On the Generating Function of Certain Involutions of the Symmetric Group $S_n$

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Abstract

This paper explores the generating function for certain subset of symmetric group $S_n$ which has a deep geometric interpretation in type $A$. In specific, we give a closed expression for the sequence of these functions via the transition matrix between the complete symmetric functions and elementary symmetric functions, the two of the bases of the algebra of symmetric functions.

Mathematics Subject Classification: 14M15, 14N15, 05E05

Keywords: Peterson variety, Symmetric functions, Composition, Brick tabloid

1. INTRODUCTION

A partition $\lambda \vdash n$ is a list of positive integers whose sum is $n$. The parts of $\lambda = (\lambda_1, \lambda_2, \ldots)$ are usually ordered in nonincreasing way, that is, $\lambda_1 \geq \lambda_2 \geq \ldots$. We shall denote the number of parts in $\lambda$ by $\ell(\lambda)$ and call it the length of $\lambda$. Another notation is to denote the number of parts of size $k$ in $\lambda$ by $m_k$ and write $\lambda = 1^{m_1}2^{m_2}\cdots$. For example, $\lambda = (4, 4, 3, 3, 3, 2, 1, 1, 1, 1, 1) = 1^523^34^2$ and $\ell(\lambda) = 11$. We shall denote by $P_n$, the set of partitions of $n$ on which the reverse lexicographic order is defined, that is, given $\lambda, \mu \vdash n$, we say $\lambda \preceq \mu$, if $\lambda_i > \mu_i$ or $\lambda_{i+1} > \mu_{i+1}$. The partitions of 4 in the reverse lexicographic order is $(4)$, $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, $(1, 1, 1, 1)$. Let $\wedge_n$ denote the space of homogeneous symmetric functions of degree $n$. The dimension of $\wedge_n$ is $|P_n|$. There are six standard bases of $\wedge_n$ which are usually considered [see [19]]. For any given
pair of bases \( \{c_\lambda : \lambda \vdash n\} \) and \( \{d_\mu : \mu \vdash n\} \) of \( \wedge_n \), there is a transition matrix \( M(c, d) \) such that

\[ c_\lambda = \sum M(c, d)d_\mu \]  

(1.1)

That is, \( M(c, d) \) is the coefficient of \( d_\mu \) in the expansion of \( c_\lambda \). Eğecioğlu and Remmel [9] introduced brick tabloid to give combinatorial interpretation of the entries of \( M(c, d) \). The focus of this work will be on only two standard bases of \( \wedge_n \) namely, elementary symmetric functions \( \{e_\lambda : \lambda \vdash n\} \) and complete symmetric functions \( \{\eta_\mu : \mu \vdash n\} \). A composition \( \alpha \) of a nonnegative integer \( n \) is a list \( (\alpha_1, \ldots, \alpha_k) \) of positive integers (the parts of the composition) with total sum \( n \). We denote it by \( \alpha \models n \). Composition respects order. That is, \((3, 3, 2)\) and \((3, 2, 3)\) are two distinct compositions of \( 8 \). \( \ell^*(\alpha) \) will mean the number of parts of a composition \( \alpha \). The number of compositions of \( n \) which have \( k \) parts is \( \binom{n-1}{k-1} \). Let \( \mathcal{C}(n) \) denote the set of all compositions of the nonnegative integer \( n \). Then \( \#\mathcal{C}(n) \) is \( 2^{n-1} \). The associated set to \( \alpha \) is defined \( \text{Set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_k\} \). Let \( [n] := \{1, 2, \ldots, n\} \). A partition of an \( n \)-set \( [n] \) is a set of non-empty disjoint subsets of \( [n] \) called blocks whose union is \( [n] \). A set partition of \( [8] := \{1, 2, 3, 4, 5, 6, 7, 8\} \) into three blocks is \( \{\{1, 4, 7\}, \{2, 3, 8\}, \{5, 6\}\} \). This is a set of sets. In general, sets are not usually ordered, so both the blocks and their constituents can be put in different order. That is, \( \{\{1, 4, 7\}, \{2, 3, 8\}, \{5, 6\}\} = \{\{6, 5\}, \{7, 4, 1\}, \{8, 3, 2\}\} \). The number of partitions of \( n \)-set into \( k \)-blocks is denoted by \( S(n,k) \) and called the Stirling number of the second kind. A permutation of a set \( [n] := \{1, 2, \ldots, n\} \) is a bijection \( \sigma : [n] \rightarrow [n] \). The symmetric group \( S_n \) is the group of all permutations of \( [n] \) under the composition of functions. We shall represent a permutation in a one-line notation, \( \sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n \) if and only if \( i \) is mapped to \( \sigma_i \), \( 1 \leq i \leq n \), so \( \sigma = 35124 \) is the permutation, \( 1 \mapsto 3, 2 \mapsto 5, 3 \mapsto 1, 4 \mapsto 2 \) and \( 5 \mapsto 4 \). The length \( \ell(\sigma) \) of a permutation \( \sigma \) is the number of inversions of \( \sigma \), i.e., the pairs \( i < j \) such that \( \sigma_i > \sigma_j \). The permutations \( 123 \cdots n \) and \( mn-1 \cdots 21 \) are the shortest and longest permutations in \( S_n \) respectively. Each permutation \( \sigma \) partitions \( [n] \) into blocks called orbits. A cycle is a permutation which has at most one orbit which contains more than one element. Every permutation \( \sigma \in S_n \) is a cycle or can be expressed as a unique product of cycles where cycles are disjoints and arranged in the decreasing order of lengths. A cycle of length 2 or a 2-cycle is a transposition. A transposition \( s_i := (i, i+1) \) is a simple transposition. The set of simple transpositions \( s_1, s_2, \ldots, s_{n-1} \) generates the symmetric group \( S_n \). A subgroup of \( S_n \) generated by simple transpositions is called parabolic (Young) subgroup. An involution is a permutation \( \sigma \) such that \( \sigma^2 \) is the identity. There are only 2-cycles and fixed points in the disjoint cycle decomposition of involutions and the 2-cycles need not be adjacent. Trivially, the identity permutation is an involution. The number of involutions of \( S_n \) is equal to the number of partitions of \( n \)-set in which every block has 1 or 2 elements [22], [24], [25]. For a permutation \( \sigma \in S_n \), we define \( \text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} \) and call it the descent of \( \sigma \),
On the generating function of certain involutions of ...

\[ \text{des}(\sigma) = |\text{Des}(\sigma)| \]. The ascent set of \( \sigma \) is given by \( \text{Asc}(\sigma) = \{ i : \sigma_i < \sigma_{i+1} \} \), \( \text{asc}(\sigma) = |\text{Asc}(\sigma)| \). There are a number of papers dedicated to the connection between permutation statistics and generating functions. One of the elegant techniques to construct such generating functions is to define a ring homomorphism on the space of symmetric functions, see [20]. The goal of this paper is to construct a generating function for certain subset of involutions which encodes geometric information by using the combinatorial interpretation of the transition matrix between the bases \( \{ e_\lambda : \lambda \vdash n \} \) and \( \{ h_\mu : \mu \vdash n \} \). We review some important backgrounds in section 2 and the results in section 3.

2. Combinatorial description of \( S^1 \) fixed points in the Peterson variety

The Peterson variety Pet\(_n\) in type \( A_{n-1} \) is the projective variety of dimension
\[
\sum_{i=1}^{n} (h(i) - i)
\]
defined as
\[
\text{Pet}_n = \{ V_\bullet \in F\ell(\mathbb{C}^n) : XV_i \subset V_{h(i)} \}
\]
where \( X \) is a regular nilpotent matrix, \( h(i) = i + 1, 1 \leq i \leq n-1 \) and \( h(n) = n \). The flag variety \( F\ell(\mathbb{C}^n) \) is the set of flags
\[
V_\bullet : \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \quad \text{such that} \quad \dim V_i = i
\]
There is a torus action on the flag variety but this action does not preserve the Peterson variety, it turns out that there is a one-dimensional subtorus \( S^1 \) which preserves the variety. See [1], [6], [13], [15], [17], [18] and [21]. The \( S^1 \)-fixed points of the Peterson variety Pet\(_n\) are permutations of the form
\[
\sigma_i \leq \sigma_{i+1} + 1, \quad 1 \leq i < n, \quad \sigma \in S_n
\]

Remark 2.1 In one-line notation of \( \sigma \in S_n \), there are two possibilities for the pattern of the entries from left to right: (i) either increase or (ii) decrease in the natural descending order, that is, decrease by 1.

The general pattern of \( \sigma \) in one-line notation is of the form
\[
\sigma = q_1q_1 - 1 \cdots q_2q_2 - 1 \cdots q_1 + 1 \cdots nn - 1 \cdots q_k + 1
\]
where \( 1 \leq q_1 < q_2 < \cdots < q_k < n \) is any strictly increasing sequence bounded by 1 and \( n \), therefore the \( S^1 \)-fixed points of the Pet\(_4\) are 1234, 2134, 1324, 1243, 2143, 3214, 1432 and 4321. We shall denote the set of these permutations in \( S_n \) by \( U_n \)

Lemma 2.1 Each \( \sigma \in U_n \) is an involution.

Proof. From its configuration, there are only 2-cycles and fixed points.
It turns out that each of the $\sigma \in U_n$ constitutes the longest element in the Young subgroup $S^\sigma_n$ which is represented by

$$S_{\{q_1,q_1-1,\ldots,1\}} \times S_{\{q_2,q_2-1,\ldots,q_1+1\}} \times \cdots \times S_{\{n,n-1,\ldots,q_k+1\}}$$

(2.5)

where the indices are the cardinalities of the partitions of $n$-set with respect to the pattern of $\sigma$ which sum up to $n$, that is

$$|\{q_1,q_1-1,\ldots,1\}| + |\{q_2,q_2-1,\ldots,q_1+1\}| + \cdots + |\{n,n-1,\ldots,q_k+1\}| = n$$

(2.6)

for example, $\sigma = 321549876$ is the longest element in the Young subgroup $S_3 \times S_2 \times S_4$ of $S_9$.

Lemm 3.2 The cardinality of $U_n$ in $S_n$ is given by the ordinary generating function

$$C_n(t) = \frac{t}{1-2t}$$

Proof. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of combinatorial class of all positive integers and let the size of each integer be its value. Its ordinary generating function $N(t) = \frac{t}{1-t}$. From the union and product relations, $C_n$ coincides with the sequence class $\text{SEQ}(N(t))$. So

$$C_n(t) = \frac{1}{1-N(t)} - 1 = \frac{t}{1-2t} = \sum_{n \geq 1} 2^{n-1}t^n$$

Theorem 2.1 There is a bijection $\phi$ between the set of compositions $\mathcal{C}(n)$ of $n$ and the subset $U_n$ of $S_n$ identifying each of the composition $\alpha$ with the permutation $\sigma_\alpha$.

Proof. From (2.4) and (2.6), it is obvious that every permutation in $U_n$ has a unique representation in $\mathcal{C}(n)$ and vice versa.
Example 2.1

<table>
<thead>
<tr>
<th>S/N</th>
<th>C(5)</th>
<th>U_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1,1,1)</td>
<td>12345</td>
</tr>
<tr>
<td>2</td>
<td>(2,1,1,1)</td>
<td>21345</td>
</tr>
<tr>
<td>3</td>
<td>(1,2,1,1)</td>
<td>13245</td>
</tr>
<tr>
<td>4</td>
<td>(1,1,2,1)</td>
<td>12435</td>
</tr>
<tr>
<td>5</td>
<td>(1,1,1,2)</td>
<td>12354</td>
</tr>
<tr>
<td>6</td>
<td>(2,2,1)</td>
<td>21435</td>
</tr>
<tr>
<td>7</td>
<td>(2,1,2)</td>
<td>21354</td>
</tr>
<tr>
<td>8</td>
<td>(1,2,2)</td>
<td>13254</td>
</tr>
<tr>
<td>9</td>
<td>(3,1,1)</td>
<td>32145</td>
</tr>
<tr>
<td>10</td>
<td>(1,3,1)</td>
<td>14325</td>
</tr>
<tr>
<td>11</td>
<td>(1,1,3)</td>
<td>12543</td>
</tr>
<tr>
<td>12</td>
<td>(3,2)</td>
<td>32154</td>
</tr>
<tr>
<td>13</td>
<td>(2,3)</td>
<td>21543</td>
</tr>
<tr>
<td>14</td>
<td>(4,1)</td>
<td>43215</td>
</tr>
<tr>
<td>15</td>
<td>(1,4)</td>
<td>15432</td>
</tr>
<tr>
<td>16</td>
<td>(5)</td>
<td>54321</td>
</tr>
</tbody>
</table>

|**Table 1. The bijection of C(5) and U_5**|

There are several elegant combinatorial interactions between these two sets. For instance, the ascent set $\text{Asc}_\alpha(\sigma)$ and descent set $\text{Des}_\alpha(\sigma)$ of the permutation $\sigma \in U_n$ identified with composition $\alpha$ have elegant description in terms of $\text{Set}(\alpha)$. More precisely, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in C(n)$ then the ascent set is given by

$$\text{Asc}_\alpha(\sigma) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} = \text{Set}(\alpha)$$  \hspace{1cm} (2.7)

and the descent set is given by the complement of $\text{Set}(\alpha)$ in $[n-1]$, that is

$$\text{Des}_\alpha(\sigma) = \{1, 2, 3, \ldots, n-1\} \setminus \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$$  \hspace{1cm} (2.8)

Lemma 2.2 Let $\sigma_\alpha \in U_n$ be the permutation identified with $\alpha$ such that $\alpha \models n$ and let $\text{asc}_\alpha(\sigma)$ and $\text{des}_\alpha(\sigma)$ count the number of elements in $\text{Asc}_\alpha(\sigma)$ and $\text{Des}_\alpha(\sigma)$ respectively. Then

(i.) $\text{asc}_\alpha(\sigma) = \ell^*(\alpha) - 1$

(ii.) $\text{des}_\alpha(\sigma) = n - \ell^*(\alpha)$

(iii.) $\text{des}_\alpha(\sigma) + \text{asc}_\alpha(\sigma) = n - 1$

Proof. (i) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$. Consider the sequence $1, 2, \ldots, \alpha_1, \alpha_1 + 1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 1, \ldots, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} + 1, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} + \alpha_k$ bounded by $1$ and $n$. By applying a function $\sigma \in U_n$ to the sequence and take the ordering, we have $\sigma(1) > \sigma(2) > \cdots > \sigma(\alpha_1)$, $\sigma(\alpha_1 + 1) > \cdots > \sigma(\alpha_1 + \alpha_2)$ and so on. From the configuration of $\sigma$ there is an ascent between $\sigma(\alpha_1)$ and $\sigma(\alpha_1 + 1)$, that is, $\sigma(\alpha_1) < \sigma(\alpha_1 + 1)$ since $\sigma(\alpha_1 + 1) > \sigma(\alpha_1) + \sigma(1)$, so there is an ascent at $\alpha_1$. So also at
\[ \alpha_1 + \alpha_2 + \ldots + \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}. \] Notice that \( \alpha_k \), the last part does not contribute to the ascents since \( \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} + \alpha_k + 1 \) does not exist in the sequence and hence the cardinality of the ascent set is \( \ell^*(\alpha) - 1 \).

(ii) By applying \( \sigma \in U_n \) to the sequence (i) and taking the ordering, a chain of \( n-1 \) links is realized out of which \( \ell^*(\alpha) - 1 \) of them are ascents, since \( \sigma \) is a bijection on \([n]\), so the remaining links are descents and there are \([(n - 1) - (\ell^*(\alpha) - 1)]\) of them.

(iii) Follows from (i) and (ii).

**Remark 2.2** Notice that \( \text{asc}_{\alpha}(\sigma) = \text{asc}(\sigma) \) and \( \text{des}_{\alpha}(\sigma) = \text{des}(\sigma) \)

**Definition 2.1** Two permutations \( \sigma_{\alpha_1}, \sigma_{\alpha_2} \in U_n \) associated to \( \alpha_1 \) and \( \alpha_2 \) respectively such that \( \alpha_2, \alpha_2 |\!\! n \) are said to be equivalent if \( \ell^*(\alpha_1) = \ell^*(\alpha_2) \).

**Proposition 2.1** The property of being equivalent is an equivalence relation.

The implication of Proposition 2.1 is that the equivalence classes in \( U_n \) are characterised by \( \text{des}_{\alpha, \bullet}(-) \) and \( \text{asc}_{\alpha, \bullet}(-) \) and hence each class \( U_{n}^{\ell^*(-)} \) is indexed by its defining composition length \( \ell^*(-) \).

**Corollary 2.1** If two permutations belong to the same class then \( \text{des}_{\alpha, \bullet}(-) = \text{des}_{\alpha, \bullet}(-) \) and \( \text{asc}_{\alpha, \bullet}(-) = \text{asc}_{\alpha, \bullet}(-) \).

The class size of \( U_{n}^{\ell^*(-)} \) denoted by \( |U_{n}^{\ell^*(-)}| \) is given by \( |U_{n}^{\ell^*(-)}| = \binom{n-1}{\ell^*(\alpha)-1} \)

### 3. The Generating Function for \( U_n \)

In this section we give a generating function \( E_n(x) \) of \( U_n \) in \( S_n \) which in principle counts the distribution of ascents \( \text{asc}_{\alpha}(\sigma) \) and descents \( \text{des}_{\alpha}(\sigma) \) of permutations \( \sigma \) in \( U_n \). That is,

\[
E_n(x) = \sum_{\sigma \in U_n} x^{\text{des}_{\alpha}(\sigma)} y^{\text{asc}_{\alpha}(\sigma)}
\]

(3.1)

For convenience, we set \( y = 1 \) so that the first few of these polynomials are

\[
\begin{align*}
E_1(x) &= 1 \\
E_2(x) &= 1 + x \\
E_3(x) &= 1 + 2x + x^2 \\
E_4(x) &= 1 + 3x + 3x^2 + x^3 \\
E_5(x) &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
E_6(x) &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\
&\vdots
\end{align*}
\]

(3.2)
Lemma 3.1 For all $n$, the recurrence relation is given by
\[ E_{n+1}(x) = (1 + x)E_n(x) \]

The goal is to give a closed expression $\sum_{n=1}^{\infty} E_n(x)t^n$ for (3.2). We shall use the technique of ring homomorphism on elementary symmetric functions developed by Remmel and others, see [8], [9] [20] to construct it. This is deeply connected with the transition matrix from the basis $\{h_\lambda : \lambda \vdash n\}$ to the basis $\{e_\mu : \mu \vdash n\}$ of the space of homogeneous symmetric functions of degree $n$.

Theorem 3.1 ([20], Theorem 2.18). The coefficient of $e_\lambda$ in $h_\mu$ is $(-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}|$. In other words
\[ h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda. \]

$B_{\lambda,\mu}$ is the set brick tabloids of content $\lambda$ and shape $\mu$. That is, all the realizable Young diagrams of $\mu$ for which the rows of $\mu$ are partitioned into the bricks of lengths giving the integer partition $\lambda$. This precisely gives a combinatorial interpretation of the entries of the $P_n \times P_n$ transition matrix. These entries are $(-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}|$, the rows are indexed by the content $\lambda$ while the columns are indexed by the shape $\mu$.

Example 3.1 If $n=5$ then the transition matrix is given by the entries inside the table below.

<table>
<thead>
<tr>
<th></th>
<th>(5)</th>
<th>(4, 1)</th>
<th>(3, 2)</th>
<th>(2, 1^2)</th>
<th>(2, 1^3)</th>
<th>(1^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(3, 1^2)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2^2, 1)</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(2, 1^3)</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1^5)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Theorem 3.2 Let $\lambda^1, \lambda^2, \ldots, \lambda^k \vdash n$ be partitions of $n$ such that $\ell(\lambda^1) = \ell(\lambda^2) = \cdots = \ell(\lambda^k) = d$. Then $\sum_{i=1}^{k} |B_{\lambda^i,n}| = \binom{n-1}{n-d}$.

Theorem 3.2 says that the sum of the entries in the first column of the transition matrix for which the partition lengths are the same gives rise to binomial coefficients. This is consistent with the Corollary 2.1 in that the compositions of the same length belong to the same class and the partition length representatives of subclasses within a given class can be added to realise the binomial coefficients. It would be nice to find the interpretation of other
columns in the matrix. We will denote the sum $\sum_{i=1}^{k} | B_{\lambda',n} |$ by $( | B_{\lambda',n} | )_{\ell(\lambda)}^*$ and write each $E_n(x)$ as

$$E_n(x) = \sum_{\ell(\lambda)=1}^{n} ( | B_{\lambda',n} | )_{\ell(\lambda)}^* x^{n-\ell(\lambda)} \quad (3.3)$$

Given a composition $\alpha$ of $n$, there is an associated partition $\lambda(\alpha)$ of $n$ obtained by re-arranging the parts of $\alpha$ in nonincreasing way. The associated partitions to all the compositions of 5 are displayed in the table below.

<table>
<thead>
<tr>
<th>S/N</th>
<th>$C(5)$</th>
<th>$\lambda(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1,1,1,1)</td>
<td>(1,1,1,1,1)</td>
</tr>
<tr>
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<td>(2,1,1,1)</td>
<td>(2,1,1,1)</td>
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<tr>
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<td>(1,2,1,1)</td>
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<td>(1,4)</td>
<td>(4,1)</td>
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<tr>
<td>16</td>
<td>(5)</td>
<td>(5)</td>
</tr>
</tbody>
</table>

**Table 2. Identification $C(5)$ with $\lambda(5)$**

Observe in Table 2 that certain different compositions underline the same partition whose length is called partition length representative $\ell^*(\alpha)$ of the composition $\alpha$ of $n$ coincides with the length $\ell(\lambda(\alpha))$ of the associated partition $\lambda(\alpha)$. This length will henceforth be simply be denoted by $\ell(\lambda)$ . These two simple observations will be important in what follows. The technique of using homomorphism ring $\phi$ on the elementary symmetric functions to find the generating function reduces to counting certain descent pattern distribution in $U_n$.

**Theorem 3.3** Let $\wedge = \oplus \wedge_n$ be the ring of symmetric functions and $\Psi : \wedge \rightarrow \mathbb{Q}(x)$ a ring homomorphism defined by $\Psi(e_n) = (-1)^{n-1}(x+1)^{n-1}$, $n \geq 1$ where $\Psi(e_0) = 1$. Then $\Psi(h_n) = E_n(x)$.

**Proof.** By the expansion of the complete symmetric function $h_n$ in terms of elementary symmetric functions which can be given in terms of brick tabloids
Theorem 3.1, we have
\[ \Psi(h_n) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \mid B_{\mu,n} \mid \Psi(e_{\mu}) \]
\[ = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \mid B_{\mu,(n)} \mid \Psi(e_{\mu_1,\mu_2,\mu_3,\ldots}) \]
\[ = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \mid B_{\mu,(n)} \mid (-1)^{\mu_1-1} (x+1)^{\mu_1-1} (-1)^{\mu_2-1} (x+1)^{\mu_2-1} (-1)^{\mu_3-1} (x+1)^{\mu_3-1} \ldots \]
\[ = \sum_{\mu \vdash (n)} \mid B_{\mu,(n)} \mid (x+1)^{n-\ell(\mu)} \]

The fixed points correspond to the permutations in \( U_n \) with an \( x \) for each descent and hence 
\[ \Psi(h_n) = \sum_{\sigma \in U_n} x^{\text{des}_{\alpha}(\sigma)} = E_n(x). \]  
(3.2)

The above result calls for some explanations. In order to construct a combinatorial object, we use (3.2) to select a partition \( \mu \vdash n \) and a brick tabloid in \( B_{\mu,(n)} \). If the bricks in the brick tabloid are given by the lengths \( k_1, k_2, \ldots, k_d \) then \([n]\) is partitioned into \( d \) subsets of size \( k_1, k_2, \ldots, k_d \) such that the subset \( k_1 \) contains \( 1, 2, \ldots, k_1 \) as elements, the subset \( k_2 \) has elements \( k_1+1, \ldots, k_1+k_2 \) and so on. The elements of these disjoint subsets are arranged in their natural decreasing order within the brick tabloid, that is, each integer from 1 to \( n \) appears once within the bricks. We follow [20] and station a "r" at the last cell of each brick and a choice of "x" or 1 at every other cells. This gives a description of a collection \( \mathcal{X} \) of signed, weighted objects. To each \( Z \in \mathcal{X} \) we associate a weight \( w(Z) \) given by the product of 1s and \( x \)s.

**Example 3.2** For \( n = 5 \), an element \( Z \in \mathcal{X} \) with weight \( (1)^2 x \) is of the form

\[
\begin{array}{ccc}
  x & 1 & r \\
 3 & 2 & 1 & 5 & 4
\end{array}
\]

**Lemma 3.2** The number of weights \( w(Z) \) for any given \( Z \) is \( 2^{n-\ell(\mu)} \).

An involution is defined on \( \mathcal{X} \) by looking at the element \( Z \in \mathcal{X} \) from left to right for the appearance of 1. If not found then \( Z \) is a fixed point, otherwise the brick that contains a 1 is broken into two smaller pieces after the appearance of the 1 and is which is later changed to \( r \). The fixed points on the involution do not contain 1. Therefore the fixed points on this sign reversing and weight reversing involution correspond to the permutations in \( U_n \) since the process changes from appearance of a 1 into a descent.
Theorem 3.4 An ordinary generating function for the sequence \( \{E_n(x)\} \) is given by

\[
\sum_{n=1}^{\infty} E_n(x)t^n = \frac{1}{1+x} \left[ \frac{1}{1 - (1 + x)t} - 1 \right].
\]

Proof.

\[
\sum_{n=1}^{\infty} E_n(x)t^n = \sum_{n=1}^{\infty} t^n \sum_{\ell(\lambda)=1}^{n} \left( B_{\lambda, n}^* \right)^* x^{n - \ell(\lambda)}.
\]

\[
= \frac{t}{1 + x} \left[ \frac{1 - (x + 1)t}{1 - (1 + x)t} - 1 \right].
\]

Acknowledgements. The author would like to thank the Fields Institute and Perimeter Institute for support.

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On the generating function of certain involutions of ...


Received: April 27, 2017; Published: June 1, 2017