The Standard Polynomial in Verbally Prime Algebras

Geraldo de Assis Junior
Departamento de Ciências Exatas e Tecnológicas
Universidade Estadual de Santa Cruz
Ilhéus, BA, Brasil

Sérgio M. Alves
Departamento de Ciências Exatas e Tecnológicas
Universidade Estadual de Santa Cruz
Ilhéus, BA, Brasil

Abstract
In this paper we discuss the minimality of the degree of the standard polynomial as a polynomial identity for verbally prime algebras and its tensor products.

Mathematics Subject Classification: 16R10, 16R20, 16R40, 15A75

Keywords: PI Theory, Standard polynomial, Verbally prime algebra.

1 Introduction
Verbally prime algebras play a prominent role in the PI theory. Recall that an algebra A is verbally prime if its T-ideal is prime in the class of all T-ideals in the free associative algebra. Let V be an vector space over K of countable infinite dimension with basis \{e_1, e_2, ...\}. The Grassmann algebra E of V is the associative algebra with K-basis consisting of 1 and all products of the form \(e_{i_1}e_{i_2}...e_{i_m}\) with \(i_1 < i_2 < ... < i_m\), \(m \geq 1\) and with multiplication induced
by \(e_i^2 = 0\) and \(e_i e_j + e_j e_i = 0\). When \(\text{char } K = p = 2\), then obviously \(E\) is commutative and hence are not very "interesting" from the PI point of view. Therefore, we restrict our attention the case \(p > 2\).

We denote by \(M_n(E)\) the algebra of \(n \times n\) matrices over \(E\). The algebra \(M_{a,b}(E)\) is the subalgebra of \(M_{a+b}(E)\) that consists of the block matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

where \(A \in M_a(E_0), B \in M_{a \times b}(E_1), C \in M_{b \times a}(E_1)\) and \(D \in M_b(E_0)\).

If two algebras \(A\) and \(B\) satisfy the same polynomial identities we say that \(A\) is PI-equivalent to \(B\) and we denote by \(A \sim B\). An important consequence of the Kemer’s structure theory is the Tensor Product Theorem (TPT).

**Theorem 1 (TPT)**. Let \(\text{char } K = 0\). Then we have \(M_{1,1}(E) \sim E \otimes E\), \(M_{a,b}(E) \otimes E \sim M_{a+b}(E)\) and \(M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)\).

In [4] it has been proved that Kemer’s tensor product theorem can not be transposed to fields of positive characteristic. However we have the following multilinear version. Let \(P(I)\) be the set of multilinear polynomials in the \(T\)-ideal \(I\):

**Theorem 2 ([6], theorem 5)**. Let \(\text{char } K = p \neq 2\). Then we have \(P(T(M_{1,1}(E))) = P(T(E \otimes E)), P(T(M_{a,b}(E) \otimes E)) = P(T(M_{a+b}(E)))\) and \(P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E)))\).

Next we will define the algebras the type \(S^1\), \(S^2\) and \(S^3\).

## 2 Algebras of the type \(S^1\), \(S^2\) and \(S^3\)

We denote by

\[
s_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \epsilon(\sigma)x_{\sigma(1)}\ldots x_{\sigma(k)}
\]

the polynomial standard of degree \(k\) where \(\epsilon(\sigma)\) is the sign of \(\sigma\). Note that if \(A\) satisfies \(s_k\) with \(k\) minimal, then \(k\) must be even. For if \(k\) were odd, then \(s_k(x_1, \ldots, x_k, 1) = s_{k-1}(x_1, \ldots, x_{k-1})\) and thus \(k\) would not be minimal. The same remark holds for powers of standard identities. We write \(\mu(A) = n\) if \(A\) satisfies \(s_{2n}\) and no \(s_k\) with \(k < 2n\). If \(A\) satisfies no standard identity we write \(\mu(A) = \infty\). We write in addition \(\mu^*(A) = n\) if \(A\) satisfies a power of \(s_{2n}\), and no power of \(s_k\) with \(k < 2n\).

Let \(A\) be an algebra.

**Definition 2.1**. We say that \(A\) is a \(S^1\)-algebra if \(A \sim M_n(K)\) and a \(S^1\)-algebra if \(A\) is a \(S^1\)-algebra for some \(n\).
Definition 2.2 We say that $A$ is a $S^n_2$-algebra if $A \sim M_n(E)$ and a $S^2$-algebra if $A$ is a $S^n_2$-algebra for some $n$.

Definition 2.3 We say that $A$ is a $S^3_{ab}$-algebra if $A \sim M_{a,b}(E)$ and a $S^3$-algebra if $A$ is a $S^3_{ab}$-algebra for some $a$ and $b$.

Thus, according to Kemer’s Theory, all verbally prime PI-algebra over a field of characteristic zero are of the type $S^1$, $S^2$ or $S^3$. The purpose of this paper is to study this classification for algebras over a field of characteristic positive and its tensor products.

3 The $S^1$, $S^2$ and $S^3$ algebras in char $K = 0$

The Amitsur-Levitzki theorem says that $\mu(M_n(K)) = n$. Now if $M_n(K)$ satisfy a standard polynomial then satisfy a power itself. Therefore $\mu(M_n(K)) \leq n$. Now let’s show that if $k < n$ then $M_n(K)$ does not satisfy $s_{2k}$. Consider

$$Q = s_{2k}(E_{12}, E_{23}, ..., E_{k,k+1}, E_{k+1,k}, ..., E_{32}, E_{21})$$

where the $E_{ij}$ are the matrices units. Then the non-zero terms $Q$ of the form $E_{ii}$, and for $i = 1$ we will have only one non-zero term $E_{11}$. It follows that $Q$ is an diagonal matrix with 1 in the upper left corner. Then $Q$ is not nilpotent, and therefore no power of $s_{2k}$ satisfies $M_n(K)$. Thus we have the following

$$\mu(M_n(K)) = \mu^*(M_n(K)) = n.$$  

Theorem 3 Let $A$ be a $S^1$-algebra, Then $\mu(A) = \mu^*(A)$ and if $A \sim M_n(K)$ we have $\mu(A) = n$.

Proof. Since $A$ is a $S^1$-algebra exists one positive integer $n$ such that $A \sim M_n(K)$. Therefore $\mu(A) = \mu(M_n(K))$ and $\mu^*(A) = \mu^*(M_n(K))$. From what has been stated above $\mu(A) = \mu^*(A) = n$.

Lemma 3.1 Let char $K = 0$. The Grassmann algebra $E$ does not satisfy any standard polynomial, that is, $\mu(E) = \infty$.

Proof. Let $e_1, ..., e_n$ we have that $e_i e_j = -e_j e_i$. Therefore $e_{\sigma(1)}...e_{\sigma(n)} = \epsilon(\sigma)e_1...e_n$. Follow that

$$s_n(e_1, ..., e_n) = \sum_{\sigma \in S_n} \epsilon(\sigma)e_{\sigma(1)}...e_{\sigma(n)} = \sum_{\sigma \in S_n} \epsilon(\sigma)\epsilon(\sigma)e_1...e_n = n!e_1...e_n$$

The lemma below was proved in [1] for zero characteristic fields and extended in [5] for any characteristic. In this article we will use the following isomorphisms $M_k(A) \simeq M_k(K) \otimes A$ and $M_k(M_{1,1}(E)) \simeq M_{k,k}(E)$ ([4], lemma 1).
Lemma 3.2 \cite[lemma 2.2.1]{5} On a field \( K \) of arbitrary characteristic we have:

1. \( M_n(E) \) satisfies \( s_{2n}^k \) for some \( k > 1 \), but does not satisfy \( s_{2n} \) nor \( s_{2n-1}^k \) for any \( k \)

2. If \( a \geq b \). Then \( M_{a,b}(E) \) satisfies \( s_{2a}^k \) for some \( k > 1 \) but does not satisfy \( s_{2a} \) or \( s_{2a-1}^k \) for any \( k \)

Theorem 4 Let \( A \) be a \( S^2 \)-algebra. Then \( \mu(A) = \infty \) and if \( A \sim M_n(E) \) we have \( \mu^*(A) = n \).

Proof. Since \( A \) is a \( S^2 \)-algebra there is one positive integer \( n \) such that \( A \sim M_n(E) \). Therefore, \( \mu(A) = \mu(M_n(E)) \) and \( \mu^*(A) = \mu^*(M_n(E)) \). According to the lemma 3.2 we have \( \mu^*(M_n(E)) = n \) and since \( T(M_n(E)) \subset T(E) \) follows from the lemma 3.1 that \( \mu(M_n(E)) = \infty \).

Theorem 5 Let \( A \) be a \( S^3 \)-algebra. Then \( \mu(A) = \infty \) and if \( A \sim M_{a,b}(E) \) we have \( \mu^*(A) = \max\{a, b\} \).

Proof. Since \( A \) is a \( S^3 \)-algebra there is positive integers \( a \) and \( b \) such that, \( A \sim M_{a,b}(E) \). However

\[
\mu(A) = \mu(M_{a,b}(E)) \quad \text{and} \quad \mu^*(A) = \mu^*(M_{a,b}(E))
\]

According to the Lemma 3.2 we have \( \mu^*(M_{a,b}(E)) = \max\{a, b\} \). And if \( d = \min\{a, b\} \) we have

\[
P(T(M_{a,b}(E))) \subset P(T(M_{d,d}(E))) = P(T(M_d(E) \otimes E)) \subset P(T(M_d(E))) \subset P(T(E)).
\]

These inclusions follow by natural immersions and equality follows from

\[
M_{d,d}(E) \simeq M_d(M_{1,1}(E)) \simeq M_d(K) \otimes M_{1,1}(E) \simeq [M_d(K) \otimes E] \otimes E \simeq M_d(E) \otimes E.
\]

The following lemma will be very useful in what follows.

Lemma 3.3 If \( A \sim B \) and \( C \sim D \) then \( A \otimes C \sim B \otimes D \).

Proof. \cite[teorema 1]{7}.

Theorem 6 If \( A \) and \( B \) are \( S^1 \)-algebras then \( A \otimes B \) is a \( S^1 \)-algebra. Then \( \mu(A \otimes B) = \mu(A) \mu(B) \), \( \mu^*(A \otimes B) = \mu^*(A) \mu^*(B) \) and \( \mu(A \otimes B) = \mu^*(A \otimes B) \).
On the power of standard polynomial to $M_{a,b}(E)$

Proof. There are integers $a$ and $b$ such that $A \sim M_a(K)$ and $B \sim M_b(E)$ respectively. Thus, by lemma 3.3

$$A \otimes B \sim M_a(K) \otimes M_b(K) \simeq M_{ab}(K).$$

Therefore $A \otimes B$ is a $S^1$-algebra. Then,

$$\mu(A) = \mu^*(A) = a, \quad \mu(B) = \mu^*(B) = b \quad \text{and} \quad \mu(A \otimes B) = ab.$$

Theorem 7 If $A$ is a $S^1$-algebra and $B$ is a $S^2$-algebra then $A \otimes B$ is a $S^2$-algebra. Then $\mu(A \otimes B) = \infty$ and $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B) = \mu(A)\mu^*(B)$

Proof. There are integers $a$ and $b$ such that $A \sim M_a(K)$ and $B \sim M_b(E)$ respectively. Thus, by lemma 3.3 we have $A \otimes B \sim M_a(K) \otimes M_b(E)$.

Theorem 8 If $A$ is a $S^1$-algebras and $B$ is a $S^3$-algebra then $A \otimes B$ is a $S^3$-algebra. Then $\mu(A \otimes B) = \infty$ and $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B) = \mu(A)\mu^*(B)$

Proof. We have $A \sim M_a(K) \otimes M_{b,c}(E)$. Thus, by lemma 3.3 $A \otimes B \sim M_a(K) \otimes M_{b,c}(E)$. On the other hand, $M_a(K) \otimes M_{b,c}(E) \simeq M_a(M_{b,c}(E)) \simeq M_{ab,ac}(E)$. Therefore $A \otimes B$ is a $S^3$-algebra. Thus $\mu(B) = \mu(A \otimes B) = \infty$, $\mu(A) = \mu^*(A) = a$, $\mu^*(B) = \max\{b,c\}$ and $\mu^*(A \otimes B) = \max\{ab,ac\} = a\max\{b,c\} = \mu^*(A)\mu^*(B)$.

Theorem 9 If $A$ and $B$ are $S^2$-algebras then $A \otimes B$ is a $S^3$-algebra. Then $\mu(A \otimes B) = \infty$ and $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$.

Proof. Let $A \sim M_a(E)$ and $B \sim M_b(E)$. Thus by lemma 3.3 we have $A \otimes B \sim M_a(E) \otimes M_b(E)$. On the other hand, $M_a(E) \otimes M_b(E) \simeq [M_a(K) \otimes E] \otimes [M_b(K) \otimes E] \simeq M_{ab}(K) \otimes (E \otimes E) \sim M_{ab}(K) \otimes M_{1,1}(E) \simeq M_{ab}(M_{1,1}(E)) \simeq M_{ab,ab}(E)$ because $E \otimes E \simeq M_{1,1}(E)$. Then $A \otimes B$ is a $S^3$-algebra. Thus $\mu(A \otimes B) = \infty$, $\mu^*(A) = a$, $\mu^*(B) = b$ and $\mu^*(A \otimes B) = ab = \mu^*(A)\mu^*(B)$.

Theorem 10 If $A$ is a $S^2$-algebras and $B$ is a $S^3$-algebra then $A \otimes B$ is a $S^2$-algebra. Then $\mu(A \otimes B) = \infty$ and if $B \sim M_{b,c}(E)$ then $\mu^*(A \otimes B) = \mu^*(A)(b + c)$. 
Theorem 11 If $A$ and $B$ are $S^3$-algebras then $A \otimes B$ is a $S^3$-algebra. Then $\mu(A \otimes B) = \infty$ and $\mu^*(A \otimes B) > \mu^*(A)\mu^*(B)$.

Proof. Let $A \sim M_n(K)$ and $B \sim M_m(K)$. Then by lemma 3.3 we have $A \otimes B \sim M_{n \times m}(K)$. On the other hand, $M_n(K) \otimes M_n(K) \simeq [M_n(K) \otimes K] \otimes M_m(K) \simeq M_{n \times m}(K)$. Therefore $A \otimes B$ is a $S^3$-algebra. Moreover $\mu(A) = \mu(B) = \infty$, so $\mu^*(A) = \mu^*(B) = \mu^*(A \otimes B)$.

4 The $S^1$, $S^2$ and $S^3$ algebras in char $K = p > 2$

In this section we will transpose the results of the previous section into algebras over a infinite field characteristic field $p \neq 2$.

4.1 On the valour of $\mu^*(A)$

If $A$ is a $S^1$-algebra then there is an integer $n$ such that $A \sim M_n(K)$ and therefore $\mu^*(A) = n$ because the theorem 3 is free of characteristic. Besides that, if $A$ is a $S^2_n$ or $S^3_{ab}$-algebra then $\mu^*(A) = n$ or $\mu^*(A) = \max\{a, b\}$, because the lemma 3.2 is free of characteristic. For the tensor products we have the following.

Theorem 12 If $A$ is a $S^1$-algebra and $B$ is a $S^2$-algebra, then $A \otimes B$ is a $S^2$-algebra and $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$.

Proof. Idem to the proof of the theorem 7.

Theorem 13 If $A$ is a $S^1$-algebra and $B$ is a $S^3$-algebra then $A \otimes B$ is a $S^3$-algebra and $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$.

Proof. Idem to the proof of the theorem 8.

Theorem 14 Let $A$ be a $S^2$-algebra and let $B$ be a $S^3$-algebra. Then

$$\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$$
On the power of standard polynomial to $M_{a,b}(E)$

Proof. Since $A$ is a $S^2$-algebra and $B$ is a $S^3$-algebra there are $a$, $b$ and $c$ such that $A \sim M_a(E)$ and $B \sim M_{b,c}(E)$, thus $\mu^*(A) = a$ and $\mu^*(B) = \max\{b,c\}$. Now note that $M_a(E) \otimes M_{b,c}(E) \simeq [M_a(K) \otimes E] \otimes M_{b,c}(E) \simeq [M_a(K) \otimes M_{b,c}(E)] \otimes E \simeq M_{ab,ac}(E) \otimes E$. Let $d = \max\{ab, ac\}$, according to ([5], Lemma 27), there exists integer $t > 1$ such that $s_{2d}$ satisfies $M_{ab,ac}(E) \otimes E$, thus $\mu^*(M_{ab,ac}(E) \otimes E) \leq d$. On the other hand, since $M_{ab,ac}(E) \otimes E$ has an isomorphic copy of $M_d(K)$ follow that $\mu^*(M_{ab,ac}(E) \otimes E) \geq d$. Therefore
\[\mu^*(A \otimes B) = \mu^*(M_{ab,ac}(E) \otimes E) = \max\{ab, ac\} = a \max\{b, c\} = \mu^*(A)\mu^*(B).\]

Theorem 15 If $A$ and $B$ are $S^2$-algebras then $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$.

Proof. Since $A$ and $B$ are $S^2$-algebras there are $a$ and $b$ such that $A \sim M_a(E)$ and $B \sim M_b(E)$, thus $\mu^*(A) = a$ and $\mu^*(B) = b$. Now note that $M_a(E) \otimes M_b(E) \simeq [M_a(K) \otimes E] \otimes [M_b(K) \otimes E] \simeq [M_{ab}(K) \otimes E] \otimes E \simeq M_{ab}(E) \otimes E$. Now, using ([5], Lemma 26) we obtain that there exists an integer $t > 1$ with $t = t(a, b, p)$ such that $s_{2ab}$ satisfies $M_{ab}(E) \otimes E$. Thus $\mu^*(M_{ab}(E) \otimes E) \leq ab$. On the other hand, since $M_{ab}(E) \otimes E$ has an isomorphic copy of $M_{ab}(K)$, follows that $\mu^*(M_{ab}(E) \otimes E) \geq ab$. Therefore,
\[\mu^*(A \otimes B) = \mu^*(M_{ab}(E) \otimes E) = ab = \mu^*(A)\mu^*(B).\]

Theorem 16 If $A$ and $B$ are $S^3$-algebras then $\mu^*(A \otimes B) = \mu^*(A)\mu^*(B)$.

Proof. Since $A$ and $B$ are $S^3$-algebras there are integers $a$, $b$, $c$ and $d$ such that $A \sim M_{a,b}(E)$ and $B \sim M_{c,d}(E)$. Without loss of generality, let us suppose that $a \geq b$ and $c \geq d$. Thus $\mu^*(A) = a$ and $\mu^*(B) = c$. According to ([4], lemma 12) $s_{2ac}$ satisfies $M_{a,b}(E) \otimes M_{c,d}(E)$ for some $t = t(a, b, p)$. Since $T(M_{a,b}(E) \otimes M_{c,d}(E)) \subset T(M_{a,b}(E) \otimes M_{c,d}(E))$ then $s_{2ac}$ also satisfies $M_{a,b}(E) \otimes M_{c,d}(E)$. On the other hand, since $M_{a,b}(E) \otimes M_{c,d}(E)$ has an isomorphic copy of $M_a(K) \otimes M_c(K) \simeq M_{ac}(K)$ so it does not satisfy $s_k^t$ for all $t$ is $k < 2ac$. Therefore,
\[\mu^*(A \otimes B) = \mu^*(M_{a,b}(E) \otimes M_{c,d}(E)) = ac = \mu^*(A)\mu^*(B).\]
4.2 On the valour of $\mu(A)$

In this section we study the valour of the $\mu(A)$ when $A$ is an $S^1$, $S^2$ and $S^3$-algebra or tensor product it. First, we observe that Theorem 3 is free of characteristic.

In [2], Lemma 5] the authors proved the following result.

**Lemma 4.1** The standard polynomial $s_m$ is a polynomial identity of the $E$ if and only if $m \geq p + 1$.

In [3], Theorem 5.5] the authors proved the following result.

**Theorem 17** If the algebra $A$ satisfies polynomial identity $s_{2n}$, then the algebra $M_k(A)$ satisfies the polynomial identity $s_{2nk^2-k^2+1}$.

In [7], Theorem 15] the authors proved the following result.

**Theorem 18** If the algebra $M_k(A)$ satisfies the polynomial identity $s_n$, then the algebra $M_{k-1}(A)$ satisfies the polynomial identity $s_{n-2}$. Moreover, the algebra $A$ satisfies the polynomial identity $s_{n-2(k-1)}$.

Now we can prove the following:

**Theorem 19** If $A$ is a $S^2_h$-algebra then

$$\frac{2n + p - 1}{2} \leq \mu(A) \leq \frac{pn^2 + 1}{2}.$$ 

**Proof.** Since $A$ is $S^2$-algebra, we see that $A \sim M_n(E)$. We have that, $\mu(A) = \mu(M_n(E))$. If the algebra $M_n(E)$ satisfies $s_m$, by theorem 18, we obtain that the algebra $E$ satisfies $s_{m-2(n-1)}$. By lemma 4.1, we have that

$$m - 2(n - 1) \geq p + 1 \equiv m \geq 2n + p - 1.$$ 

Thus, we obtain that $\mu(A) \geq \frac{(2n+p-1)}{2}$.

Now, since $E$ satisfies $s_{2m}$ with $2m = p + 1$, by theorem 17, we have that $M_n(E)$ satisfies $s_{2mn^2-n^2+1}$. We observe that

$$2mn^2 - n^2 + 1 = (p+1)n^2 - n^2 + 1 = pn^2 + 1.$$ 

Therefore $2\mu(A) \leq pn^2 + 1$. 

This result will be the basis for the next theorems.

**Theorem 20** Let $A$ be a $S^3_{ab}$-algebra and $d = \min\{a,b\}$. Then

$$\frac{2d + p - 1}{2} \leq \mu(A) \leq \frac{p(a+b)^2 + 1}{2}.$$
Theorem 23

If \( A \sim M_{a,b}(E) \) and \( \mu(A) = \mu(M_{a,b}(E)) \), we obtain that

\[
P(T(M_{a,b}(E))) \subset P(T(M_{d,d}(E))) \subset P(T(M_d(E) \otimes E)) \subset P(T(M_d(E))).
\]

Now, if \( s_m \in T(M_{a,b}(E)) \) then \( s_m \in T(M_d(E)) \), and by theorem 19, we obtain that \( \mu(A) \geq \frac{2d+p-1}{2} \). Then by theorem 17 \( s_{p(a+b)^2+1} \) satisfies \( M_{a,b}(E) \), thus \( \mu(A) \leq p(a+b)^2 + 1 \).

\[\square\]

Theorem 21

If \( A \) is a \( S_2^3 \)-algebra and \( B \) is a \( S_{bc}^{3/2} \) then

\[
\frac{2(ab + ac) + p - 1}{2} \leq \mu(A \otimes B) \leq \frac{p(ab + ac)^2 + 1}{2}
\]

Proof. We know that \( M_a(E) \otimes M_{b,c}(E) \simeq M_{ab,ac}(E) \otimes E \). By theorem 2 \( P(T(M_{ab,ac}(E) \otimes E)) = P(T(M_{ab+ac}(E))) \). However \( \mu(A \otimes B) = \mu(M_{ab+ac}(E)) \).

\[\square\]

Theorem 22

If \( A \) is a \( S_2^3 \)-algebra and \( B \) is a \( S_{bc}^{3/2} \)-algebra then

\[
\frac{2ab + p - 1}{2} \leq \mu(A) \leq 2(ab)^2p.
\]

Proof. Since \( A \sim M_a(E) \) and \( B \sim M_b(E) \) then \( A \otimes B \sim M_a(E) \otimes M_b(E) \simeq M_{ab}(K) \otimes [E \otimes E] \). Since \( T(M_{1,1}(E)) \subset T(E \otimes E) \) then:

\[
T(M_a(E) \otimes M_b(E)) \supset T(M_{ab}(K) \otimes M_{1,1}(E)) = T(M_{ab}(M_{1,1}(E))) = T(M_{ab,ab}(E)).
\]

follows that \( M_a(E) \otimes M_b(E) \) satisfies \( s_{p(ab+ab)^2+1} \). Since \( p(ab+ab)^2 + 1 \) is odd then \( M_a(E) \otimes M_b(E) \) satisfies \( s_{4(ab)^2} \) and therefore \( \mu(A \otimes B) \leq 2(ab)^2 \). On the other hand, \( M_a(E) \otimes M_b(E) \simeq M_{ab}(E) \otimes E \) contains an isomorphic copy of \( M_{ab}(E) \) and therefore \( \mu(A \otimes B) \geq \frac{2ab+p-1}{2} \).

\[\square\]

Theorem 23

If \( A \) is a \( S_{ab}^3 \)-algebra, \( B \) is a \( S_{cd}^3 \)-algebra and \( d = \text{min}\{ac + bd, ad + bc\} \) then

\[
\frac{2d + p - 1}{2} \leq \mu(A \otimes B) \leq \frac{p(ac + bd + ad + bc)^2 + 1}{2}.
\]

Proof. By theorem 2 we have \( P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E))) \), thus \( \mu(A \otimes B) = \mu(M_{ac+bd,ad+bc}(E)) \), and by theorem 20 follows the result.

\[\square\]

Acknowledgements. Sérgio M. Alves is partially supported by PROPP/UESC.
References


Received: February 21, 2017; Published: May 12, 2017