{u} = 0$$
and then \( k = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2} \). Therefore

\[
\mathbf{p}^* = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2} \mathbf{u}.
\]  

(1)

Note that the norm \( d \) of the vector \( \mathbf{p} \) is given by

\[
d = ||\mathbf{p}|| = \frac{||\mathbf{v} \cdot \mathbf{u}||}{||\mathbf{u}||}.
\]  

(2)

It is easy to show the following result that it is true for a general vector space with a inner product.

**Lemma 2.2** Let \( V \) a vector space with an inner product \( \bullet \) and \( \mathbf{u}, \mathbf{v} \in V \). If \( \mathbf{u} \neq \mathbf{0} \), then there exist a unique multiple \( k \mathbf{u} \) such that the vector \( \mathbf{v}^* = \mathbf{v} - k \mathbf{u} \) is orthogonal to \( \mathbf{u} \).

**Proof.** We want to solve the equation \( \mathbf{v} - k \mathbf{u} \bullet \mathbf{u} = 0 \). Using the well-known properties of the inner product, we see that the previous equation is equivalent to having \( \mathbf{v} \bullet \mathbf{u} - k(\mathbf{u} \bullet \mathbf{u}) = 0 \). Since \( \mathbf{u} \bullet \mathbf{u} \) is positive, we obtain

\[
k = \frac{\mathbf{v} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}}.
\]

and the result follows. □

### 2.2 Hyperplane associated to a point and a line

The concept of hyperplane associated to a point and a line plays an important role in this paper. We need for defining this hyperplane the concept of normal line common to a point and a line.

Recall that in \( \mathbb{R}^n \), a hyperplane containing the point \( \mathbf{a} \) and is perpendicular to the vector \( \mathbf{u} \), denoted by \( H(\mathbf{a}, \mathbf{u}) \), can be defined by the following identity

\[
H(\mathbf{a}, \mathbf{u}) := (\mathbf{x} - \mathbf{a}) \cdot \mathbf{u} = 0.
\]  

(3)

From [\[6\], page 165-166] we may write the following

**Definition 2.3** Let \( \mathbf{a} \) be an external point to the line \( r \) given by

\[
\mathbf{x}^* = \mathbf{p} + \alpha \mathbf{u}, \quad \alpha \in \mathbb{R}.
\]

Drop a perpendicular line \( s \) from point \( \mathbf{a} \) to line \( r \). We denote by \( \mathbf{q} \) the intersection point of lines \( r \) and \( s \). The normal line common to the point \( \mathbf{a} \) and the line \( r \) is the line \( s \) which contains the point \( \mathbf{q} \in r \) and is perpendicular to the line \( r \).
Remark 2.4 In fact, $\vec{q}$ has the same direction of $\vec{u}$ and then $\vec{q} = \vec{p} + k \vec{u}$, where $k = \frac{(\vec{a} - \vec{p}) \cdot \vec{u}}{||\vec{u}||^2}$.

Proposition 2.5 In $\mathbb{R}^n$, let $\vec{a}$ be an external point to the line $r$ defined by 
$$\vec{x} = \vec{p} + \alpha \vec{u}, \ \alpha \in \mathbb{R}.$$ 
Let $\vec{s}$ be the projection of the vector $\vec{v} = \vec{a} - \vec{p}$ onto $\vec{u}$. Then, the normal line common to the point $\vec{a}$ and the line $r$ is the line $l$ perpendicular to the line $r$ and given by 
$$\vec{x} = \vec{q} + \beta (\vec{v} - \vec{s}),$$
for some $\vec{q} \in r$.

Proof. By Definition 2.1, $\vec{s}$ has the same direction of $\vec{u}$ with $(\vec{v} - \vec{s}) \perp \vec{u}$. By (1), $\vec{s} = \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||^2} \vec{u}$. Consider the norm $d$ of $\vec{s}$ given by the equality (2) and let $\vec{q} = \vec{p} + d \vec{u}$. Clearly, $\vec{q} \in r$ and consider the line $l$ passing through $\vec{q}$ whose director vector is $\vec{u} - \vec{s}$. Then by Definition 2.3, $\vec{q}$ is the intersection point of the lines $r$ and $l$ and $l$ is the required normal line common to the point $\vec{a}$ and the line $r$. □

Remark 2.6 The point $\vec{q}$ is the intersection of line $r$ and hyperplane $H(\vec{a}, \vec{u})$ that contains the point $\vec{a}$ and is perpendicular to the director vector $\vec{u}$ of the line $r$. Then by (3),
$$\vec{q} = r \cap H(\vec{a}, \vec{u}) = \{ \vec{x} = \vec{p} + \alpha \vec{u}, \ (\alpha \in \mathbb{R}), \ (\vec{x} - \vec{a}) \cdot \vec{u} = 0 \}.$$ 
By Definition 2.3 and Proposition 2.5, we have

Proposition 2.7 Let $\vec{a}$ be an external point to the line $r$ given by 
$$\vec{x} = \vec{p} + \alpha \vec{u}, \ \alpha \in \mathbb{R}.$$ 
The normal line common to the point $\vec{a}$ and the line $r$ is the line $s$ given by 
$$\vec{x} = \vec{a} + \beta (\vec{a} - \vec{q}), \ \beta \in \mathbb{R},$$
where 
$$\vec{q} = \vec{p} + \frac{(\vec{a} - \vec{p}) \cdot \vec{u}}{||\vec{u}||^2} \vec{u}.$$ 
(4)

Proof. By Definition 2.3, let $\vec{v} = \vec{a} - \vec{p}$ and by Definition 2.1 and the equality (1), the projection of $\vec{v}$ onto $\vec{u}$ is given by $\frac{\vec{v} \cdot \vec{u}}{||\vec{u}||^2} \vec{u}$. The director vector of the line $s$ is the vector $\vec{v} - \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||^2} \vec{u}$ and, by Lemma 2.2, this vector is orthogonal to $\vec{u}$. The point $\vec{q}$ is the point of the intersection of line $s$ and line $r$ and so $\vec{q} = \vec{p} + \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||^2} \vec{u}$. □

Next we present the concept of a hyperplane associated to a point and a line given by Ruiz in [6]. From [[6], page 166] we may write the following
Definition 2.8 In $\mathbb{R}^n$, consider a point $\overrightarrow{a}$ and a line $r$. The hyperplane associated to the point $\overrightarrow{a}$ and the line $r$, denoted by $H^\ast(\overrightarrow{a}, r)$ is defined as being the hyperplane which contains a point $\overrightarrow{p}$ of $r$ and is perpendicular to the normal line common to the point $\overrightarrow{a}$ and the line $r$.

Proposition 2.9 Let $\overrightarrow{a}$ be an external point to the line $r$ given by $\overrightarrow{x} = \overrightarrow{p} + \alpha \overrightarrow{u}$, $\alpha \in \mathbb{R}$.

The hyperplane associated to the point $\overrightarrow{a}$ and the line $r$ is given by

$$(\overrightarrow{x} - \overrightarrow{p}) \cdot (\overrightarrow{a} - \overrightarrow{q}) = 0,$$

where

$$\overrightarrow{q} = \overrightarrow{p} + \frac{(\overrightarrow{a} - \overrightarrow{p}) \cdot \overrightarrow{u}}{||\overrightarrow{u}||^2} \overrightarrow{u}. \quad (5)$$

Proof. By Definition 2.8, Proposition 2.7 and using (3) the result follows immediately. $\square$

2.3 Hyperplane associated to two skew lines

The concept of hyperplane associated to two skew lines serves the main role when studying the distance between two skew lines. For defining this hyperplane, we need the concept of normal line common to two lines. From [[6], page 168] and [[4], page 296, Definition 415], we may write

Definition 2.10 In $\mathbb{R}^n$, let us consider two skew lines $r_1$ and $r_2$ defined, respectively, by

$\overrightarrow{x} = \overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1$ and $\overrightarrow{x} = \overrightarrow{p}_2 + \beta \overrightarrow{u}_2$.

Then the normal line common to the lines $r_1$ and $r_2$ is the line $l$ that intersects $r_1$ and $r_2$ and is perpendicular to $r_1$ and $r_2$.

Remark 2.11 In [[4], page 297], we find, in the case of the ordinary vector space $\mathbb{R}^3$, two different geometric constructions of a normal line common to two lines. Also, for the case of $\mathbb{R}^3$, in [[5], pages 73 – 75] the author gives equations of the normal line common to two skew lines.

Proposition 2.12 Let $r_1$ and $r_2$ be two skew lines given, respectively, by $\overrightarrow{x} = \overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1$ and $\overrightarrow{x} = \overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2$, $\alpha_1$, $\alpha_2 \in \mathbb{R}$.

The normal line common to the lines $r_1$ and $r_2$ is the line $l$ given either by

$\overrightarrow{x} = \overrightarrow{p}_1 + \gamma (\overrightarrow{p}_1 - \overrightarrow{p}_2)$, $\gamma \in \mathbb{R}$

or by
Best approximation pair of two skew lines

\[ \overrightarrow{x} = \overrightarrow{p}_2 + \delta(\overrightarrow{p}_1 - \overrightarrow{p}_2), \quad \delta \in \mathbb{R}, \]

where \( \overrightarrow{p}_1 = \overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1, \overrightarrow{p}_2 = \overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2 \) with

\[
\alpha_1^* = \frac{\begin{vmatrix} \overrightarrow{u}_1 \cdot \overrightarrow{u}_2 & ||\overrightarrow{u}_2||^2 \end{vmatrix}}{G(\overrightarrow{u}_1, \overrightarrow{u}_2)} \quad \text{and} \quad \alpha_2^* = \frac{\begin{vmatrix} ||\overrightarrow{u}_1||^2 & (\overrightarrow{p}_2 - \overrightarrow{p}_1) \cdot \overrightarrow{u}_2 \\ \overrightarrow{u}_1 \cdot \overrightarrow{u}_2 & ||\overrightarrow{u}_1||^2 \end{vmatrix}}{G(\overrightarrow{u}_1, \overrightarrow{u}_2)}.
\]

**Proof.** The following system

\[
\begin{cases}
((\overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1) - (\overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2)) \cdot \overrightarrow{u}_1 = 0 \\
((\overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1) - (\overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2)) \cdot \overrightarrow{u}_2 = 0
\end{cases}
\]
gives us the perpendicularity referred to in the Definition 2.10. Since \( r_1 \) and \( r_2 \) are skew lines, the vectors \( \overrightarrow{u}_1 \) and \( \overrightarrow{u}_2 \) are linearly independent and then the system has a unique solution. The Gram determinant \( G(\overrightarrow{u}_1, \overrightarrow{u}_2) \) is different from zero ([3], page 129). Then the points \( \overrightarrow{p}_1 \in r_1 \) and \( \overrightarrow{p}_2 \in r_2 \) are unique. \( \square \)

In what follows we deal with the concept of a hyperplane associated to two skew lines given by Ruiz. From [[6], page 168-169] we may write

**Definition 2.13** In \( \mathbb{R}^n \), let us consider two skew lines lines \( r_1 \) and \( r_2 \). The hyperplane associated to \( r_1 \) and \( r_2 \), denoted by \( H^*(r_1, r_2) \), is defined as being the hyperplane that contains a point \( \overrightarrow{p}_1 \) of \( r_1 \) and is perpendicular to the normal common line to \( r_1 \) and \( r_2 \).

According to Definition 2.13, the hyperplane associated to the pair \( (r_1, r_2) \) is different from the hyperplane associated to the pair \( (r_2, r_1) \).

**Proposition 2.14** Let \( r_1 \) and \( r_2 \) be two skew lines given, respectively, by

\[ \overrightarrow{x} = \overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1 \quad \text{and} \quad \overrightarrow{x} = \overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2, \quad \alpha_1, \alpha_2 \in \mathbb{R}. \]

The hyperplane \( H^*(r_1, r_2) \) associated to the skew lines \( r_1 \) and \( r_2 \) is defined by the equation

\[ (\overrightarrow{x} - \overrightarrow{p}_1) \cdot (\overrightarrow{p}_1 - \overrightarrow{p}_2) = 0. \]

The hyperplane \( H^*(r_2, r_1) \) associated to the skew lines \( r_2 \) and \( r_1 \) is defined by the equation

\[ (\overrightarrow{x} - \overrightarrow{p}_2) \cdot (\overrightarrow{p}_1 - \overrightarrow{p}_2) = 0, \]

where \( \overrightarrow{p}_1 = \overrightarrow{p}_1 + \alpha_1 \overrightarrow{u}_1, \overrightarrow{p}_2 = \overrightarrow{p}_2 + \alpha_2 \overrightarrow{u}_2 \) with

\[
\alpha_1^* = \frac{\begin{vmatrix} \overrightarrow{u}_1 \cdot \overrightarrow{u}_2 & ||\overrightarrow{u}_2||^2 \end{vmatrix}}{G(\overrightarrow{u}_1, \overrightarrow{u}_2)} \quad \text{and} \quad \alpha_2^* = \frac{\begin{vmatrix} ||\overrightarrow{u}_1||^2 & (\overrightarrow{p}_2 - \overrightarrow{p}_1) \cdot \overrightarrow{u}_1 \\ \overrightarrow{u}_1 \cdot \overrightarrow{u}_2 & ||\overrightarrow{u}_1||^2 \end{vmatrix}}{G(\overrightarrow{u}_1, \overrightarrow{u}_2)}.
\]
3 Distances

In this section, as we already mentioned an important role is played by a specific instance of the hyperplane associated to a point and a line and the hyperplane associated to two skew lines. The orthogonal projection of a point onto a line also comes into play.

3.1 Orthogonal projection of a point onto a line

Formulae (4) and (5) are expressions of orthogonal projections of a point onto a line.

**Proposition 3.1** Let $\vec{p}$ be an external point to the line $l$ given by $\vec{x} = \vec{m} + \alpha \vec{u}$. Let $\vec{s}$ be the orthogonal projection of the point $\vec{p}$ onto the line $l$. Then,

$$\vec{s} = \vec{m} + \frac{(\vec{p} - \vec{m}) \cdot \vec{u}}{||\vec{u}||^2} \vec{u}. \quad (6)$$

**Proof.** Let us consider $l$ as a line in $\mathbb{R}^n$, where $\vec{m}$ is a given point of $l$ and $\vec{u}$ a director vector of $l$. Take $\vec{p}$ an external point to the line $l$. If $r$ is the unique line containing $\vec{p}$ which is perpendicular to $l$, we obtain the point $\vec{s}$ which is the orthogonal projection of $\vec{p}$ onto $l$. Hence the point $\vec{s}$ belongs to $l$ and satisfies the equation $\vec{s} = \vec{m} + \beta \vec{u}$, for some $\beta \in \mathbb{R}$. Next we find an expression for the scalar $\beta$. By Lemma 2.2, we get $\beta = \frac{\vec{u} \cdot (\vec{p} - \vec{m})}{\vec{u} \cdot \vec{u}}$ and the result follows. □

3.2 Distance from a point to a hyperplane

The distance from a point $\vec{p}$ to a hyperplane

$$\pi = H(\vec{m}, \vec{u}) := (\vec{x} - \vec{m}) \cdot \vec{u} = 0,$$

is the norm of the vector $(\vec{p} - \vec{s})$ where $\vec{s}$ is the projection of the point $\vec{p}$ onto the hyperplane $H(\vec{m}, \vec{u})$.

The point $\vec{s}$ is obtained from $l \cap H(\vec{m}, \vec{u})$ where the line $l$, perpendicular to $H(\vec{m}, \vec{u})$, is given by $\vec{x} = \vec{p} + \beta \vec{u}$.

The projection $\vec{s}$ is given in the following

**Lemma 3.2** Let $\vec{p}$ an external point to the hyperplane

$$\pi = H(\vec{m}, \vec{u}) := (\vec{x} - \vec{m}) \cdot \vec{u} = 0.$$

Then the projection $\vec{s}$ of $\vec{p}$ onto $H(\vec{m}, \vec{u})$ is given by

$$\vec{s} = \vec{p} + \frac{(\vec{m} - \vec{p}) \cdot \vec{u}}{||\vec{u}||^2} \vec{u}. \quad (7)$$
Best approximation pair of two skew lines

Proof. We have \( \overrightarrow{s} = r \cap H(\overrightarrow{m}, \overrightarrow{u}) \) which is equivalent to

\[
\begin{cases}
\overrightarrow{x} = \overrightarrow{p} + \beta \overrightarrow{u} \\
(\overrightarrow{x} - \overrightarrow{m}) \cdot \overrightarrow{u} = 0
\end{cases}
\tag{7}
\]

and then from the system (7) we obtain \( \beta^* = \frac{(\overrightarrow{m} - \overrightarrow{p}) \cdot \overrightarrow{u}}{||\overrightarrow{u}||^2} \), and the result follows. \( \square \)

From [Ruiz, [6], page 165], [Borsuk, [1], page 76] and [Deutsch, [3], page 98], we may write

**Proposition 3.3** Let \( H(\overrightarrow{m}, \overrightarrow{u}) \) the hyperplane that contains the point \( \overrightarrow{m} \) and is perpendicular to the vector \( \overrightarrow{u} \). Let \( \overrightarrow{p} \) an external point to the hyperplane \( H(\overrightarrow{m}, \overrightarrow{u}) \). Then the distance \( d(\overrightarrow{p}, H(\overrightarrow{m}, \overrightarrow{u})) \) from the point \( \overrightarrow{p} \) to the hyperplane \( H(\overrightarrow{m}, \overrightarrow{u}) \) is given by

\[
d(\overrightarrow{p}, H(\overrightarrow{m}, \overrightarrow{u})) = \frac{|(\overrightarrow{m} - \overrightarrow{p}) \cdot \overrightarrow{u}|}{||\overrightarrow{u}||}.
\]

### 3.3 Distance from a point to a line

In this section, use is made of the hyperplane associated to a point and a line. From [[6], page 166], we may write

**Definition 3.4** The distance \( d(\overrightarrow{a}, r) \) from a point \( \overrightarrow{a} \) to a line \( r \) is defined as the distance from the point \( \overrightarrow{a} \) to the hyperplane \( H^*(\overrightarrow{a}, r) \) associated to the point \( \overrightarrow{a} \) and the line \( r \).

If \( \overrightarrow{a} \) is an external point of the line \( r \) defined by \( \overrightarrow{x} = \overrightarrow{m} + \alpha \overrightarrow{u}, \alpha \in \mathbb{R} \), invoking Ruiz [6], we put

\[
d(\overrightarrow{a}, r) = d(\overrightarrow{a}, H^*(\overrightarrow{a}, r)) = \frac{|(\overrightarrow{a} - \overrightarrow{m}) \cdot (\overrightarrow{a} - \overrightarrow{q})|}{||\overrightarrow{a} - \overrightarrow{q}||},
\]

where \( \overrightarrow{q} = \overrightarrow{m} + \frac{(\overrightarrow{a} - \overrightarrow{m}) \cdot \overrightarrow{u}}{||\overrightarrow{u}||^2} \overrightarrow{u} \). In (6) we have the best approximation point to a line.

Now we have

**Proposition 3.5** Let \( \overrightarrow{p} \) an external point to the line \( l \) given by \( \overrightarrow{x} = \overrightarrow{m} + \alpha \overrightarrow{u}, \alpha \in \mathbb{R} \). Let \( d(\overrightarrow{p}, l) \) be the distance from the point \( \overrightarrow{p} \) to the line \( l \). Then

\[
d(\overrightarrow{p}, l) = \frac{|(\overrightarrow{s} - \overrightarrow{p}) \cdot (\overrightarrow{m} - \overrightarrow{p})|}{||\overrightarrow{s} - \overrightarrow{p}||},
\]

where \( \overrightarrow{s} \) is the orthogonal projection of the point \( \overrightarrow{p} \) onto the line \( l \).
Proof. The relation (8), considered in Ruiz [[6], page 166] is related to the hyperplane $H^*(\vec{p}, l)$ that contains $\vec{m}$ and is perpendicular to the director vector $(\vec{s} - \vec{p})$ of the normal line common to the point $\vec{p}$ and the line $l$. According Proposition 3.3, we have

$$d(\vec{p}, l) = d(\vec{p}, H(\vec{m}, \vec{s} - \vec{p}))$$

as, by Definition 3.4 and Definition 2.8, we obtain

$$d(\vec{p}, l) := d(\vec{p}, H^*(\vec{p}, l)) := d(\vec{p}, H(\vec{m}, \vec{s} - \vec{p})). \square$$

Remark 3.6 In [[5], pages 72–73], for the case of the ordinary vector space $\mathbb{R}^3$, the author presents two analytical methods for the distance from a point to a line.

3.4 Distance between two skew lines
The distance between two skew lines is the minimum of the distances between two points: one point belonging to one line, one point belonging to the other line.

In the other words,

$$d(r_1, r_2) = \min\{d(\vec{p}, \vec{q}) : \vec{p} \in r_1, \vec{q} \in r_2\} = \min\{|\vec{p} - \vec{q}| : \vec{p} \in r_1, \vec{q} \in r_2\}$$

Proposition 3.7 Let be given, in $\mathbb{R}^n$, the skew lines defined by

$$r_1 := \vec{x} = \vec{p_1} + \alpha \vec{u} \quad \text{and} \quad r_2 := \vec{x} = \vec{p_2} + \beta \vec{v}.$$ 

Then there exist points $\vec{p_1} \in r_1$, $\vec{p_2} \in r_2$ given by Proposition 2.12 such that the vector $\vec{p_1p_2} = \vec{p_2} - \vec{p_1}$ is perpendicular to the lines $r_1$ and $r_2$.

Furthermore, this vector $\vec{p_1p_2} = \vec{p_2} - \vec{p_1}$ gives the direction of the unique normal line common to the lines $r_1$ and $r_2$.

The normal line common to the lines $r_1$ and $r_2$ is given both by

$$\vec{x} = \vec{p_1} + \gamma(\vec{p_2} - \vec{p_1})$$

and by

$$\vec{x} = \vec{p_2} + \delta(\vec{p_2} - \vec{p_1}).$$

Proof. We look for points $\vec{p_1} \in r_1$ and $\vec{p_2} \in r_2$ such that

$$\begin{cases} (\vec{p_2} - \vec{p_1}) \cdot \vec{u} = 0 \\ (\vec{p_2} - \vec{p_1}) \cdot \vec{v} = 0 \end{cases}$$

(9)
The system (9) is equivalent to the system

\[ - \begin{bmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (\vec{p}_1 - \vec{p}_2) \cdot \vec{u} \\ (\vec{p}_1 - \vec{p}_2) \cdot \vec{v} \end{bmatrix} \tag{10} \]

The vectors \( \vec{u} \) and \( \vec{v} \) are linearly independent so the coefficient matrix is non-singular [7], [3]. This fact guarantees the existence and uniqueness of the solution \( \begin{bmatrix} \alpha^* \\ \beta^* \end{bmatrix} \) of the system (10) and hence the existence and unicity of the points \( \vec{p}_1 = \vec{p}_1' + \alpha^* \vec{u}, \vec{p}_2 = \vec{p}_2' + \beta^* \vec{v} \). □

The next result is used in the proof of Proposition 3.10.

**Proposition 3.8** Let be the skew lines \( r_1 \) and \( r_2 \) given by

\( r_1 := \vec{x} = \vec{p}_1 + \alpha \vec{u} \) and \( r_2 := \vec{x} = \vec{p}_2 + \beta \vec{v} \).

Let \( \vec{p}_1' \vec{p}_2 = \vec{p}_2 - \vec{p}_1' \) be the direction vector of the normal line common to the lines \( r_1 \) and \( r_2 \), where \( \vec{p}_1', \vec{p}_2 \) are given by Proposition 2.12.

Then

\( (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 = (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 = (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 \).

**Proof.** As \( \vec{p}_1 = \vec{p}_1' + k \vec{u}, \vec{p}_2 = \vec{p}_2' + p \vec{v} \), for some \( k, p \in \mathbb{R} \), we have

\( (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 = (\vec{p}_2 - \vec{p}_1' - k \vec{u}) \cdot \vec{p}_1' \vec{p}_2 = (\vec{p}_2 - \vec{p}_1') - k \vec{u} \cdot \vec{p}_1' \vec{p}_2 \).

Now by the linearity of the inner product, we obtain

\( (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 = (\vec{p}_2 - \vec{p}_1') \cdot \vec{p}_1' \vec{p}_2 - k(\vec{u} \cdot \vec{p}_1' \vec{p}_2) \)

and the result follows as the vectors \( \vec{u} \) and \( \vec{p}_1' \vec{p}_2 \) are orthogonal. □

In the next result, we go further in the knowledge of the best approximation points \( \vec{p}_1' \) and \( \vec{p}_2' \): a characterization of these points is given

**Proposition 3.9** Let \( r_1 := \vec{x} = \vec{p}_1 + \alpha \vec{u}, r_2 := \vec{x} = \vec{p}_2 + \beta \vec{v} \) two skew lines in \( \mathbb{R}^n \). Let \( \vec{p}_1' \vec{p}_2 = \vec{p}_2 - \vec{p}_1', \vec{p}_1' \in r_1, \vec{p}_2' \in r_2, \) be the direction vector of the normal line common to the lines \( l_1 \) and \( l_2 \), where \( \vec{p}_1', \vec{p}_2 \) are given by Proposition 2.12.

Then \( \vec{p}_1' \) is the orthogonal projection of the point \( \vec{p}_2 \) onto the line \( r_1 \) and \( \vec{p}_2' \) is the orthogonal projection of the point \( \vec{p}_1' \) onto the line \( r_2 \).

**Proof.** We have

\[ \vec{p}_1' + \frac{(\vec{p}_2 - \vec{p}_1') \cdot \vec{u}}{||\vec{u}||^2} \vec{u} = (\vec{p}_1' - \alpha^* \vec{u}) + \frac{(\vec{p}_2 - \vec{p}_1') \cdot \vec{u}}{||\vec{u}||^2} \vec{u}. \]

The
orthogonal projection of the point $\mathbf{p}'_2$ onto the line $r_1$. *Mutatis mutandis* it is shown that $\mathbf{p}'_2$ is the orthogonal projection of the point $\mathbf{p}'_1$ onto the line $r_2$.  □

In the next result, use is made of the associated hyperplane for obtaining the distance $d(r_1, r_2)$ between two skew lines $r_1$ and $r_2$.

**Proposition 3.10** Let be the skew lines, in $\mathbb{R}^n$, given by $r_1 := \mathbf{x} = \mathbf{p}'_1 + \alpha \mathbf{u}$ and $r_2 := \mathbf{x} = \mathbf{p}'_2 + \beta \mathbf{v}$.

Let $\mathbf{p}'_1 \in r_1$ and $\mathbf{p}'_2 \in r_2$ be points given by Proposition 2.12 such that the vector $\mathbf{p}'_1\mathbf{p}'_2 = \mathbf{p}'_2 - \mathbf{p}'_1$ gives the direction of the normal line common to the lines $r_1$ and $r_2$.

Then, the distance $d(r_1, r_2)$ between the lines $r_1$ and $r_2$ is given by:

1. $d(r_1, r_2) = d(\mathbf{p}'_2, H^*(r_1, r_2)) = d(\mathbf{p}'_2, H(\mathbf{p}'_1, \mathbf{p}'_2)) = \frac{|\mathbf{p}'_1 - \mathbf{p}'_2| \cdot \mathbf{p}'_1 \mathbf{p}'_2|}{||\mathbf{p}'_1 \mathbf{p}'_2||}$;

2. $d(r_1, r_2) = d(\mathbf{p}'_1, H^*(r_2, r_1)) = d(\mathbf{p}'_1, H(\mathbf{p}'_2, \mathbf{p}'_1\mathbf{p}'_2)) = \frac{|\mathbf{p}'_2 - \mathbf{p}'_1| \cdot \mathbf{p}'_2 \mathbf{p}'_1|}{||\mathbf{p}'_1 \mathbf{p}'_2||}$.

**Proof.** From Proposition 3.1 and Proposition 3.3, we obtain 1.; For 2., we use Proposition 3.3 considering the hyperplane $H^*(r_2, r_1)$. □

In [5], (on pages 75 – 77), the author displays, in the context of the Euclidean space $\mathbb{R}^3$, analytical methods for the calculation of the distance from a point to a plane or to a line and the minimum distance between two lines.

## 4 Final Remarks and Conclusions

In this paper, we aimed at getting better understood the geometric approach by Ruiz for some geometric problems. This geometric approach fosters the creation of spacial abilities. These ideas are well documented in the works by [4] and [5] in the context of the ordinary space $\mathbb{R}^3$.

**Acknowledgements.** This research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Strategic Projects UID/MAT/00013/2013 and UID/CEI/00194/2013.
References


Received: March 11, 2017; Published: March 29, 2017